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TWO STATIONARITY CONDITIONS IN THE $GI/r_0/M/r_1$ DAM

Abstract. We study the infinite dam in which the content process alternatively decreases or increases continuously at random time-intervals. The output and input rates are content-dependent. We prove the equivalence of two stationarity conditions in the case of exponential output phase and we find the limiting distribution of the content process. We use the methods of investigations of Markov processes.

1. Introduction. We have considered in [4] the dam in which the content process alternatively decreases (output phase) or increases (input phase) continuously at random time-intervals and the output and input rates depend on the state of the process. We have investigated there, among others, the stationarity of the content process and we have obtained two conditions in the case of the exponential output phase. Similar conditions have been obtained by Çinlar [2], Harrison and Resnick [3] and by Brockwell [1] for the dam with a pure-jump input process and with content-dependent output rate. In [1] it was shown that the conditions obtained there for the input process, being a pure-jump Lévy process, are equivalent. In this paper, using the methods of investigations of Markov processes applied in [3], we show that the stationarity conditions obtained in [4] are equivalent. Moreover, we find the limiting distribution of the content process with exponential output phase.

2. Stationarity conditions. Let $(\Omega, \mathscr{F}, P)$ be a probability space, $\mathscr{R}_0 = [0, \infty)$, $\mathscr{R}_+ = (0, \infty)$, let $\mathscr{B}_0$, $\mathscr{B}_+$ stand for the σ-algebras of Borel subsets of these half-lines, respectively, and $\mathrm{b}\mathscr{B}_0$ be the class of real bounded $\mathscr{B}_0$-measurable functions. Introduce the notation Y for a stochastic process $\{Y(t),\ t \geqslant 0\}$, Y for a random sequence $\{Y_n,\ n = 0, 1, \ldots\}$, and I_A for the indicator of a set A.

We give now the construction of the content process in the general model with content-dependent input and output.

Suppose we are given the following:

(a) sequences of times τ_{2m+a}, $m = 0, 1, \ldots$ $(\tau_0 = 0)$, such that the intervals $t_m^a = \tau_{2m+2-a} - \tau_{2m+1-a}$ are positive independent random variables with common distribution function H_a and finite mean $1/\mu_a$ $(a = 1, 0)$; the sequences t^a $(a = 1, 0)$ are also independent;

(b) real functions r_a $(a = 1, 0)$ strictly positive continuous in $\mathscr{R}_+$ and such that $r_1(0) = 0$, $r_0(0) > 0$, r_1 non-decreasing, r_0 non-increasing in $\mathscr{R}_+$,

$$\int_0^x (1/r_1(u))du < \infty \qquad \text{for every } x > 0;$$

(c) the random variable $X_0^1\colon \Omega \to \mathscr{R}_+$.

We call τ_{2m+a} $(m = 0, 1, \ldots; a = 1, 0)$ the *moments of change of the phases* and r_a $(a = 1, 0)$ the *intensity functions.*

We define the content process X for the distances t^a and the intensity functions r_a $(a = 1, 0)$. We assume that the contents $X(t)$ at the time t decrease with intensity $r_1(X(t))$ or alternatively increase with intensity $r_0(X(t))$ with regard to whether the process is in the output or input phase at time t. Thus for fixed a for

$$t \in [\tau_{2m+1-a}, \tau_{2m+2-a}), \qquad m = 0, 1, \ldots,$$

the content process X is given by the functions $q_a = q_a(x, t)$, $x, t \geqslant 0$, fulfilling the differential equation

$$\frac{dq_a}{dt} = (-1)^a r_a(q_a) \tag{1}$$

with the initial condition $q_a(x, 0) = x$. Hence we obtain the solution in the implicit form

$$q_a(x, t) = x + (-1)^a \int_0^t r_a(q_a(x, u))du. \tag{2}$$

With our assumptions the equation (1) can be also solved in an explicit form. Write

$$R_a(x) = \int_0^x (1/r_a(u))du, \qquad a = 1, 0.$$

Then the solutions are of the form

$$\begin{aligned} q_1(x, t) &= \begin{cases} R_1^{-1}(R_1(x) - t) & \text{for } x > 0,\ 0 \leqslant t < R_1(x), \\ 0 & \text{for } x > 0,\ t \geqslant R_1(x), \end{cases} \\ q_0(x, t) &= R_0^{-1}(R_0(x) + t) \qquad \text{for } x, t \geqslant 0. \end{aligned} \tag{3}$$

From the formula (1) for $a = 1$ it is easy to see that the assumption

$$\int_0^x (1/r_1(u))du < \infty$$

means the finiteness of the time to emptiness of the contents x provided we are in the output phase. For $a = 0$ and by the assumption that r_0 is non-increasing, the integral

$$\int_0^x (1/r_0(u))du$$

is finite for every x, which means the finiteness of the time to reach the contents x starting from zero provided we are in the input phase.

Define the chains X^a $(a = 1, 0)$ as follows:

$$\begin{aligned} X_0^0 &= q_1(X_0^1, t_0^1), \\ X_{m+1}^a &= q_{1-a}\big(q_a(X_m^a, t_m^a), t_{m+1-a}^{1-a}\big) \quad \text{for } m = 0, 1, \ldots \end{aligned} \tag{4}$$

Then

$$X(t) = q_a(X_m^a, t-\tau_{2m+1-a}) \tag{5}$$
$$\text{for } \tau_{2m+1-a} \leqslant t < \tau_{2m+2-a},\ a = 1, 0,\ m = 0, 1, \ldots$$

From the construction (5) we see that $X_m^a = X(\tau_{2m+1-a})$, i.e., the chain X^1 gives the contents at the initial times of the output phases, and the chain X^0 gives the contents at the initial times of the input phases. Simultaneously, because the distances t_m^a are independent for $m = 0, 1, \ldots,$ it follows from (4) that X^a $(a = 1, 0)$ are Markov chains.

The content process X is defined by the quadruplet $\{t^0, r_0, t^1, r_1\}$. Under these assumptions we will therefore use the symbol $GI/r_0/GI/r_1$ for this model applying the notation related to Kendall's notation in the theory of queues.

We introduce two conditions obtained in [4]. Let $\mathscr{F}_0$ be the σ-algebra generated by the random variable X_0^1, let P_x be the conditional probability $P\{\cdot|\mathscr{F}_0\}$ on the set $\{X_0^1 = x\}$, and let E_x be the appropriate expectation. Next, let Q^a be the operators defined by the transition probabilities for the Markov chains X^a $(a = 1, 0)$. Thus from (4) for $f \in \mathrm{b}\mathscr{B}_0$ we have

$$Q^a f(x) = \int_0^\infty \int_0^\infty f\big(q_{1-a}(q_a(x, u), v)\big)dH_a(u)dH_{1-a}(v), \qquad a = 1, 0. \tag{6}$$

Write

$$\beta_0 = \inf_{x \geqslant 0} r_0(x), \qquad \beta_1 = \sup_{x \geqslant 0} r_1(x)$$

(β_1 may be infinite).

THEOREM 1. *In the model* $GI/r_0/GI/r_1$ *the condition*

$$\beta_0/\mu_0 < \beta_1/\mu_1 \tag{i}$$

is sufficient for the Markov chains X^a *to have the invariant probability measures* N_a^+, $a = 1, 0$.

For the completeness of the paper we quote the proof of this theorem.

Proof. We prove the existence of the invariant measure N_0^+. Then the existence of N_1^+ follows from (4).

Choose c_0 such that $\beta_0 < c_0 < \beta_1 \mu_0/\mu_1$ and put

$$\mathscr{D}_0 = \{x\colon r_0(x) < c_0\}.$$

Since r_0 is positive continuous non-increasing, the set $\mathscr{D}_0$ is non-empty and it is of the form (d_0, ∞). In addition, if $x \in \mathscr{D}_0$, then from (1) for $a = 0$ we have

$$q_0(x, t) < x + c_0 t, \quad t > 0. \tag{7}$$

Choose c_1 such that $c_0\mu_1/\mu_0 < c_1 < \beta_1$ and put

$$\mathscr{D}_1 = \{x\colon r_1(x) > c_1\}.$$

The set $\mathscr{D}_1$ is also non-empty and it is of the form (d_1, ∞). In addition, if $q_1(x, t) \in \mathscr{D}_1$, then from (1) for $a = 1$ we have

$$q_1(x, t) < x + c_1 t, \quad t > 0. \tag{8}$$

Consider the set $\mathscr{D} = \mathscr{D}_0 \cap \mathscr{D}_1$. If $d = \max(d_0, d_1)$, then $\mathscr{D} = (d, \infty)$. Let us define the chain S by the formula

$$S_0 = X_0^0, \quad S_{m+1} = S_m + c_0 t_m^0 - c_1 t_{m+1}^1, \ m = 0, 1, \ldots$$

Let also

$$U = \min_{m \geq 0}\{m\colon S_m \notin \mathscr{D}\}, \quad V = \min_{m \geq 0}\{m\colon X_m^0 \notin \mathscr{D}\}.$$

It follows from (7) and (8) that for fixed $\omega \in \Omega$ we have: if

$$X_0^0, X_1^0, \ldots, X_m^0 > d,$$

then $X_1^1, X_2^1, \ldots, X_{m+1}^1 > d$, and

$$X_m^0 < X_{m-1}^0 + c_0 t_{m-1}^0 - c_1 t_m^1.$$

Hence, if $X_0^0, X_1^0, \ldots, X_m^0 > d$, then $S_m > d$ and we obtain

$$\begin{aligned} \{V > m\} &= \{X_0^0 > d, X_1^0 > d, \ldots, X_m^0 > d\} \\ &\subset \{S_0 > d, S_1 > d, \ldots, S_m > d\} = \{U > m\}. \end{aligned} \tag{9}$$

Since the random variables t_m^a ($a = 1, 0$, $m = 0, 1, \ldots$) are independent, the chain S forms a random walk. The constants c_a ($a = 1, 0$) have been chosen such that

$$\mathrm{E}_x\{S_{m+1} - S_m\} = c_0/\mu_0 - c_1/\mu_1 < 0.$$

Therefore, the random walk S drifts to $-\infty$ and for each $x \geqslant 0$ we have $\mathrm{E}_x\{U\} < \infty$. Define the operator $\mathscr{G}$ such that

$$\mathscr{G}f(x) = Q^0(I_{\mathscr{D}} f)(x) \quad \text{for each } f \in \mathrm{b}\mathscr{B}_0.$$

From (9) we obtain

$$\sum_{n=0}^{\infty} \mathscr{G}^n 1(x) = \mathrm{E}_x\{V\} = \sum_{n=0}^{\infty} P_x\{V > n\} \leqslant \sum_{n=0}^{\infty} P_x\{U > n\} = \mathrm{E}_x\{U\}.$$

Thus

$$\sum_{n=0}^{\infty} \mathscr{G}^n 1(x) < \infty \quad \text{for each } x \geqslant 0.$$

Now the same arguments as in the proof of Theorem (4.4) in [2] allow to prove the existence and finiteness of N_0^+.

Define the processes Y and α by the formulas

(10) $$Y(t) = \tau_{2m+1+a} - t, \quad \alpha(t) = a$$
$$\text{for } t \in [\tau_{2m+a}, \tau_{2m+1+a}),\ a = 1, 0,\ m = 0, 1, \ldots$$

Y is called a *residual-time process*, and α is called an *alternating process*.

One obtains the second condition for the model $GI/r_0/M/r_1$ investigating the extended Markov process

$$\bar{X}(t) = [X(t), Y(t), \alpha(t)], \quad t \geqslant 0,$$

valued in $\bar{\mathscr{R}} = \mathscr{R}_0 \times \mathscr{R}_+ \times \{0, 1\}$. We assume that $\bar{X}$ has the unique stationary distribution $\bar{N}$ and that the Markov chains X^a have the unique invariant probability measures N_a^+. Putting

$$N_a(A) = \bar{N}(A \times \mathscr{R}_+ \times \{a\}),$$
$$N_a(x) = N_a([0, x]), \quad N_a^+(x) = N_a^+([0, x]),$$

we assume moreover that there exist derivatives

$$n_a(x) = \frac{d}{dx} N_a(x) \quad \text{and} \quad n_a^+(x) = \frac{d}{dx} N_a^+(x) \quad \text{for } x > 0.$$

Before we formulate the theorem introducing the second condition, define the functions $K^{*n} = K^{*n}(x, y)$, $0 \leqslant y < x$, $n = 1, 2, \ldots$, by the recurrent formulas

(11) $$K^{*1}(x, y) = \mu_1\big(1 - H_0(R_0(x) - R_0(y))\big)/r_1(x),$$

(12) $$K^{*(n+1)}(x, y) = \int_y^x K(x, u) K^{*n}(u, y)\, du, \quad n = 1, 2, \ldots,$$

where $K = K^{*1}$.

It is easy to verify by mathematical induction that the iterates K^{*n} can be evaluated as follows:

$$K^{*(n+1)}(x, y) \leqslant \mu_1^{n+1}\big(R_1(x)-R_1(y)\big)^n/\big(r_1(x)n!\big), \quad 0 \leqslant y < x, \; n = 0, 1, \ldots \tag{13}$$

Hence

$$\sum_{n=1}^{\infty} K^{*n}(x, y) \leqslant \mu_1 \exp\{\mu_1\big(R_1(x)-R_1(y)\big)\}/r_1(x). \tag{14}$$

From (14) we define the function $K^* = K^*(x, y)$, $0 \leqslant y < x$, by the formula

$$K^*(x, y) = \sum_{n=1}^{\infty} K^{*n}(x, y). \tag{15}$$

Putting

$$v^{-1} = 1/\mu_0 + 1/\mu_1, \qquad k = 1/\big(1+\int_0^{\infty} K^*(u, 0)du\big)$$

we formulate the second stationarity condition.

THEOREM 2. *In the model $GI/r_0/M/r_1$ the condition*

$$\int_0^{\infty} K^*(u, 0)du < \infty \tag{ii}$$

is necessary and sufficient for the equation

$$n_0^+(x) = N_0^+(0)K(x, 0) + \int_0^x K(x, u)n_0^+(u)du, \qquad x > 0, \tag{16}$$

to have a unique solution of the form

$$N_0^+(x) = k\big(1+\int_0^x K^*(u, 0)du\big), \qquad x \geqslant 0. \tag{17}$$

Moreover, the distributions N_1^+, N_0, N_1 are then of the form

$$N_1^+(x) = \int_0^x H_0\big(R_0(x)-R_0(u)\big)dN_0^+(u), \tag{18}$$

$$N_0(x) = (v/\mu_1)N_0^+(x), \tag{19}$$

$$N_1(x) = (v/\mu_0)\int_0^x \hat{H}_0\big(R_0(x)-R_0(u)\big)dN_0^+(u), \tag{20}$$

where $\hat{H}_0(x) = \mu_0 \int_0^x \big(1-H_0(u)\big)du, \; x \geqslant 0.$

Theorem 2 is a direct consequence of Theorems 5–7 contained in [4]. Here we give a sketch of the proof of the theorem. It is based on the paper [5] in which one considers extended piecewise Markov processes with general state space. The process X is a simple example of such a process with regenerative moments τ_{2m+a} and with continuous trajectories. On the interval $[\tau_{2m+a}, \tau_{2m+a+1})$ (m, a fixed) it is a Markov process with transition probabilities P_a of the form

(21) $$P_a(t, x, A) = I_A(q_{1-a}(x, t)), \quad t, x \geqslant 0, \ A \in \mathscr{B}_0.$$

Applying Theorems 2 and 3 from [5] we obtain the following relations between the distributions N_a and N_a^+:

(22) $$N_{1-a}(A) = v \int_{0-}^{\infty} N_a^+(du) \int_{0}^{\infty} P_{1-a}(t, u, A)(1-H_a(t))dt,$$

(23) $$r_{1-a}(x) n_a(x) = v(N_0^+(x) - N_1^+(x)), \quad x > 0, \ A \in \mathscr{B}_0, \ a = 1, 0.$$

To verify the assumptions of these theorems, it suffices to show that for each function f on $\mathscr{R}_0$, real, continuous, vanishing at infinity and for $a = 1, 0$ the following holds:

(24) $$\lim_{t \downarrow 0} \sup_{x \geqslant 0} \Big| \int_{0-}^{\infty} f(u) P_a(t, x, du) - f(x) \Big| = 0.$$

By (21) we have

$$\int_{0-}^{\infty} f(u) P_a(t, x, du) = f(q_{1-a}(x, t)).$$

Simultaneously from equation (1) we obtain

$$\lim_{t \downarrow 0} q_a(x, t) = x,$$

and so

$$\lim_{t \downarrow 0} f(q_a(x, t)) = f(x)$$

uniformly on $\mathscr{R}_0$, which proves (24). Relation (22) follows from Theorem 2 (b) in [5] and relation (23) follows from Theorem 3 in [5] after the calculation of the value of the infinitesimal operator $\mathscr{A}(P_a)$ induced by transition probabilities P_a, namely

$$\mathscr{A}(P_a) N_a((x, \infty)) = (-1)^a r_{1-a}(x) \frac{d}{dx} N_a((x, \infty)).$$

One can transform relations (22) and (23) substituting (21) into (22) and using (4). For our purpose it suffices to remark that

$$N_1^+(x) = P\{q_0(X_m^0, t_m^0) \leqslant x\} = \int_{0-}^{x} H_0(R_0(x) - R_0(u)) dN_0^+(u)$$

and to substitute this into (23) for $a=0$. Hence we obtain

$$n_0(x) = v\int_{0-}^{\infty} \big(1-H_0(R_0(x)-R_0(u))\big)/r_1(x)\,dN_0^+(u), \quad x>0. \tag{25}$$

The next step is to show that relations (22) become simple when one assumes that one of the distribution functions H_a is exponential. For fixed a, if H_a is exponential, then

$$N_{1-a}(x) = (v/\mu_a)N_{1-a}^+(x), \quad x \geqslant 0. \tag{26}$$

In the proof of this step one uses the relation known in the theory of semi-groups of contraction operators $\mathscr{R}_\lambda(P_a) = \big(\lambda I - \mathscr{A}(P_a)\big)^{-1}$, where I is the identity operator and $\mathscr{R}_\lambda(P_a)$, $\lambda>0$, is the resolvent induced by the transition probabilities P_a.

In our case the distribution function H_1 is exponential, so we substitute (26) for $a=1$ into (25) and we obtain (16) for N_0^+. Iterating (16) $n-1$ times we have

$$n_0^+(x) = N_0^+(0)\sum_{j=1}^{n} K^{*j}(x,0) + \int_0^x K^{*n}(x,u)n_0^+(u)\,du, \quad x>0. \tag{27}$$

Letting $n\to\infty$ in (27), using (13) and the bounded convergence theorem we establish that the unique non-zero solution of (16) is $n_0^+(x) = N_0^+(0)K^*(x,0)$, which is a density iff condition (ii) is fulfilled. Hence we have (17) and from (27) we have also (19). Formula (18) follows from (17) and (4), whereas we obtain formula (20) substituting (17) and (18) into (23) for $a=1$ and integrating. It is easy to verify that $N_1(\infty)+N_0(\infty)=1$.

3. Limiting distribution. In the former section we have defined the content process X in the model $GI/r_0/GI/r_1$. Now we consider two other auxiliary models. Let the following be given for $i=0,1$:

(1) a sequence of moments $\boldsymbol{\sigma}^i$ ($\sigma_0^i=0$) such that the distances $s_n^i = \sigma_{n+1}^i - \sigma_n^i$ ($n=0,1,\ldots$) are positive independent random variables with common distribution function H_{1-i} and finite mean $1/\mu_{1-i}$;

(2) a sequence $\boldsymbol{S}^i$ of positive independent random variables with common distribution function H_i and finite mean $1/\mu_i$;

(3) real functions r_a ($a=1,0$) with properties such as in the model $GI/r_0/GI/r_1$.

We assume that in the first model ($i=0$) the contents $Z^0(t)$ at the moment $t\in(\sigma_n^0, \sigma_{n+1}^0)$, $n=0,1,\ldots$, decrease with intensity $r_1(Z^0(t))$ whereas σ_n^0 are the moments of jumps up of the form $q_0(x, S_n^0)$, where x is the contents just before the jump. Analogously, in the second model ($i=1$) the contents $Z^1(t)$ at the moment $t\in(\sigma_n^1, \sigma_{n+1}^1)$, $n=0,1,\ldots$, increase with intensity $r_0(Z^1(t))$ whereas σ_n^1 are the moments of jumps down of the form $q_1(x, S_n^1)$, where x is the contents just before the jump. In accordance with this description we construct, for fixed i, the content process Z^i as follows: let $\boldsymbol{Z}^i$ be the chain defined by the formula

$$Z_{n+1}^i = q_{1-i}\big(q_i(Z_n^i, S_n^i), s_{n+1}^i\big), \quad n=0,1,\ldots, \tag{28}$$

where we take $Z_0^1 = X_0^1$, $Z_0^0 = q_1(X_0^1, s_0^0)$, respectively, and the functions q_a, $a = 1, 0$, are defined by formulas (1)–(3). Thus we have

(29) $$Z^i(t) = q_{1-i}(Z_n^{1-i}, t-\sigma_n^i) \quad \text{for } t \in [\sigma_n^i, \sigma_{n+1}^i),\ n = 0, 1, \ldots,$$

where $Z_n^{1-i} = q_i(Z_n^i, S_{n-1+i}^i)$ for $n = 1, 2, \ldots$ and we take $Z_0^1 = X_0^1$, $Z_0^0 = q_1(X_0^1, s_0^1)$, respectively.

One can see from this construction that $\mathbf{Z}^i$ are Markov chains and they give the contents just before the jumps of the processes Z^i. The content process Z^0 is defined by the quadruplet $\{S^0, r_0, s^0, r_1\}$. With our assumptions we will use the notation $GI^{r_0}/GI/r_1$ for this model. The content process Z^1 is defined by the quadruplet $\{S^1, r_1, s^1, r_0\}$ and we will use the symbol $GI^{r_1}/GI/r_0$ for this model.

Comparing the definitions (4) of the Markov chains $\mathbf{X}^a$, $a = 1, 0$, in the model $GI/r_0/GI/r_1$ with the definitions (28) of the Markov chains $\mathbf{Z}^i$ in the models $GI^{r_i}/GI/r_{1-i}$, $i = 0, 1$, we obtain the equalities

(30) $$P_x\{X_n^i \in A\} = P_x\{Z_n^i \in A\}, \quad x \in \mathscr{R}_+,\ A \in \mathscr{B}_0,\ i = 0, 1.$$

Similarly, comparing the construction (5) of the process X with the constructions (29) of the processes Z^i, $i = 0, 1$, we obtain the equalities

(31) $$P_x\{X(t) \in A \mid \alpha(t) = i\} = P_x\{Z^i(t) \in A\},$$
$$x \in \mathscr{R}_+,\ A \in \mathscr{B}_0,\ i = 0, 1,\ t > 0,$$

where α is the alternating process defined in (10). Equalities (30) and (31) will be useful to find the limiting distribution of the process X. First we show a lemma concerning the limiting distributions of the processes Z^i, $i = 0, 1$.

LEMMA 1. *For every i ($i = 0, 1$), if the Markov chain $\mathbf{Z}^{1-i}$ has the unique invariant probability measure N_{1-i}^+, then for every $x \in \mathscr{R}_+$ and $A \in \mathscr{B}_0$ we have*

(32) $$\lim_{t\to\infty} P_x\{Z^i(t) \in A\} = \mu_{1-i} \int_{0-}^{\infty} N_{1-i}^+(dz) \int_0^{\infty} (1-H_{1-i}(u)) I_A(q_{1-i}(z, u))du.$$

Proof. The proof proceeds analogously to the proofs of Lemma (4.12) and Theorem (4.14) in [2] where one assumed a condition analogous to condition (i). In our model $GI/r_0/GI/r_1$, condition (i) is sufficient for the existence of invariant probability measures of the Markov chains $\mathbf{X}^a$, $a = 1, 0$, but we have no proof for it to be the sufficient condition for the uniqueness of these measures. Hence the assumption of the lemma. The renewal process defined in Lemma (4.12) in [2] is here the process

$$M_i(t) = \sup\{n:\ \sigma_n^i \leqslant t\}$$

and the Markov process is the process

$$[Z_{M_i(t)}^{1-i}, t-\sigma_{M_i(t)}^i], \quad t \geqslant 0.$$

The assumptions of the theorem due to Orey (see Theorem 4 in [11] cited in [2]) are satisfied for the state space $\mathscr{R}_0$ and by the assumption of the lemma the integral (0.1) is equal to

$$\int_{0-}^{\infty} N_{1-i}^{+}(dy) \int_{0}^{\infty} u dH_{1-i}(u) = 1/\mu_{1-i}$$

and, by (1) at the beginning of Section 3, it is finite. Thus, as in Lemma (4.12), for any open sets $A \in \mathscr{B}_0$, $B \in \mathscr{B}_+$ and for any $x \in \mathscr{R}_+$ we obtain the following:

$$\lim_{t \to \infty} P_x\{Z_{M_i(t)}^{1-i} \in A,\ t - \sigma_{M_i(t)}^{i} \in B\} = \mu_{1-i} N_{1-i}^{+}(A) \int_{B} (1 - H_{1-i}(u)) du. \tag{33}$$

Next, analogously to Theorem (4.14) in [2], the process Z^i can be presented in the form

$$Z^i(t) = q_{1-i}(Z_{M_i(t)}^{1-i},\ t - \sigma_{M_i(t)}^{i})$$

(see formula (29)). Hence and from (33) we obtain (32).

By Lemma 1 and equalities (30) and (31) we have the following limiting theorem:

THEOREM 3. *In the model $GI/r_0/GI/r_1$, let the Markov chains X^a have the unique invariant probability measures N_a^+ $(a = 1, 0)$ and let the distributions H_a be aperiodic. Then for every $x \in \mathscr{R}_+$ and $y \in \mathscr{R}_0$ we have*

$$\lim_{t \to \infty} P_x\{X(t) \leqslant y\} = v\Big[\int_{0}^{\infty} N_1^{+}(dz) \int_{\max(0, R_1(z) - R_1(y))}^{\infty} (1 - H_1(u) du) + \int_{0-}^{y} N_0^{+}(dz) \int_{0}^{R_0(y) - R_0(z)} (1 - H_0(u)) du\Big]. \tag{34}$$

Proof. From equalities (30)–(32) and from the main renewal theorem we have

$$\lim_{t \to \infty} P_x\{X(t) \in A\} = v\Big[\int_{0}^{\infty} N_1^{+}(dz) \int_{0}^{\infty} (1 - H_1(u)) I_A(q_1(z, u)) du + \int_{0-}^{\infty} N_0^{+}(dz) \int_{0}^{\infty} (1 - H_0(u)) I_A(q_0(z, u)) du\Big].$$

Hence we obtain directly (34).

The proof of Theorem 3 will be used to show that, in the model $GI/r_0/M/r_1$, condition (i) (also condition (ii)) is necessary and sufficient for the existence of the limiting distribution of the process X and we will find the explicit form of this distribution. For this purpose in the next part we prove the equivalence of (i) and (ii).

4. Equivalence of the conditions. Consider the model $GI/r_0/M/r_1$. Condition (ii) obtained in Theorem 2 is difficult to verify. Simultaneously, we have here also condition (i). Thus the question arises what is the relation between them. The next theorem contains the answer.

THEOREM 4. *In the model* $GI/r_0/M/r_1$, *condition* (i) *is equivalent to condition* (ii).

We divide the proof of Theorem 4 into 5 lemmas. The first of them contains the implication (i)$\Rightarrow$(ii). Next we consider the content process Z^0 in the model $GI/r^0/M/r_1$. By the assumption that the distances s^0 are exponential, the process Z^0 is a Markov process. In Lemmas 3 and 4 we calculate the generator of this process and in Lemma 5 we show that condition (ii) is necessary and sufficient for the process Z^0 to have a unique stationary distribution identical with the distribution N_0^+. In the end, we prove in Lemma 6 that condition (i) is necessary for the existence of a stationary distribution of the process Z^0.

LEMMA 2. *Condition* (i) *implies condition* (ii).

Proof. Let f_n $(n = 1, 2, \ldots)$ be the sequence of densities defined recursively by the formulas

$$f_1(x) = \mu_0(1-H_0(x)),$$
$$f_{n+1}(x) = \int_0^x f_1(x-u)f_n(u)du, \qquad x \geqslant 0,\ n = 1, 2, \ldots$$

By condition (i), definitions (11), (12) and with r_1 assumed to be non-decreasing and r_0 non-increasing, one can verify, using mathematical induction, that there exist x_0 and ϱ_0 $(0 \leqslant x_0 < \infty,\ 0 < \varrho_0 < 1)$ such that for $x_0 \leqslant y < x$ we have

$$K^{*n}(x, y) \leqslant \varrho_0^n f_n(R_0(x)-R_0(y))/r_0(x), \qquad n = 1, 2, \ldots \tag{35}$$

Taking into account definition (15) and dealing identically as in [3] (see p. 357, formulas (30)–(32)), we obtain the identity

$$K^*(x, 0) = k(x)+\int_{x_0}^{x} K^*(x, u)k(u)du, \qquad x_0 \leqslant x < \infty, \tag{36}$$

$$\text{where } k(x) = K(x, 0)+\int_0^{x_0} K(x, u)K^*(u, 0)du.$$

Identity (36) gives us, by Fubini's Theorem, the following:

$$\int_{x_0}^{\infty} K^*(u, 0)du = \int_{x_0}^{\infty} k(u)du+\int_{x_0}^{\infty} k(u)\Big(\int_u^{\infty} K^*(v, u)dv\Big)du. \tag{37}$$

For the estimation of the integrals occurring on the right-hand side of (37) we use (35) and we calculate

$$\int_{x}^{\infty} K^*(u, x)du \leqslant \varrho_0/(1-\varrho_0), \qquad x \geqslant x_0, \tag{38}$$

$$\int_{x_0}^{\infty} k(u)du \leqslant \varrho_0 \int_{R_0(x_0)}^{\infty} f_1(u)du + \varrho_0 \int_{0}^{x_0} K^*(v, 0)\Big(\int_{R_0(x_0)-R_0(v)}^{\infty} f_1(u)du\Big)dv \tag{39}$$

$$\leqslant \varrho_0 + \varrho_0 \int_{0}^{x_0} K^*(u, 0)du.$$

Combining (37)–(39) we complete the proof because the function $K^*(\cdot, 0)$ is integrable in the finite interval (see (14)).

Remark 1. It follows from the proof of Lemma 2 that instead of (i) one can assume the condition

$$\mu_1 r_0(x)/(\mu_0 r_1(x)) \leqslant \varrho_0 \quad \text{for } 0 \leqslant x_0 \leqslant x \text{ and } 0 < \varrho_0 < 1$$

without assuming the monotonicity of r_0 and r_1.

Now consider the model $GI^{r_0}/M/r_1$. The content process Z^0 constructed by (29) for $i = 0$ is a Markov process under the assumption that the distances s^0 have a common exponential distribution with parameter μ_1. In this part we omit, for convenience, the indicator 0 and we write: the process Z, the distances s, and so on. The paths of Z are right-continuous and the moments σ are its Markov moments. One can compare this process with the content process considered in [3]. The last one is also a Markov process with right-continuous paths. Namely, at the Poisson jump times the process has jumps forming a sequence of positive i.i.d. random variables. Between the jump times the process decreases with intensity dependent on the state of the process. The difference between the content process in [3] and the process Z is such that jumps of Z are not independent because they depend on the contents just before jumps. But if we take $r_0(x) = 1$, $x \geqslant 0$, then the model $GI^1/M/r_1$ is identical with the model considered in [3]. Therefore, we use the methods of investigation of Markov processes with the convergence introduced in [3] (bounded pointwise convergence) to show that condition (ii) is necessary and sufficient for the existence and the uniqueness of the stationary distribution of Z.

If γ is a probability distribution on $(\mathscr{R}_0, \mathscr{B}_0)$, then let

$$P_\gamma(\cdot) = \int_{\mathscr{R}_0} P_x(\cdot)\gamma(dx), \qquad E_\gamma(\cdot) = \int_{\mathscr{R}_0} E_x(\cdot)\gamma(dx).$$

For each $f \in b\mathscr{B}_0$ let $\varphi_t(x) = E_x\{f(Z(t))\}$. Further, let $\mathscr{L}$ and $\mathscr{D}$ be the classes of functions defined as follows:

$$\mathscr{L} = \{f \in \mathrm{b}\mathscr{B}_0\colon \lim_{t\downarrow 0} \varphi_t(x) = f(x) \text{ for each } x \geqslant 0\},$$

$$\mathscr{D} = \{f \in \mathscr{L}\colon \lim_{t\downarrow 0} (\varphi_t(x) - f(x))/t \text{ exists by bounded pointwise convergence on } \mathscr{R}_0 \text{ and belongs to } \mathscr{L}\}.$$

Bounded pointwise convergence on $\mathscr{R}_0$ means that $(\varphi_t(x) - f(x))/t$ is bounded on $\mathscr{R}_0$. If $f \in \mathscr{D}$, then the operator $\mathscr{A}$ defined by the formula

$$\mathscr{A}f(x) = \lim_{t\downarrow 0} (\varphi_t(x) - f(x))/t$$

is called a *generator* (*infinitesimal operator*).

Finally, for each $\alpha > 0$ and $f \in \mathrm{b}\mathscr{B}_0$ we define the operator $\mathscr{R}_\alpha$, called a *resolvent*, by

$$\mathscr{R}_\alpha f(x) = \int_0^\infty \exp\{-\alpha t\} \varphi_t(x) dt.$$

Remark that from the construction (29) for $i = 0$ and from the strong Markov property of the process Z we obtain the next auxiliary form of the function φ_t:

$$\varphi_t(x) = \int_0^t \mu_1 \exp\{-\mu_1 u\} \int_0^\infty \varphi_{t-u}\big(q_0(q_1(x, u), v)\big) dH_0(v) du, \qquad x \geqslant 0. \tag{40}$$

The next two lemmas contain the characterization of the domain $\mathscr{D}$ of the operator $\mathscr{A}$.

LEMMA 3. *The class $\mathscr{L}$ consists of all left-continuous functions. Every $f \in \mathscr{D}$ is absolutely continuous on $\mathscr{R}_+$ with a left-continuous density.*

Proof. From (40) it follows that

$$\varphi_t(x) = \exp\{-\mu_1 t\} f(q_1(x, t)) + o(1)$$

as $t \to 0$ for $f \in \mathrm{b}\mathscr{B}_0$. Hence we obtain the characterization of $\mathscr{L}$ because $q_1(x, t) \to x$ continuously as $t \to 0$ (see (2) for $a = 1$).

It is known (see, e.g., [3], Proposition 2) that for every $\alpha > 0$ we have $\mathscr{R}_\alpha \mathscr{L} = \mathscr{D}$. Thus, if $f \in \mathscr{D}$, then there exists a function $h \in \mathscr{L}$ such that $f = \mathscr{R}_\alpha h$. Hence by the construction (29) for $i = 0$ of the process Z and from its strong Markov property we obtain

$$\begin{aligned} f(x) &= \mathrm{E}_x\Big\{\int_0^{\sigma_1} e^{-\alpha t} h(q_1(x, t)) dt\Big\} + \mathrm{E}_x\Big\{\exp\{-\alpha\sigma_1\} \int_0^\infty e^{-\alpha u} h(Z(\sigma_1 + u)) du\Big\} \\ &= \mathrm{E}_x\Big\{\int_0^{\sigma_1} e^{-\alpha t} h(q_1(x, t)) dt\Big\} + \mathrm{E}_x\Big\{\exp\{-\alpha\sigma_1\} f\big(q_0(q_1(x, \sigma_1), S_0)\big)\Big\}. \end{aligned}$$

This formula is analogous to formula (6) in [3].

Dealing further as in the proof of Proposition 3 in [3], i.e., taking into account that σ_1 is exponential and that $q_1(x, t) = 0$ for $t \geqslant R_1(x)$, we calculate

$$f(x) = \int_0^x \exp\{-(\alpha+\mu_1)(R_1(x)-R_1(u))\}(h(u)+\mu_1 W(u))(r_1(u))^{-1}du$$
$$+(h(0)+\mu_1 W(0))\exp\{-(\alpha+\mu_1)R_1(x)\}(\alpha+\mu_1)^{-1},$$

where

$$W(x) = \int_0^\infty f(q_0(x, u))dH_0(u).$$

It follows from this form of the function f that f is absolutely continuous because R_1 is absolutely continuous and h, W are bounded. The integrand is left-continuous because h is left-continuous ($h \in \mathscr{L}$) and W, R_1, r_1 are continuous on $\mathscr{R}_+$. Hence f has a left-continuous density.

Remark 2. It follows from the proof of Lemma 3 that one can assume only that r_1 is left-continuous and has strictly positive right limits in $\mathscr{R}_+$. Such an assumption has been taken in [3].

LEMMA 4. *The domain $\mathscr{D}$ consists of all absolutely continuous functions f on $\mathscr{R}_+$ having a left-continuous density f' such that $r_1 f'$ is bounded on $\mathscr{R}_+$. Furthermore, if $f \in \mathscr{D}$, then*

$$\mathscr{A}f(0) = \mu_1 \int_0^\infty \big(f(q_0(0, u))-f(0)\big)dH_0(u), \tag{41}$$

$$\mathscr{A}f(x) = \mu_1 \int_0^\infty \big(f(q_0(x, u))-f(x)\big)dH_0(u)-r_1(x)f'(x), \quad x > 0, \tag{42}$$

Proof. The proof is analogous to the proof of Proposition 4 in [3]. So we emphasize only the differences following from the modification of the model.

By Lemma 3 we assume that f is absolutely continuous on $\mathscr{R}_+$ with a left-continuous density f'. Introduce the notation

$$a_t(x) = \int_0^t \mu_1 \exp\{-\mu_1 u\} \int_0^\infty \varphi_{t-u}\big(q_0(q_1(x, u), v)\big)dH_0(v)du,$$

$$b_t(x) = \exp\{-\mu_1 t\} f\big(q_1(x, t)\big),$$

$$c_t(x) = f(x)-f\big(q_1(x, t)\big),$$

$$d_t(x) = (1-\exp\{-\mu_1 t\}) f\big(q_1(x, t)\big).$$

Hence we can write (40) in the form

$$\varphi_t(x) = a_t(x)+b_t(x). \tag{43}$$

Comparing (43) and (40) with formula (8) in [3] we see that the only difference is the appearance of the function q_0 in the integrand. Since $q_0(\cdot, v)$ is continuous (see (2) for $a = 0$), the fact of the bounded pointwise convergence of $a_t(x)/t$ $(t \downarrow 0)$ does not change, i.e., we have

$$\lim_{t \downarrow 0} (a_t(x)/t) = \mu_1 \int_0^\infty f(q_0(x, v)) dH_0(v), \quad x \in \mathscr{R}_0. \tag{44}$$

By (43) and the notation, we can present the difference $\varphi_t(x) - f(x)$ in the form

$$\varphi_t(x) - f(x) = a_t(x) - c_t(x) - d_t(x). \tag{45}$$

The expression $b_t(x)$ in (43) is identical with the expression $b_t(x)$ in formula (8) of [3] (substituting $\lambda = \mu_1$, $q(x, t) = q_1(x, t)$). Thus, further parts of the proof of Lemma 4 and Proposition 4 in [3] are also identical and we obtain

$$\lim_{t \downarrow 0} (d_t(x)/t) = \mu_1 f(x), \quad x \in \mathscr{R}_0, \tag{46}$$

where the convergence is bounded pointwise, and

$$\lim_{t \downarrow 0} (c_t(x)/t) = r_1(x) f'(x), \quad x \in \mathscr{R}_+, \tag{47}$$

where the convergence is bounded pointwise iff $r_1 f'$ is bounded.

Taking into account formula (45) and convergences (44), (46), (47) we conclude that $f \in \mathscr{D}$ iff the function $r_1 f'$ is bounded. Formulas (41) and (42) follow from (44)–(47) and from the fact that $c_t(0) = 0$.

COROLLARY. *Let $\mathscr{D}_+$ denote the class of $f \in \mathscr{D}$ having a non-negative density f'. Then for $f \in \mathscr{D}_+$ we have*

$$\mathscr{A} f(0) = \mu_1 \int_0^\infty \big(1 - H_0(R_0(u))\big) f'(u) du, \tag{48}$$

$$\mathscr{A} f(x) = \mu_1 \int_x^\infty \big(1 - H_0(R_0(u) - R_0(x))\big) f'(u) du - r_1(x) f'(x), \quad x \in \mathscr{R}_+. \tag{49}$$

The Corollary follows immediately from (41), (42) and from the definition of $\mathscr{D}_+$.

LEMMA 5. *Condition* (ii) *is necessary and sufficient for the process Z to have a stationary distribution γ. Then the density g on $\mathscr{R}_+$ of this distribution fulfills the equation*

$$g(x) = \gamma(0) K(x, 0) + \int_0^x K(x, u) g(u) du, \quad x > 0. \tag{50}$$

Equation (50) *has a unique solution of the form*

$$g(x) = k K^*(x, 0), \quad x > 0. \tag{51}$$

Proof. We use the proofs of Propositions 5, 6 and Theorem 1 from [3]. Taking the arguments from the theory of Markov processes contained in Proposition 5 from [3] we establish that Z has a stationary distribution γ iff

$$\int_{\mathscr{R}_0} \mathscr{A}f(x)\gamma(dx) = 0 \quad \text{for all } f \in \mathscr{D}, \tag{52}$$

which is equivalent to

$$0 = \mathrm{E}_\gamma h(Z(0)) - \alpha \int_0^\infty e^{-\alpha t}\mathrm{E}_\gamma h(Z(t))dt \tag{53}$$

$$\text{for } \alpha > 0 \text{ and } h \in \mathscr{L} \text{ such that } f = \mathscr{R}_\alpha h \in \mathscr{D}.$$

We show using (53) that if γ fulfills (52) for all $f \in \mathscr{D}_+$, then it is a stationary distribution of Z. Namely, it is easy to verify that the functions $q_a = q_a(x, t)$, $a = 1, 0$, are non-decreasing with respect to x for each t because

$$dq_a/dx = r_a(q_a)/r_a(x) > 0 \quad \text{for } x > 0 \text{ and } t > 0.$$

Hence and from the construction of the process Z, considering two paths of Z with the same sequences $\boldsymbol{s}$, $\boldsymbol{S}$ and with the same functions r_a ($a = 1, 0$) but with different starting points, say x_1 for the first one and x_2 for the second one, where $x_1 \geqslant x_2$, we can establish that the first path can never cross the second path from above. Therefore, if the function h is non-decreasing, then also the function $\mathscr{R}_\alpha h$ is non-decreasing for every $\alpha > 0$. Further, if $\mathscr{R}_\alpha h$ is non-decreasing, then $\mathscr{R}_\alpha h \in \mathscr{D}_+$ and it fulfills (52) and (53). Simultaneously, if h is continuous, then by the right-continuity of paths of Z, the function $\mathrm{E}_\gamma h(Z(t))$ is right-continuous in t. Thus from (53) and by the uniqueness theorem for Laplace transforms we have

$$\mathrm{E}_\gamma h(Z(t)) = \mathrm{E}_\gamma h(Z(0)), \quad t > 0,$$

for all h that are non-decreasing and continuous, which implies the equality

$$P_\gamma(Z(t) \in A) = \gamma(A) \quad \text{for } t > 0 \text{ and } A \in \mathscr{D}_0,$$

i.e., γ is a stationary distribution of Z.

Thus we have shown that γ is a stationary distribution if (52) is fulfilled for all $f \in \mathscr{D}_+$. Hence and from (48) and (49) we obtain

$$0 = \mu_1 \gamma(0) \int_0^\infty \big(1 - H_0(R_0(u))\big) f'(u)du$$
$$+ \mu_1 \int_0^\infty \Big[\int_x^\infty \big(1 - H_0(R_0(u) - R_0(x))\big) f'(u)du - r_1(x) f'(x)\Big] \gamma(dx).$$

By Lemma 4 the function $r_1 f'$ is bounded, so γ is a stationary distribution iff the equation

$$\int_0^\infty r_1(x) f'(x)\gamma(dx) = \int_0^\infty \Big(\int_{0-}^\infty K(x, u)\gamma(du)\Big) r_1(x) f'(x)dx \tag{54}$$

is fulfilled for all $f \in \mathscr{D}_+$. It is obvious that any probability measure γ which is absolutely continuous on $\mathscr{R}_+$ and satisfies (50) also satisfies (54), and hence it is stationary. To prove the converse it suffices to take f absolutely continuous with $f'(x) = 1/r_1(x)$ for $x \in (a, b]$, $0 \leqslant a < b$, and $f'(x) = 0$ otherwise. By Lemma 4 and by the definition of $\mathscr{D}_+$, such a function f belongs to $\mathscr{D}_+$ and from (54) we have

$$\int_a^{b+} \gamma(dx) = \int_a^{b+} \Big(\int_{0-}^{x} K(x, u)\gamma(du)\Big)dx.$$

Since a, b are arbitrary, this implies that γ is absolutely continuous with density satisfying (50).

Remark that (50) is identical with (16), and therefore by Theorem 2 condition (ii) is necessary and sufficient for equation (50) to have a unique solution of the form

$$\gamma([0, x]) = k\Big(1 + \int_0^x K^*(u, 0)du\Big),$$

i.e., $g(x) = kK^*(x, 0)$, $x > 0$.

To complete the proof of Theorem 4 it remains to show that condition (i) is necessary for the existence of a stationary distribution of the process Z.

LEMMA 6. *If condition* (i) *is not fulfilled, then the process Z has no stationary distribution.*

Proof. Assume that

$$\beta_0/\mu_0 \geqslant \beta_1/\mu_1 \tag{55}$$

and suppose that the process Z has a stationary distribution γ. We show that this leads to a contradiction.

If $\beta_0 = 0$, then by (55) also $\beta_1 = 0$, and we have a contradiction with the assumption that r_1 is strictly positive on $\mathscr{R}_+$. Therefore, let $0 < \beta_0, \beta_1 < \infty$. Consider the special case of the model $GI^{r_0}/M/r_1$ in which the sequences s, S have means $1/\mu_0$, $1/\mu_1$ and the functions r_a $(a = 1, 0)$ are constant and equal to $r_0(x) = \beta_0$, $x \geqslant 0$, $r_1(x) = \beta_1$, $x > 0$. Denote the content process in this model by Y. Since for the process Z we have $r_0(x) \geqslant \beta_0$ for $x \geqslant 0$ and $r_1(x) \leqslant \beta_1$ for $x \geqslant 0$, it is not difficult to remark that by the same sequences s, S and the same starting points the paths of Y are below the paths of Z. Hence, taking for both processes the initial distribution γ, we obtain

$$P\big(Y(t) \leqslant y\big) \geqslant \gamma([0, y]) \quad \text{for } t > 0,\ y \in \mathscr{R}_0,$$

whence

$$\liminf_{t\to\infty} P\big(Y(t) \leqslant y\big) > 0 \quad \text{for some } y \in \mathscr{R}_0. \tag{56}$$

On the other hand, the process Y is also a special case of the content process considered in [3], therefore by (55) we have

$$\lim_{t\to\infty} P\big(Y(t) \leqslant y\big) = 0 \quad \text{for all } y \in \mathscr{R}_0$$

(see Proposition 8, Theorems 2 and 7 in [3]).

5. Limiting distribution for the exponential output phase. Consider again the model $GI/r_0/M/r_1$. Using the results obtained hitherto we find the limiting distribution of the process X in this model. In Theorem 3 we have obtained the limiting distribution of the content process in the model $GI/r_0/GI/r_1$ under the assumption that the imbedded Markov chains X^a have unique invariant probability measures N_a^+ $(a = 1, 0)$. Therefore, first we determine the problem of the existence and uniqueness of measures N_a^+ $(a = 1, 0)$ in our particular case. Introduce the notation

$$\alpha_0 = \sup_{x\geqslant 0}\{x\colon H_0(x) < 1\}.$$

LEMMA 7. *Let* $\alpha_0 = \infty$. *Then in the model* $GI/r_0/M/r_1$ *condition* (i) *is necessary and sufficient for the Markov chains* X^a $(a = 1, 0)$ *to have unique invariant probability measures* N_a^+ *defined by formulas* (17) *and* (18).

Proof. If (i) is fulfilled, then from Theorem 1 there exist invariant probability measures N_a^+ for X^a $(a = 1, 0)$. The uniqueness of N_0^+ follows from the irreducibility of the chain X^0 for the exponential H_1 and $\alpha_0 = \infty$. For this it suffices to show that, for each $x \in \mathscr{R}_0$ and each set $A \in \mathscr{B}_0$ with positive Lebesgue measure, we have $Q^0(x, A) > 0$, where Q^0 is defined by (6) for $a = 0$. Let $A = [a, b]$, $a < b$, $a, b \in \mathscr{R}_0$ and let

$$B_u = \{v \in \mathscr{R}_0\colon q_1(q_0(x, u), v) \in A\}.$$

From definition (3) of the functions q_a it follows that the sets B_u are non-empty for $u \in \mathscr{R}_0$ such that $u > R_0(a) - R_0(x)$. If $\alpha_0 = \infty$, then

$$\int_M^\infty dH_0(u) > 0 \quad \text{for } M > R_0(a) - R_0(x).$$

Therefore it suffices to estimate

$$Q^0(x, A) \geqslant \int_M^\infty dH_0(u) \int_{B_u} \mu_1 \exp\{-\mu_1 v\}\, dv > 0.$$

The uniqueness of N_1^+ follows now from relations (4). The forms of the measures follow from Theorems 4 and 2. Conversely, if there exist unique invariant probability measures for X^a $(a = 1, 0)$, then, by Theorems 2 and 4, condition (i) is satisfied.

The next lemma is a consequence of Lemmas 1 and 7.

LEMMA 8. *Let the distribution H_1 be exponential, let the distribution H_0 be aperiodic, and let $\alpha_0 = \infty$. Then condition* (i) *is necessary and sufficient for the process Z^i to have, for every i ($i = 0, 1$), the limiting distribution*

(57) $$v_i(z) = \lim_{t\to\infty} P_x(Z^i(t) \leqslant z), \qquad x, z \in \mathscr{R}_0,$$

defined, respectively, by the equality

(58) $$v_0(z) = N_0^+(z)$$

or

(59) $$v_1(z) = \mu_0 N_1(z)/v,$$

where N_0^+ is of the form (17) *and N_1 is of the form* (20).

Proof. If (i) is fulfilled, then by Lemma 7 and equality (30) the Markov chain Z^{1-i} has a unique invariant probability measure N_{1-i}^+. Applying Lemma 1 we obtain equality (32) for fixed i. Now, using Theorems 4 and 2 and taking into account relations (22) and (26) for $a = 1$, we establish that the right-hand side of (32) for $i = 0$ equals $\mu_1 N_0(A)/v = N_0^+(A)$, whence we obtain (58). Similarly, applying Theorems 4 and 2 and relation (22) for $a = 0$, we establish that the right-hand side of (32) for $i = 1$ equals $\mu_0 N_1(A)/v$, so we have (59).

Conversely, if for every i the limiting distribution v_i exists, then the distribution v_0 is a stationary distribution of the process Z^0. Therefore, by Lemma 5 and Theorem 4, condition (i) is satisfied.

Having these lemmas we can obtain the following limit theorem:

THEOREM 5. *In the model $GI/r_0/M/r_1$, let the distribution H_0 be aperiodic and let $\alpha_0 = \infty$. Then condition* (i) *is necessary and sufficient for the content process X to have a limiting distribution of the form*

(60) $$\lim_{t\to\infty} P_x(X(t) \leqslant y) = N_0(y) + N_1(y),$$

where the functions N_a ($a = 1, 0$) are defined by formulas (20) *and* (19), *respectively.*

Proof. If (i) is fulfilled, then from Lemma 7, equality (30) and Lemma 1 we have (32) for $i = 0, 1$. Hence, from (31) and from the main renewal theorem we obtain

$$\lim_{t\to\infty} P_x(X(t) \leqslant y) = v\Big[\int_0^\infty N_1^+(dv) \int_0^\infty (1-H_1(u)) I_{[0,y]}(q_1(v, u))\,du$$
$$+ \int_{0-}^\infty N_0^+(dv) \int_0^\infty (1-H_0(u)) I_{[0,y]}(q_0(v, u))\,du\Big].$$

Thus by Theorems 4 and 2 and by (22) the above implies (60).

On the other hand, the right-hand side of (60) is a probability distribution iff (ii) is satisfied, i.e., by Theorem 4 also (i) is satisfied.

Remark 3. In Theorem 5 we do not solve the problem of the existence of $\lim_{t\to\infty} P_x(X(t) \leqslant y)$ in the case where (i) is not satisfied. It follows from Lemma 6 that in this case there does not exist a stationary distribution of the process Z^0. Dealing similarly as in [3] (see Proposition 8 and Theorem 2 in [3]) one can show that if (ii) is not satisfied, then

$$\lim_{t\to\infty} P_x(Z^0(t)\in A) = 0$$

for $A \in \mathscr{B}_0$ bounded. Hence and from Theorem 4 it follows that if (i) is not satisfied, then even when $\lim_{t\to\infty} P_x(Z^1(t)\in A)$ exists and it is a probability distribution, there does not exist a limiting probability distribution of X because in this case we have

$$\lim_{y\to\infty}\lim_{t\to\infty} P_x(X(t) \leqslant y) = \lim_{y\to\infty}\lim_{t\to\infty} P_x(Z^1(t) \leqslant y)P_x(\alpha(t) = 1) = v/\mu_0 < 1.$$

The probability distribution N_0^+ given by (17) is difficult to calculate in general. We give some special cases where one can calculate the series K^* in the explicit form. We put $\varrho = \mu_1/\mu_0$.

Exponential input phase. Assuming

$$H_0(x) = 1-\exp\{-\mu_0 x\}, \qquad x \geqslant 0,$$

we calculate

$$K^{*(n+1)}(x, y) = \mu_1^{n+1}\exp\{-\mu_0(R_0(x)-R_0(y))\}(R_1(x)-R_1(y))(r_1(x)n!)^{-1}$$
$$0 \leqslant y < x,\ n = 0, 1, \ldots,$$

and hence

$$K^*(x, y) = \mu_1\exp\{-\mu_0(R_0(x)-R_0(y))+\mu_1(R_1(x)-R_1(y))\}/r_1(x),$$
$$0 \leqslant y < x.$$

By Theorem 4 the integral

$$\int_0^\infty (\mu_1\exp\{-\mu_0 R_0(u)+\mu_1 R_1(u)\}/r_1(u))du$$

is finite iff

$$\inf_{x\geqslant 0} r_0(x)/\mu_0 < \sup_{x\geqslant 0} r_1(x)/\mu_1.$$

Then by Theorem 2 we obtain

$$N_0^+(x) = k\left[1+\mu_1 \int_0^x \left(\exp\{-\mu_0 R_0(u)+\mu_1 R_1(u)\}/r_1(u)\right)du\right],$$

$$N_1^+(x) = k\Big[1-\exp\{-\mu_0 R_0(x)+\mu_1 R_1(x)\}$$
$$+\mu_1 \int_0^x \left(\exp\{-\mu_0 R_0(u)+\mu_1 R_1(u)\}/r_1(u)\right)du\Big],$$

$$N_0(x) = N_0^+(x)/(1+\varrho),$$
$$N_1(x) = \varrho N_1^+(x)/(1+\varrho).$$

Constant input intensity function. Assuming $r_0(x) = 1$ we have from (11) the equality

$$K^{*1}(x, y) = \mu_1\big(1-H_0(x-y)\big)/r_1(x), \quad 0 \leqslant y < x.$$

The kernel K^{*1} of this form has been considered in [3] and the probability distribution N_0^+ is here identical with the stationary distribution of the content process given in [3]. We can obtain the remaining distributions applying formulas (18)–(20) and substituting $R_0(x) = x$. The calculations of N_0^+ for some output intensity functions can be found in [7].

Constant input and output intensity functions. Assuming $r_0(x) = 1$ and $r_1(x) = 1$ we obtain the particular model of the above case. For $\varrho < 1$ we have

$$N_0^+(x) = (1-\varrho)\sum_{n=0}^{\infty} \varrho^n (\hat{H}_0)^{*n}(x),$$

$$N_1^+(x) = \int_0^x H_0(x-u)\,dN_0^+(u), \qquad N_0(x) = N_0^+(x)/(1+\varrho),$$

$$N_1(x) = \varrho \int_0^x \hat{H}_0(x-u)\,dN_0(u), \qquad x \geqslant 0.$$

The distributions N_0 and N_1 agree with the limiting distributions given in [6] (p. 280).

References

[1] P. J. Brockwell, *Stationary distributions for dams with additive input and content-dependent release rate*, Adv. in Appl. Probab. 9 (1977), pp. 645–663.

[2] E. Çinlar, *On dams with continuous semi-Markovian inputs*, J. Math. Anal. Appl. 35 (1971), pp. 434–448.

[3] J. M. Harrison and S. I. Resnick, *The stationary distribution and first exit probabilities of a storage process with general release rule*, Math. Oper. Res. 1 (1976), pp. 347–358.

[4] M. Jankiewicz, *The dam with content-dependent input and output*, Zastos. Mat. 19 (1987), pp. 13–29.
[5] — *Extended piecewise Markov processes in continuous time*, ibidem 16 (1978), pp. 175–195.
[6] R. G. Miller, Jr., *Continuous time stochastic storage processes with random linear inputs and outputs*, J. Math. Mech. 12 (1963), pp. 275–291.
[7] G. Yeo, *A dam with general release rule*, J. Austral. Math. Soc. Ser. B 19 (1976), pp. 469–477.

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