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A NOISY DUEL UNDER ARBITRARY MOVING. II

1. Definitions and suppositions. In paper [19]–[23] of the author and in this paper an m versus n bullets noisy duel is considered in which duelists can move at will.

In this paper we solve the duels in which players have one, two or three bullets.

Let us define the game which will be called the *game* (m, n). Two Players I and II fight in a duel. They can move as they like. Maximal velocity of Player I is v_1, maximal velocity of Player II is v_2, and let $v_1 > v_2 \geqslant 0$. Player I has m bullets (or rockets), Player II has n bullets (rockets).

Assume that at the moment $t = 0$ the players are in the distance 1 off from each other and that $v_1 + v_2 = 1$.

Denote by $P(s)$ the probability that Player I (II) achieves a success (destroys the opponent) if he fires in the distance $1-s$. We assume that the function $P(s)$ is increasing and continuous in the interval $[0, 1]$ and has a continuous second derivative inside this interval, $P(s) = 0$ for $s \leqslant 0$, $P(1) = 1$.

Player I gains 1 if he only achieves the success, gains -1 if Player II only achieves the success, and gains 0 in the remaining cases. It is assumed that the duel is a zero-sum game.

The duel is *noisy* – the player hears the shot of his opponent.

Without loss of generality we can assume that Player II is motionless. Then $v_1 = 1$, $v_2 = 0$.

When Player I has fired all his bullets, his motion in the direction of the opponent loses sense. Then we shall always assume that Player I evades with maximal speed after firing all his bullets.

Suppose that Player I has fired all his bullets and he evades. In this case Player II will do the best (if he survives) if he fires all his bullets immediately after the last shot of Player I. If, on the other hand, Player II has fired all his bullets and Player I survives and has yet bullets, the best what he can do is to reach the opponent and to achieve the success surely.

We shall keep the above assumptions throughout the paper. We suppose also that the reader knows paper [19] and remembers the definitions, notation and assumptions given there.

For the definitions and notions concerning duels see [1], [5], [6], [24]. For other games of timing see [2], [3], [8], [9], [11]–[13], [15], [25].

2. Duel $(m, 1)$. Let a point $a_{mn} \in [0, 1]$ and let $\langle a_{mn} \rangle$ be the earliest moment when Player I reaches this point (at the moment 0 he is at the point 0). Denote by a_{mn}^{ε} the random moment,

$$\langle a_{mn} \rangle \leqslant a_{mn}^{\varepsilon} \leqslant \langle a_{mn} \rangle + \alpha(\varepsilon),$$

distributed according to an absolute continuous probability distribution (ACPD) in the above interval, $\alpha(\varepsilon) \to 0$ if $\varepsilon \to 0$.

Let us consider the case where Players I and II have one bullet each. Define the following strategies ξ and η of these players:

Strategy of Player I. Reach the point a_{11}, and if Player II did not fire before, fire the shot at a_{11}^{ε}.

Strategy of Player II. Fire at the earliest moment when Player I reaches the point a_{11} (at the moment $\langle a_{11} \rangle$). If he did not reach this point, do not fire.

The number a_{11} satisfies the equation

$$P(a_{11}) = \sqrt{2} - 1. \tag{1}$$

Now, let us consider the case where Player I has m bullets, $m \geqslant 2$, and Player II has one bullet. In this case we define strategies ξ and η of these players as follows:

Strategy of Player I. Reach the point a_{m1}, and if Player II did not fire before, fire a shot at $\langle a_{m1} \rangle$ and play ε-optimally the obtained duel $(m-1, 1)$.

Strategy of Player II. If Player I reaches the point a_{m1}, fire a shot at a_{m1}^{ε}. If he does not reach this point, do not fire.

The number a_{m1} satisfies the equation

$$P(a_{m1}) = \frac{P(a_{11})}{1 + (m-1)P(a_{11})}. \tag{2}$$

In [19] it is proved that in the case $m = 1$ the strategy ξ is ε-maximin and the strategy η is minimax. In the case $m \geqslant 2$ the strategy ξ is ε-maximin and η is ε-minimax (for a properly chosen $\alpha(\varepsilon)$). In both cases the value of the game is given by the formula

$$v_{m1} = \frac{1 + (m-3)P(a_{11})}{1 + (m-1)P(a_{11})}. \tag{3}$$

3. Further definitions and assumptions. Suppose now that the duel (m, n) begins when the distance between the players is $1 - a$. This duel will be denoted by (m, n), $\langle a \rangle$. To simplify considerations we calculate the time also from $t = a$. All other suppositions about the duel (m, n) made at the beginning of the paper holds also for the duel (m, n), $\langle a \rangle$. Thus Player I, after firing all his bullets, evades with maximal speed, etc.

In the further part of the paper (and in the forthcoming papers [20]–[23] we assume that between successive shots of the same player the time $\hat{\varepsilon}$ has to pass.

We say that *Player* I *assures in limit the value* u_1 if for each $\varepsilon > 0$, $\hat{\varepsilon} > 0$ he has a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ such that

$$K(\xi_{\varepsilon\hat{\varepsilon}};\, \hat{\eta}) \geqslant u_1 - k_1(\varepsilon, \hat{\varepsilon}) \tag{4}$$

for any strategy $\hat{\eta}$ of Player II, where $k_1(\varepsilon, \hat{\varepsilon})$ is a function tending to zero when $\varepsilon \to 0$, $\hat{\varepsilon} \to 0$.

Similarly, *Player* II *assures in limit the value* u_2 if for each $\varepsilon > 0$, $\hat{\varepsilon} > 0$ he has a strategy $\eta_{\varepsilon\hat{\varepsilon}}$ such hat

$$K(\hat{\xi};\, \eta_{\varepsilon\hat{\varepsilon}}) \leqslant u_2 + k_2(\varepsilon, \hat{\varepsilon}) \tag{5}$$

for any strategy $\hat{\xi}$ of Player I, where $k_2(\varepsilon, \hat{\varepsilon})$ is a function tending to zero when $\varepsilon \to 0$, $\hat{\varepsilon} \to 0$.

Other notions defined below can be defined wider. Since I want to be understood also by people not working in the theory of games, I define these notions in a simpler way but under the following additional assumption (satisfied in the paper):

(C) Assume that Players I and II assure in limit the same value v_{mn}^a.

The number v_{mn}^a will be called the *limit value of the game.*

Suppose that there is a strategy $\xi_{\hat{\varepsilon}}$ of Player I assuring in limit, in the duel (m, n), $\langle a \rangle$, the value v_{mn}^a, where $k_1(\varepsilon, \hat{\varepsilon}) = k_1(\hat{\varepsilon})$. This strategy $\xi_{\hat{\varepsilon}}$ will be called *optimal* or *maximin in limit.*

Similarly we define the *optimal* or *minimax in limit* strategy of Player II.

If, however, instead of the condition $k_1(\hat{\varepsilon}) \to 0$ for $\hat{\varepsilon} \to 0$ we have

$$\lim_{\varepsilon \to 0} k_1(\hat{\varepsilon}) \leqslant \varepsilon, \tag{6}$$

then such a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ is called *ε-optimal in limit.*

Let us consider a family $\mathscr{F}$ of strategies such that for each $\varepsilon \geqslant 0$, $\hat{\varepsilon} > 0$ there is a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ belonging to this family and being ε-optimal in limit. In the paper we consider only families $\mathscr{F}$ of strategies containing for each $\hat{\varepsilon} > 0$ a strategy $\xi_{\varepsilon\hat{\varepsilon}}$ such that $\varepsilon \leqslant \delta(\hat{\varepsilon})$ and

$$\lim_{\hat{\varepsilon} \to 0} \delta(\hat{\varepsilon}) = 0. \tag{7}$$

If Player I has at his disposal such a family of strategies, then he has a strategy $\xi_{\hat{\varepsilon}}$ optimal in limit.

A similar corollary is true also for Player II.

4. Duel $(1, 2)$, $\langle a \rangle$. In this section we give a solution for the duel $(1, 2)$, $\langle a \rangle$ when a is small. This solution will be necessary to solve other duels with m and n bullets and, in particular, the duels (m, n), $\langle 0 \rangle$, which is the main aim of papers [19]–[23] and of this article.

The duel (1, 2), $\langle a \rangle$, and generally (m, n), $\langle a \rangle$, has some pecularities. Firstly, when $a = 0$, Player I assures the value zero simply evading at the moment zero. Then the value of the game (or the limit value of the game) v_{mn}^0 has to be nonnegative even in the case where Player I has less bullets! Secondly, as can be seen, for example, in the duel considered in this section, Player II has, in general, for $m < n$, infinitely many optimal in limit strategies different in this sense that he has not to fire only at $\langle a_{12} \rangle$, defined below, to assure the value v_{12}^a.

Case 1. Denote by ξ and η the following strategies of Players I and II:

Strategy of Player I. Evade if Player II did not fire a shot. If he fired, play ε-optimally the duel (1, 1).

Strategy of Player II. Do not fire as long as Player I does not reach the point a_{12}. If he reaches this point, fire a shot at $\langle a_{12} \rangle$ and play optimally the duel (1, 1).

The number a_{12} is given by the equation

$$P(a_{12}) = \frac{v_{11}}{1+v_{11}} = \frac{2-\sqrt{2}}{4}. \tag{8}$$

We prove that if $a \leqslant a_{12}$, then the strategies ξ and η defined above are optimal in limit and the value of the game is

$$v_{12}^a = 0. \tag{9}$$

Suppose that Player II playing against ξ fires at the point $a' \leqslant a$. We have

$$K(\xi;\, a', \hat{\eta}_0) \geqslant -P(a') + \big(1-P(a')\big)(v_{11}-\varepsilon) \geqslant -P(a) + \big(1-P(a)\big)v_{11} - \varepsilon$$

for any strategy $\hat{\eta}_0$ (maybe dependent on a') in the subsequent duel. Then Player I assures in limit the value 0 (see Section 3).

Throughout the paper, by $k(\hat{\varepsilon})$, $k_i(\hat{\varepsilon})$ we denote functions tending to zero if $\hat{\varepsilon} \to 0$.

Suppose that Player I playing against η fires before he reaches the point a_{12}. For such a strategy (and the point) a', $a' < a_{12}$, we have

$$\begin{aligned} K(a';\eta) &\leqslant P(a') - \big(1-P(a')\big)\big(1-(1-P(a'))^2\big) + k(\hat{\varepsilon}) \\ &= 1 - 2Q(a') + Q^3(a') + k(\hat{\varepsilon}) \\ &= \big(Q(a')-1\big)\big(Q^2(a') + Q(a') - 1\big) + k(\hat{\varepsilon}) \leqslant k(\hat{\varepsilon}) \end{aligned}$$

if

$$Q(a') \geqslant \frac{\sqrt{5}-1}{2} = 0{,}618\ldots,$$

where $Q(a') = 1 - P(a')$. Since $a' < a_{12}$, the above condition holds.

Suppose that Player I reaches the point a_{12} but does not fire the shot. For such a strategy $\hat{\xi}$

$$K(\hat{\xi};\, \eta) \leqslant -P(a_{12})+\big(1-P(a_{12})\big)v_{11}+k(\hat{\varepsilon}) = k(\hat{\varepsilon}).$$

Suppose that Player I fires at $\langle a_{12}\rangle$ (if Player II did not fire before) i.e., at the earliest moment when he reaches the point a_{12}. For such a strategy $\hat{\xi}$

$$K(\hat{\xi};\, \eta) \leqslant -\big(1-P(a_{12})\big)^2 P(a_{12})+k(\hat{\varepsilon}) < k(\hat{\varepsilon}).$$

Suppose, finally, that (if Player II did not fire) Player I never reaches the point a_{12} and never fires. For such a strategy $\hat{\xi}$

$$K(\hat{\xi};\, \eta) = 0.$$

Then also Player II applying the strategy η assures in limit the value 0. This completes the proof of the proposition.

Let us notice that here and in the other cases it is sufficient to consider only nonrandom strategies $\hat{\xi}$, $\hat{\eta}$ (and $(a', \hat{\xi}_0)$, $(a', \hat{\eta}_0)$).

Case 2. In the previous case, the strategies ξ and η of Players I and II were optimal in limit when $a \leqslant a_{12}$. Now, let us assume that $a \geqslant a_{12}$. Let now ξ and η denote strategies of Players I and II defined as follows:

Strategy of Player I. Evade if Player I did not fire a shot. If he fired, play ε-optimally the duel (1, 1).

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally afterwards.

We prove that if

$$0.730842 \cong Q(\hat{a}_{12}) \leqslant Q(a) \leqslant Q(a_{12}), \tag{10}$$

where $\hat{a}_{12}$ is the root of the equation $S\big(Q(\hat{a}_{12})\big) = 0$ (S given in (12)), then ξ and η defined above are optimal in limit strategies of Player I and II and the limit value of the game (1, 2), $\langle a\rangle$ is

$$v_{12}^a = -1+(1+v_{11})Q(a). \tag{11}$$

Assume that Player II fires at $a' \leqslant a$. Then for such a strategy $(a', \hat{\eta}_0)$

$$\begin{aligned} K(\xi;\, a', \hat{\eta}_0) &\geqslant -P(a')+\big(1-P(a')\big)v_{11}-k(\hat{\varepsilon}) \\ &\geqslant -P(a)+\big(1-P(a)\big)v_{11}-k(\hat{\varepsilon}) = v_{12}^a-k(\hat{\varepsilon}). \end{aligned}$$

Assume that Player II does not fire if Player I evades. For such a strategy $\hat{\eta}$

$$K(\xi;\, \hat{\eta}) = 0 \geqslant v_{12}^a$$

if

$$Q(a) \leqslant \frac{1}{1+v_{11}} = Q(a_{12}).$$

Otherwise, assume that Player I did not fire at $\langle a\rangle$. For such ξ

$$K(\xi;\eta) \leqslant -P(a)+(1-P(a))v_{11}+k(\hat{\varepsilon}) = v_{12}^{a}+k(\hat{\varepsilon}).$$

If he fires at $\langle a\rangle$, we have

$$K(\xi;\eta) \leqslant -(1-P(a))^2 P(a)+k(\hat{\varepsilon}) \leqslant v_{12}^{a}+k(\hat{\varepsilon})$$

if

$$S(Q(a)) = Q^3(a)-Q^2(a)-(1+v_{11})Q(a)+1 \leqslant 0. \tag{12}$$

The function $S(Q)$ is a decreasing function of the variable Q and is equal to zero for $Q = Q(\hat{a}_{12}) \cong 0.730842$. Then under the condition (10) the proposition is proved.

5. Definition of the duels (m, n), $\langle 1, a\wedge c, a\rangle$ **and** (m, n), $\langle 2, a, a\wedge c\rangle$. **Discussion.** We have suppossed that between successive shots of the same player the time $\hat{\varepsilon}$ has to pass. Let

$$(m, n),\ \langle 2, a, a\wedge c\rangle, \qquad 0 < c \leqslant \hat{\varepsilon},$$

be the duel in which Player I has m bullets, Player II has n bullets but if $c < \hat{\varepsilon}$, Player I can fire his bullets beginning from the moment $\langle a\rangle$ and Player II from the moment $\langle a\rangle+c$. If $c = \hat{\varepsilon}$, the rule is the same with the only exception that Player I is not allowed to fire at $\langle a\rangle$.

Similarly we define the duel (m, n), $\langle 1, a\wedge c, a\rangle$.

All other definitions and suppositions made for the duel (m, n), $\langle a\rangle$ hold also for the above two duels.

If in the duel (m, n), $\langle a\rangle$ Player II fires as the first the bullet at the point a' and misses and Player I does not fire at the same moment, then the duel (m, n), $\langle a\rangle$ reduces to the game $(m, n-1)$, $\langle 2, a', a'\wedge\hat{\varepsilon}\rangle$.

Moreover, if in the duel (m, n), $\langle 1, a\wedge c, a\rangle$ Player I fires as the first at the moment $t \geqslant \langle a\rangle+c$ and misses, $t = \langle\langle a'\rangle\rangle$, and Player II did not fire at this moment, then the duel (m, n), $\langle 1, a\wedge c, a\rangle$ reduces to the game $(m-1, n)$, $\langle 1, a'\wedge\hat{\varepsilon}, a'\rangle$.

The symbol $\langle\langle a'\rangle\rangle$ means a moment when Player I is at the point a' (not necessarily the earliest one which is denoted by $\langle a'\rangle$).

If as the first fires Player II and misses, then:

(a) If he fires at $t < \langle a\rangle+c$, $t = \langle\langle a'\rangle\rangle$, the duel considered reduces to the game $(m, n-1)$, $\langle 2, a', a'\wedge c_1\rangle$ for some c_1, $c_1 < \hat{\varepsilon}$.

(b) If he fires at $t = \langle a\rangle+c$, the duel considered reduces to the game $(m, n-1)$, $\langle a_2\rangle$, where a_2 is the point where Player I will be at the moment $\langle a\rangle+c+\hat{\varepsilon}$.

(c) If he fires at $t > \langle a\rangle+c$, $t = \langle\langle a'\rangle\rangle$, the duel considered reduces to the game $(m, n-1)$, $\langle 2, a', a'\wedge\hat{\varepsilon}\rangle$.

Similar situations arise also when the players fire at the same moments.

Then if the player which has only one bullet does not fire, each of the duels

$$(m, n), \langle a \rangle; \quad (m, n), \langle 1, a \wedge c, a \rangle; \quad (m, n) \langle 2, a, a \wedge c \rangle$$

reduces after a shot to a duel of the above three kinds.

The same conclusion holds when the players fire simultaneously.

6. Duel (1, 2), $\langle 1, a \wedge c, a \rangle$.

Case 1. Define the strategies ξ and η of Players I and II.

Strategy of Player I. Evade if Player II did not fire before. If he fired, play optimally the duel (1, 1).

Strategy of Player II. Do not fire a shot as long as Player I does not reach the point a_{12}. If he reaches this point, fire at $\langle a_{12} \rangle$ and play optimally the duel (1, 1).

From this section "play optimally" means "apply a strategy optimal in limit".

Denote by $\overset{1a}{v}_{mn}$ the limit value of the game (m, n), $\langle 1, a \wedge c, a \rangle$, and by $\overset{2a}{v}_{mn}$ the limit value of the game (m, n), $\langle 2, a, a \wedge c \rangle$.

Let us return to the duel (1, 2), $\langle 1, a \wedge c, a \rangle$. Comparing with the duel (1, 2), $\langle a \rangle$ it is easy to see that if $a \leqslant a_{12}$, then the strategies ξ and η defined above are optimal in limit and the limit value of the game is $\overset{1a}{v}_{12} = 0$.

Case 2. Now let the strategies ξ and η of Players I and II be defined as follows:

Strategy of Player I. Evade if Player I did not fire. If he fired, play optimally the duel (1, 1).

Strategy of Player II. Fire at the moment t, $\langle a \rangle < t < \langle a \rangle + c$, and play optimally the duel (1, 1).

For each strategy ξ of Player I, if $a < a_{11}$, then

$$K(\xi; \eta) \leqslant -P(a) + (1 - P(a))v_{11} + k(\varepsilon).$$

Comparing with the duel (1, 2), $\langle a \rangle$ it is easy to prove that Player I assures in limit the same value if

$$-P(a) + (1 - P(a))v_{11} \leqslant 0,$$

i.e., if $a \geqslant a_{12}$. Then if $a_{12} \leqslant a \leqslant a_{11}$, the constant

$$\overset{1a}{v}_{12} = -1 + (1 + v_{11})Q(a) \tag{13}$$

is the limit value of the game (1, 2), $\langle 1, a \wedge c, a \rangle$ and the strategies ξ and η defined above are optimal in limit.

7. Duel (1, 2), $\langle 2, a, a \wedge c \rangle$.

Case 1. Let the strategies ξ and η of Players I and II be the same as in Case 1 of the previous two duels. It is easy to prove that if $a \leqslant a_{12}$, then also now the strategies ξ and η are optimal in limit. The limit value of the game is $\overset{2a}{v}_{12} = 0$.

Case 2. Define ξ and η.

Strategy of Player I. Evade if Player II did not fire. If he fires a shot, play optimally the duel (1, 1).

Strategy of Player II. If Player I did not fire before, fire a shot at $\langle a \rangle + c$ and play optimally afterwards.

When

$$Q(\check{a}_{12}) \leqslant Q(a) \leqslant Q(a_{12}), \tag{14}$$

$Q(\check{a}_{12}) \cong 0.780539$ is the root of the equation

$$Q^3(\check{a}_{12}) - (3 + v_{11}) Q(\check{a}_{12}) + 2 = 0, \tag{15}$$

the strategies ξ and η are optimal in limit and the limit value of the game is

$$\overset{2a}{v}_{12} = -1 + (1 + v_{11}) Q(a). \tag{16}$$

The proof that Player I assures in limit a value a satisfying the conditions (14) is the same as in the duel (1, 2), $\langle a \rangle$. We shall prove that the same is true for Player II.

Assume that Player I applying $\hat{\xi}$ fires before $\langle a \rangle + c$. We have

$$K(\hat{\xi}; \eta) \leqslant P(a) - Q(a)\big(1 - Q^2(a)\big) + k(\hat{\varepsilon}) = 1 - 2Q(a) + Q^3(a) + k(\hat{\varepsilon}) \leqslant \overset{2a}{v}_{12} + k(\hat{\varepsilon})$$

if

$$S\big(Q(a)\big) = Q^3(a) - (3 + v_{11}) Q(a) + 2 \leqslant 0.$$

The function S is a decreasing function of the variable Q and has the root $Q = Q(\check{a}_{12})$. The the inequality holds for $a < \check{a}_{12}$.

Assume that Player I applying $\hat{\xi}$ fires at $\langle a \rangle + c$. We have

$$K(\hat{\xi}; \eta) \leqslant -Q^2(a) P(a) + k(\hat{\varepsilon}) \leqslant -1 + (1 + v_{11}) Q(a) + k(\hat{\varepsilon})$$

if $a \leqslant \hat{a}_{12}$ (see (12)).

Assume that Player I applying $\hat{\xi}$ does not fire a shot before or at $\langle a \rangle + c$. We obtain

$$K(\hat{\xi}; \eta) \leqslant -P(a) + \big(1 - P(a)\big) v_{11} + k(\hat{\varepsilon}) = \overset{2a}{v}_{12} + k(\hat{\varepsilon}).$$

The proposition is proved.

We put

$$C_{12} = \{a\colon\ Q(a) \geqslant Q(a_{12})\}, \qquad D_{12} = \{a\colon\ Q(a_{12}) \geqslant Q(a) \geqslant Q(\check{a}_{12})\}.$$

8. Results for the duel (1, 2). We have

$$\overset{1}{v}{}^{a}_{12} = \begin{cases} 0 & \text{if } Q(a) \geqslant Q(a_{12}) \cong 0.853553, \\ -1+(1+v_{11})Q(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(a_{11}) \cong 0.585787; \end{cases}$$

$$v^{a}_{12} = \begin{cases} 0 & \text{if } Q(a) \geqslant Q(a_{12}), \\ -1+(1+v_{11})Q(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(\hat{a}_{12}) \cong 0.730842; \end{cases}$$

$$\overset{2}{v}{}^{a}_{12} = \begin{cases} 0 & \text{if } Q(a) \geqslant Q(a_{12}), \\ -1+(1+v_{11})Q(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(\check{a}_{12}) \cong 0.780539. \end{cases}$$

9. Duel (2, 2). Define ξ and η.

Strategy of Player I. Reach the point a_{22} and if Player II did not fire before, fire a shot at a^{ε}_{22} and play optimally afterwards.

Strategy of Player II. If Player I did not fire before, fire at $\langle a_{22}\rangle$ and play optimally afterwards. If Player I did not reach the point a_{22} do not fire.

The numbers v_{22} and a_{22} are determined from the equations

$$v_{22} = -P(a_{22})+Q(a_{22})v_{21} = P(a_{22})+Q(a_{22})\overset{1}{v}{}^{a_{22}}_{12}.$$

Let $a_{22} \in D_{12}$. In this case

$$\overset{1}{v}{}^{a_{22}}_{12} = -1+(1+v_{11})Q(a_{22})$$

and from the above we obtain

$$(1+v_{11})Q^2(a_{22})-(3+v_{21})Q(a_{22})+2 = 0. \tag{17}$$

Since $v_{21} \cong 0.414214$ (see [23]), we obtain

$$Q(a_{22}) \cong 0.812085. \tag{18}$$

Then $a_{22} \in D_{12}$ as was assumed. Moreover,

$$v_{22} = -1+(1+v_{21})Q(a_{22}) \cong 0.148461. \tag{19}$$

We prove that for a_{22} defined by (17) the strategies ξ and η are optimal in limit if $a \leqslant a_{22}$.

Firstly, let us notice that from (2) it follows that

$$Q(a_{21}) = \frac{1}{1+P(a_{11})} = \frac{\sqrt{2}}{2} = 0.707107.$$

Then $a_{22} < a_{21}$ and the strategies ξ and η are well defined.

Suppose that Player II fired when Player I has been at the point $a' \leqslant a_{22}$. We obtain

$$K(\xi; a', \hat{\eta}_0) \geqslant -P(a')+Q(a')v_{21}-k(\hat{\varepsilon})$$
$$-P(a_{22})+Q(a_{22})v_{21}-k(\hat{\varepsilon}) = v_{22}-k(\hat{\varepsilon}).$$

Suppose that Player II did not fire before $\langle a_{22}\rangle+\alpha(\varepsilon)$. For such a strategy $\hat{\eta}$ we have

$$K(\xi;\, \hat{\eta}) \geqslant P(a_{22})+Q(a_{22})\,{}^{1}v^{a_{22}}_{12}-k(\hat{\varepsilon}) = v_{22}-k(\hat{\varepsilon}).$$

Then

$$K(\xi;\, \hat{\eta}) \geqslant v_{22}-k(\hat{\varepsilon})$$

for each strategy $\hat{\eta}$ of Player II. Obviously, in all these cases the functions k are, is general, different.

Otherwise, let us assume that Player I applying $\hat{\xi}$ never reaches the point a_{22} and never fires. Then

$$K(\hat{\xi};\, \eta) = 0 < v_{22}.$$

If Player I fires at $a' < a_{22}$, then

$$\begin{aligned} K(a',\, \hat{\xi}_0;\, \eta) &\leqslant P(a')+Q(a')\,{}^{1}v^{a'}_{12}+k(\hat{\varepsilon}) \\ &= \begin{cases} 1-Q(a')+k(\hat{\varepsilon}) & \text{if } a' \in C_{12}, \\ 1-2Q(a')+(1+v_{11})Q^2(a')+k(\hat{\varepsilon}) & \text{if } a' \in D_{12}. \end{cases} \end{aligned}$$

The first function is an increasing function of the variable a'. The second one has the only minimum at the point $a' = a_{12}$ (i.e., if $Q(a') = 1/(1+v_{11}) = Q(a_{12})$). Then it is also increasing for $a' \in D_{12}$, $a' \in a_{22}$. For $a' = a_{12}$, that is to say for a' being at the end of the intervals C_{12} and D_{12}, both functions take the same value. Then

$$\begin{aligned} K(a',\, \hat{\xi}_0;\, \eta) &\leqslant 1-2Q(a_{22})+(1+v_{11})Q(a_{22})+k(\hat{\varepsilon}) \\ &= 2-(3+v_{21})Q(a_{22})+(1+v_{11})Q(a_{22})-1+(1+v_{21})Q(a_{22})+k(\hat{\varepsilon}) \\ &= -1+(1+v_{21})Q(a_{22})+k(\hat{\varepsilon}) = v_{22}+k(\hat{\varepsilon}). \end{aligned}$$

If Player I fires at $\langle a_{22}\rangle$, then

$$K(\hat{\xi};\, \eta) \leqslant Q^2(a_{22})v_{11}+k(\hat{\varepsilon}) \cong 0.113149+k(\hat{\varepsilon}) < v_{22}+k(\hat{\varepsilon}).$$

If, finally, Player I does not fire before or at $\langle a_{22}\rangle$ but reaches the point a_{22}, we obtain

$$K(\hat{\xi};\, \eta) \leqslant -P(a_{22})+Q(a_{22})v_{21}+k(\hat{\varepsilon}) = v_{22}+k(\hat{\varepsilon}).$$

Then if $a \leqslant a_{22}$, the strategies ξ and η are optimal in limit.

Solutions for duels $(m, 2)$, $2 \leqslant m \leqslant 25$, are given in [23].

10. Duel (1, 3), $\langle a\rangle$. Define ξ and η.

Strategy of Player I. Evade if Player II does not fire. If he fired (say at a'), play optimally the obtained duel (1, 2), $\langle 2,\, a',\, a' \wedge \hat{\varepsilon}\rangle$.

Strategy of Player II. Fire at $\langle a\rangle$ and play optimally afterwards.

Now

$$K(\xi;\eta) = -P(a)+Q(a)\overset{2a}{v}_{12}+k(\hat\varepsilon) \tag{20}$$
$$= \begin{cases} -1+Q(a)+k(\hat\varepsilon) & \text{if } a\in C_{12}, \\ -1+(1+v_{11})Q^2(a)+k(\hat\varepsilon) & \text{if } a\in D_{12}. \end{cases}$$

We prove that, for $Q(a) \geqslant Q(a_{13}) \cong 0.814115$,

$$Q^4(a_{13})-(2+v_{11})Q^2(a_{13})+1 = 0, \tag{21}$$

the strategies ξ and η are optimal in limit and

$$v_{13}^a = \begin{cases} -1+Q(a) & \text{if } a\in C_{12}, \\ -1+(1+v_{11})Q^2(a) & \text{if } a_{12} \leqslant a \leqslant a_{13}. \end{cases} \tag{22}$$

To prove this assume that Player II fires at $a' \leqslant a$. We obtain

$$K(\xi;a',\hat\eta_0) \geqslant -P(a')+Q(a')\overset{2a'}{v}_{12}-k(\hat\varepsilon)$$
$$\geqslant -P(a)+Q(a)\overset{2a}{v}_{12}-k(\hat\varepsilon) = v_{13}^a-k(\hat\varepsilon)$$

for $a\in C_{12}\cup D_{12}$ since from (20) it follows that $-P(a')+Q(a')\overset{2a'}{v}_{12}$ is a decreasing function of the variable a' for $a'\in C_{12}\cup D_{12}$.

If Player II applying $\hat\eta$ does not fire,

$$K(\xi;\hat\eta) = 0 \geqslant v_{13}^a$$

since

$$v_{13}^a = \begin{cases} -P(a) \leqslant 0 & \text{for } a\in C_{12}, \\ -1+(1+v_{11})Q^2(a) < -1+(1+v_{11})Q(a) \leqslant 0 & \text{for } a\in D_{12}. \end{cases}$$

Otherwise, if Player I applying $\hat\xi$ does not fire at the beginning of the duel,

$$K(\hat\xi;\eta) < -P(a)+Q(a)\overset{2a}{v}_{12}+k(\hat\varepsilon) = v_{13}^a+k(\hat\varepsilon)$$

for $a \leqslant a_{13}$,

If Player I also fires at $\langle a\rangle$, $a\in C_{12}$, then

$$K(\hat\xi;\eta) \leqslant -Q^2(a)\big(1-Q^2(a)\big)+k(\hat\varepsilon) \leqslant v_{13}^a+k(\hat\varepsilon)$$

for

$$-Q^2(a)\big(1-Q^2(a)\big) \leqslant -1+Q(a),$$

i.e., if

$$S\big(Q(a)\big) = Q^3(a)+Q^2(a)-1 \geqslant 0.$$

$S(Q)$ is an increasing function of the variable Q for $Q\in Q(C_{12})$ and

$$S\big(Q(a_{12})\big) \cong S(0.853553) > 0.$$

Then the inequality $S\big(Q(a)\big) \geqslant 0$ holds for $a\in C_{12}$.

If $a \in D_{12}$, we obtain the inequality

$$Q^4(a)-(2+v_{11})Q^2(a)+1 < 0$$

which is satisfied for $Q(a) \geqslant Q(a_{13}) \cong 0.814115$. Thus the proposition is proved.

Let us notice now that Case 1 does not occur as was in the duel (1, 2), $\langle a\rangle$. Player II has always to fire at $\langle a\rangle$ if $a < a_{13}$.

11. Duel (1, 3), $\langle 1, a \wedge c, a\rangle$. Define ξ and η.

Strategy of Player I. Evade if Player II did not fire. If he fired (say at a'), play optimally the obtained duel (1, 2), $\langle a_1', a_1' \wedge c_1\rangle$,

$$a_1' = \max(a', a_1), \quad a_1 = \rangle\langle a\rangle + c\langle.$$

The symbol $\rangle t\langle$ denotes the coordinate of the point at which Player I has been at the moment t.

Strategy of Player II. Fire before $\langle a\rangle + c$ and play optimally afterwards.

Now the strategies ξ and η are optimal in limit and

$$\overset{1a}{v}_{13} = \begin{cases} -1+Q(a) & \text{if } a \in C_{12}, \\ -1+(1+v_{11})Q^2(a) & \text{if } a \in D_{12}. \end{cases} \tag{23}$$

For any strategy ξ of Player I we have

$$K(\xi; \eta) \leqslant -P(a)+Q(a)\overset{2a}{v}_{12}+k(\hat{\varepsilon}) = \overset{1a}{v}_{13}+k(\hat{\varepsilon})$$

if $a \in C_{12} \cup D_{12}$.

On the other hand, if Player II fires a shot at a', then

$$K(\xi; a', \hat{\eta}_0) \geqslant -P(a')+Q(a')\overset{2a'}{v}_{12}-k(\hat{\varepsilon})$$
$$\geqslant -P(a)+Q(a)\overset{2a}{v}_{12}-k(\hat{\varepsilon}) = \overset{1a}{v}_{13}-k(\hat{\varepsilon})$$

for $a' \in C_{12} \cup D_{12}$ (see the duel (1, 3), $\langle a\rangle$).

If Player II applying $\hat{\eta}$ does not fire,

$$K(\xi; \hat{\eta}) = 0 > v_{13}^a \quad \text{for } a \in C_{12} \cup D_{12}.$$

12. Duel (1, 3), $\langle 2, a, a \wedge c\rangle$. Define ξ and η.

Strategy of Player I. Evade if Player II did not fire. If he fired a shot, play optimally the obtained duel.

Strategy of Player II. If Player I did not fire before, fire a shot at $\langle a\rangle + c$ and play optimally afterwards.

We shall prove that

$$\overset{2a}{v}_{13} = \begin{cases} -1+Q(a) & \text{if } a \in C_{12}, \\ -1+(1+v_{11})Q^2(a) & \text{if } a_{12} \leqslant a \leqslant \hat{a}_{13}, \end{cases} \tag{24}$$

where $Q(\hat{a}_{13}) \cong 0.834554$ is the root of the equation

(25) $$Q^4(\hat{a}_{13})-(1+v_{11})Q^2(\hat{a}_{13})-2Q(\hat{a}_{13})+2 = 0,$$

and we shall prove that the strategies ξ and η are optimal in limit.

The proof that Player I assures in limit the value $\overset{2}{v}{}^a_{13}$ for $a \leqslant \hat{a}_{13}$ is the same as for the duel (1, 3), $\langle a \rangle$. To prove that Player II does the same let us assume that Player I applying $\hat{\xi}$ fires before $\langle a \rangle + c$. We obtain

$$\begin{aligned} K(\hat{\xi};\eta) &\leqslant P(a)-Q(a)(1-Q^3(a))+k(\hat{\varepsilon}) \\ &= 1-2Q(a)+Q^4(a)+k(\hat{\varepsilon}) \leqslant \overset{2}{v}{}^a_{12}+k(\hat{\varepsilon}) \end{aligned}$$

if, for $a \in C_{12}$,

$$Q^4(a)-3Q(a)+2 \leqslant 0$$

or, dividing by $Q(a)-1$,

$$S_1(Q(a)) = Q^3(a)+Q^2(a)+Q(a)-2 \geqslant 0.$$

The function $S_1(Q)$ of the variable Q is increasing for $Q \in Q(C_{12})$ and

$$S_1(Q(a_{12})) \cong 0.203965 > 0.$$

Thus the inequality holds.

For $a \in D_{12}$ we obtain the condition

$$S_2(Q(a)) = Q^4(a)-(1+v_{11})Q^2(a)-2Q(a)+2 \leqslant 0.$$

This function is a decreasing one of the variable Q and $S_2(Q(\hat{a}_{13})) = 0$. Then $S_2(Q(a)) \leqslant 0$ for $a \leqslant \hat{a}_{13}$.

The proof in the situations when Player II fires a shot at $\langle a \rangle + c$ or after $\langle a \rangle + c$ or does not fire at all is the same as in the duel (1, 3), $\langle a \rangle$.

13. Results for the duels (1, 3). We have

$$\overset{1}{v}{}^a_{13} = \begin{cases} -1+Q(a) & \text{if } Q(a) \geqslant Q(a_{12}) \cong 0.853553, \\ -1+(1+v_{11})Q^2(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(\check{a}_{12}) \cong 0.780539; \end{cases}$$

$$v^a_{13} = \begin{cases} -1+Q(a) & \text{if } Q(a) \geqslant Q(a_{12}), \\ -1+(1+v_{11})Q^2(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(a_{13}) \cong 0.814115; \end{cases}$$

$$\overset{2}{v}{}^a_{13} = \begin{cases} -1+Q(a) & \text{if } Q(a) \geqslant Q(a_{12}), \\ -1+(1+v_{11})Q^2(a) & \text{if } Q(a_{12}) \geqslant Q(a) \geqslant Q(\hat{a}_{13}) \cong 0.834554. \end{cases}$$

Then

$$\overset{1}{v}{}^a_{13} = v^a_{13} = \overset{2}{v}{}^a_{13} \quad \text{if } a \leqslant \hat{a}_{13}.$$

14. Duel (2, 3), $\langle a \rangle$.

Case 1. Define ξ and η.

Strategy of Player I. If Player II did not fire before, fire a shot at a^{ε}_{23} and play optimally the resulted duel. If he fired, play optimally the duel (2, 2).

Strategy of Player II. If Player I did not fire before, fire at $\langle a_{23}\rangle$ and play optimally afterwards. If he fired (say at a'), play optimally the duel (1, 3), $\langle 1,\ a'\wedge c,\ a'\rangle$. If Player I did not reach the point a_{23}, do not fire.

We remind that the random variable a_{23}^{ξ} is distributed according to an ACPD in the time interval

$$[\langle a_{23}\rangle,\ \langle a_{23}\rangle+\alpha(\varepsilon)],\qquad \alpha(\varepsilon)\to 0\quad \text{if}\quad \varepsilon\to 0$$

(and also if $\hat{\varepsilon}\to 0$, see Section 3).

The number a_{23} satisfies the equations

$$v_{23}^{a}=P(a_{23})+Q(a_{23})\overset{1}{v}{}_{13}^{a_{23}}=-P(a_{23})+Q(a_{23})v_{22}\overset{\text{df}}{=}v_{23}^{a_1}.$$

Suppose that $a_{23}\leqslant a_{12}$. Then $\overset{1}{v}{}_{13}^{a_{23}}=-1+Q(a_{23})$ and we obtain the equation

$$Q^2(a_{23})-(3+v_{22})Q(a_{23})+2=0, \tag{26}$$

$Q(a_{23})\cong 0.882709$. Then the assumption $a_{23}<a_{12}$ holds.

Moreover,

$$v_{23}^{a_1}=-1+(1+v_{22})Q(a_{23})\cong 0.013757. \tag{27}$$

We shall prove that if $a\leqslant a_{23}$, then the strategies ξ and η are optimal in limit and $v_{23}^{a}=v_{23}^{a_1}$.

Suppose that Player II fired before Player I reaches the point a_{23} ($a'<a_{23}$). Then

$$\begin{aligned}K(\xi;\ a',\ \hat{\eta}_0)&\geqslant -P(a')+Q(a')v_{22}-k(\hat{\varepsilon})\\&\geqslant -P(a_{23})+Q(a_{23})v_{22}-k(\hat{\varepsilon})=v_{23}^{a_1}-k(\hat{\varepsilon}).\end{aligned}$$

Suppose that Player II fires after $\langle a_{23}\rangle+\alpha(\varepsilon)$. For such a strategy $\hat{\eta}$,

$$K(\xi;\ \hat{\eta})\geqslant P(a_{23})+Q(a_{23})\overset{1}{v}{}_{13}^{a_{23}}-k(\hat{\varepsilon})=v_{23}^{a_1}-k(\hat{\varepsilon}).$$

On the other hand, if Player I fired before a_{23}, $a'<a_{23}$ since $a_{23}\in C_{12}$, we obtain

$$\begin{aligned}K(a',\ \xi_0;\ \eta)&\leqslant P(a')+Q(a')\overset{1}{v}{}_{13}^{a'}+k(\hat{\varepsilon})\\&=1-2Q(a')+Q^2(a')+k(\hat{\varepsilon})=P^2(a')+k(\hat{\varepsilon})\\&\leqslant 1-2Q(a_{23})+Q^2(a_{23})+k(\hat{\varepsilon})=v_{23}^{a_1}+k(\hat{\varepsilon}).\end{aligned}$$

If Player I applying $\hat{\xi}$ fires at $\langle a_{23}\rangle$, we have

$$K(\hat{\xi};\ \eta)\leqslant Q^2(a_{23})v_{12}^{a_{23}}+k(\hat{\varepsilon})=v_{23}^{a_1}+k(\hat{\varepsilon}).$$

If Player I applying $\hat{\xi}$ did not fire before or at $\langle a_{23}\rangle$, we obtain

$$K(\hat{\xi};\ \eta)\leqslant -P(a_{23})+Q(a_{23})v_{22}+k(\hat{\varepsilon})=v_{23}^{a_1}+k(\hat{\varepsilon}).$$

If, finally, Player I never reaches the point a_{23} and never fires, then

$$K(\hat{\xi}; \eta) = 0 < v_{23}^{a_1}.$$

Thus the proposition is proved.

Case 2. Define ξ and η.

Strategy of Player I. If Player II did not fire before, fire at a^{ε} and play optimally the obtained duel $\langle 1, \rangle a^{\varepsilon} \langle \wedge c, \rangle a^{\varepsilon} \langle \rangle$. If he fired, play optimally the duel (2, 2).

Strategy of Player II. Fire at $\langle a \rangle$ and if Player II did not fire at this moment, play optimally the duel (2, 2). If he fired, play optimally the obtained duel (1, 2), $\langle a_1 \rangle$.

The strategies ξ and η are optimal in limit and

$$v_{23}^{a} = -1 + (1 + v_{22}) Q(a) \tag{28}$$

if

$$0.882709 \cong Q(a_{23}) \geqslant Q(a) \geqslant \frac{1}{1+v_{22}} = Q(\hat{a}_{23}) \cong 0.870730, \tag{29}$$

where

$$Q^2(a_{23}) - (3 + v_{22}) Q(a_{23}) + 2 = 0. \tag{30}$$

To prove this let us assume that Player II fires at $\langle a \rangle$. For such a strategy $\hat{\eta}$ we obtain

$$K(\xi; \hat{\eta}) \geqslant -P(a) + Q(a) v_{22} - k(\hat{\varepsilon}) = v_{23}^{a} - k(\hat{\varepsilon})$$

if $a \leqslant a_{22}$.

Let us assume that Player II fires after $\langle a \rangle + \alpha(\varepsilon)$. In this case, for $a \in C_{12}$ we have

$$\begin{aligned} K(\xi; \hat{\eta}) &\geqslant P(a) + Q(a) \overset{1}{v}{}_{13}^{a} - k(\hat{\varepsilon}) \\ &= 1 - 2Q(a) + Q^2(a) - k(\hat{\varepsilon}) \geqslant v_{23}^{a} - k(\hat{\varepsilon}) \end{aligned}$$

if

$$Q^2(a) - (3 + v_{22}) Q(a) + 2 \geqslant 0$$

i.e., if $a \geqslant a_{23}$.

Player II also assures in limit the value v_{23}^{a} since in the case where Player I fires at $\langle a \rangle$ we have

$$K(\xi; \eta) \leqslant Q^2(a) v_{12}^{a} + k(\hat{\varepsilon}) \leqslant -1 + (1 + v_{22}) Q(a) + k(\hat{\varepsilon})$$

for

$$Q(a) \geqslant \frac{1}{1+v_{22}},$$

and when Player I does not fire at $\langle a\rangle$ we obtain

$$K(\hat{\xi};\eta)\leqslant -P(a)+Q(a)v_{22}+k(\hat{\varepsilon})=v_{23}^{a}+k(\hat{\varepsilon}).$$

Case 3. Define ξ and η.

Strategy of Player I. Fire at $\langle a\rangle$ and if Player II did not fire, play optimally the obtained duel (1, 3), $\langle 1, a\wedge c, a\rangle$. If he fired, play optimally the obtained duel (1, 2), $\langle a_1\rangle$.

Strategy of Player II. Fire at $\langle a\rangle$ and if Player I did not fire, play optimally the duel (2, 2). If he fired, play optimally the obtained duel (1, 2), $\langle a_1\rangle$.

Now ξ and η are optimal in limit for

$$\hat{a}_{23}\leqslant a\leqslant a_{12} \tag{31}$$

and for these a's we have

$$v_{23}^{a}=0. \tag{32}$$

Proof. If Player II fires later then $\langle a\rangle$ or does not fire,

$$K(\xi;\hat{\eta})\geqslant P(a)+Q(a)\overset{1}{v}{}_{13}^{a}-k(\hat{\varepsilon})=P^2(a)-k(\hat{\varepsilon})\geqslant -k(\hat{\varepsilon}).$$

If Player I fires later than $\langle a\rangle$ or does not fire, we have

$$K(\hat{\xi};\eta)\leqslant -P(a)+Q(a)v_{22}+k(\hat{\varepsilon})\leqslant k(\hat{\varepsilon})$$

if

$$Q(a)\leqslant\frac{1}{1+v_{22}}=Q(\hat{a}_{23}).$$

The proposition is proved.

15. Duel (2, 3), $\langle 1, a\wedge c, a\rangle$.

Case 1. When $a\leqslant a_{23}$, the optimal in limit strategies of Players I and II are the same as in the duel (2, 3), $\langle a\rangle$.

Case 2. Define ξ and η.

Strategy of Player I. If Player I did not fire, fire at the random moment a_1^{ε}, $a_1=\rangle\langle a\rangle+c\langle$, and play optimally the resulting duel.

Strategy of Player II. Fire before $\langle a\rangle+c$ and play optimally the duel (2, 2).

The value of the game is

$$\overset{1}{v}{}_{23}^{a}=-1+(1+v_{22})Q(a). \tag{33}$$

The strategies ξ and η are optimal in limit and formula (33) holds surely if

$$a_{23}\leqslant a\leqslant\hat{a}_{23}. \tag{34}$$

The proof is similar to that for the duel (2, 3), $\langle a\rangle$, Case II, and is omitted.

16. Duel (2, 3), $\langle 2,\ a,\ a\wedge c\rangle$.

Case 1. Here also the optimal in limit strategies of Players I and II are the same as in the duel (2, 3), $\langle a\rangle$ if $a \leqslant a_{23}$.

Case 2. Define ξ and η.

Strategy of Player I. Fire before $\langle a\rangle+c$ and play optimally afterwards.

Strategy of Player II. If Player I did not fire before, fire at $\langle a\rangle+c$ and play optimally afterwards. If he fired, play optimally the obtained duel (1, 3), $\langle 1,\ a_1\wedge c_1,\ a_1\rangle$, where $a_1 = \rangle\langle a\rangle+c\langle$.

The value of the game is

$$\overset{2a}{v}_{23} = P(a)+Q(a)\overset{2a}{v}_{13} = P^2(a). \tag{35}$$

These strategies are optimal in limit when

$$a_{23} \leqslant a \leqslant a_{12}. \tag{36}$$

To prove this let us notice that if $a\in C_{12}$, then Player I assures the value $\overset{2a}{v}_{23}$ since he fires first and $\overset{1a}{v}_{13} = P(a)$ for $a\in C_{12}$.

On the other hand, assume that Player I fires before $\langle a\rangle+c$. We have

$$K(\xi;\ \eta) \leqslant P(a)+Q(a)\overset{1a}{v}_{13}+k(\hat{\varepsilon}).$$

If Player I fires at $\langle a\rangle+c$, we obtain

$$K(\xi;\ \eta) \leqslant Q^2(a)v^a_{12}+k(\hat{\varepsilon}) = k(\hat{\varepsilon}) \leqslant P^2(a)+k(\hat{\varepsilon}).$$

If Player I fires after $\langle a\rangle+c$ or does not fire at all, we have

$$K(\xi;\ \eta) \leqslant -P(a)+Q(a)v_{22}+k(\hat{\varepsilon}) \leqslant P^2(a)+k(\hat{\varepsilon})$$

if

$$Q^3(a)-(3+v_{22})Q(a)+2 \geqslant 0,$$

i.e., when $a \geqslant a_{23}$. The proposition is proved.

17. Results for the duels (2, 3). We have

$$\overset{1a}{v}_{23} = \begin{cases} v^{a_1}_{23} \cong 0.013757 & \text{if } Q(a) \geqslant Q(a_{23}) \cong 0.882709,\\ -1+(1+v_{22})Q(a) & \text{if } Q(a_{23}) \geqslant Q(a) \geqslant Q(\hat{a}_{23}) \cong 0.870730; \end{cases}$$

$$v^a_{23} = \begin{cases} v^{a_1}_{23} & \text{if } Q(a) \geqslant Q(a_{23}),\\ -1+(1+v_{22})Q(a) & \text{if } Q(a_{23}) \geqslant Q(a) \geqslant Q(\hat{a}_{23}),\\ 0 & \text{if } Q(\hat{a}_{23}) \geqslant Q(a) \geqslant Q(a_{12}) \cong 0.853553; \end{cases}$$

$$\overset{2a}{v}_{23} = \begin{cases} v^{a_1}_{23} & \text{if } Q(a) \geqslant Q(a_{23}),\\ P^2(a) & \text{if } Q(a_{23}) \geqslant Q(a) \geqslant Q(a_{12}). \end{cases}$$

18. Duel (3, 3). Define ξ and η.

Strategy of Player I. Reach the point a_{23} and if Player II did not fire before, fire at a_{23}^{ξ} and play optimally the resulting duel. If he fired, play optimally the duel (3.2).

Strategy of Player II. If Player I did not fire before, fire at $\langle a_{33}\rangle$ and play optimally the duel (3, 2) (or (2, 2)). If he fired (say at a'), play optimally the obtained duel (2, 3), $\langle 1, a'\wedge c, a'\rangle$. If Player I did not reach the point a_{33}, do not fire.

Assume that the number a_{33} is determined by the equations

$$v_{33} = P(a_{23})+Q(a_{23})\,{}^{1}v_{23}^{a_{23}} = -P(a_{33})+Q(a_{33})v_{32},$$

where v_{32} is given in [23], $v_{32}\cong 0.289928$, a_{23} is defined in Section 14, and

$$Q(a_{23})\cong 0.882709.$$

We have

$${}^{1}v_{23}^{a_{23}} = v_{23}^{a_1}\cong 0.013757,$$

$$v_{33} = 1-(1-v_{23}^{a_1})Q(a_{23})\cong 0.129435, \tag{37}$$

$$Q(a_{33}) = \frac{1+v_{33}}{1+v_{32}}\cong 0.875580. \tag{38}$$

To prove that the strategies ξ and η are optimal in limit and v_{33} is the value of the game assume that Player I fires before a_{23}. For such a strategy $(a', \hat{\eta}_0)$ we obtain

$$\begin{aligned} K(\xi;\, a', \hat{\eta}_0) &\geqslant -P(a')+Q(a')v_{32}-k(\hat{\varepsilon}) \\ &\geqslant -P(a_{23})+Q(a_{23})v_{32}-k(\hat{\varepsilon})\cong 0.138631-k(\hat{\varepsilon}) > v_{33}-k(\hat{\varepsilon}).\end{aligned}$$

Suppose that Player II fires after $\langle a_{23}\rangle+\alpha(\varepsilon)$. For such a strategy $\hat{\eta}$,

$$K(\xi;\hat{\eta})\geqslant P(a_{23})+Q(a_{23})\,{}^{1}v_{23}^{a_{23}}-k(\hat{\varepsilon}) = v_{33}-k(\hat{\varepsilon}).$$

If, on the other hand, Player I fires before a_{33} (strategy $(a', \hat{\xi}_0)$), then

$$\begin{aligned} K(a', \hat{\xi};\eta) &\leqslant P(a')+Q(a')\,{}^{1}v_{23}^{a'}+k(\hat{\varepsilon}) \\ &= \begin{cases} 1-(1-v_{23}^{a_1})Q(a')+k(\hat{\varepsilon}) & \text{if } a'\leqslant a_{23}, \\ 1-2Q(a')+(1+v_{22})Q^2(a')+k(\hat{\varepsilon}) & \text{if } a_{23}\leqslant a' < a_{33}.\end{cases}\end{aligned}$$

The first expression is not greater than

$$1-(1-v_{23}^{a_1})Q(a_{23})+k(\hat{\varepsilon}) = v_{33}+k(\hat{\varepsilon}).$$

The second one is not greater than

$$1-2Q(a_{23})+(1+v_{22})Q^2(a_{23})+k(\hat{\varepsilon}) = 1-(1-v_{23}^{a_1})Q(a_{23})+k(\hat{\varepsilon}) = v_{33}+k(\hat{\varepsilon})$$

since

$$\overset{1}{v}{}^{a_{23}}_{23} = v^{a_1}_{23} = -1+(1+v_{22})Q(a_{23}).$$

If Player I fires at $\langle a_{33}\rangle$,

$$K(\hat{\xi};\eta) \leqslant Q^2(a_{33})v_{22}+k(\hat{\varepsilon}) \cong 0.113816+k(\hat{\varepsilon}) < v_{33}+k(\hat{\varepsilon}).$$

If Player I does not fire at $\langle a_{33}\rangle$ or before,

$$K(\hat{\xi};\eta) \leqslant -P(a_{33})+Q(a_{33})v_{32}+k(\hat{\varepsilon}) = v_{33}+k(\hat{\varepsilon}).$$

If Player I does not reach the point a_{33} and does not fire,

$$K(\hat{\xi};\eta) = 0 < v_{33}.$$

This completes the proof of the proposition.

Let us notice that the strategies ξ and η in the duel (3, 3) are of different kind from those in the duel (2, 2). Now players fire at a^{ε}_{23} and $\langle a_{33}\rangle$, $a_{23} \neq a_{33}$, whereas in the duel (2, 2) at a^{ε}_{22} and $\langle a_{22}\rangle$.

The optimal in limit strategies for the duels (m, n), $n = 4, 5, 6$, $m \leqslant n$, are given in [20]–[23]. Solutions of the duels (m, n), $n \leqslant 6$, $n < m \leqslant 25$ are given in [23].

Noisy duels with retreat after the shots are considered by the author in [16]–[18].

For other noisy duels see [4], [10], [14], [26].

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