$$\sum_{m\mid P(z,u)} \frac{\omega(m)}{m} = \prod_{p\mid P(z,u)} \left(1+\frac{\omega(p)}{p}\right) \leqslant \prod_{p\mid P(z,u)} \left(1-\frac{\omega(p)}{p}\right)^{-1}$$
$$\leqslant \left(\frac{\ln z}{\ln u}\right)^{\varkappa} \left\{1+\frac{K}{\ln u}\right\}$$
$$\leqslant \frac{\ln z}{\ln u}\left\{1+\frac{K}{\ln u}\right\} \ll \varepsilon^{-2}\left\{1+\frac{K}{\ln u}\right\}.$$

On combining the estimates above we obtain

$$\sum_{\substack{d\mid P(z,u)\\ \Delta^{1/(1+\eta)} \leqslant d^- < \Delta^{1+\eta}}} \frac{\omega(d)}{d} \ll \varepsilon^{-4}\left\{1+\frac{K^2\varepsilon^{-4}}{\ln \Delta}\right\}\left\{\varepsilon^7+\frac{K\varepsilon^{-2}}{\ln \Delta}\right\}$$

$$\ll \varepsilon^{-4}\left\{\varepsilon^7+\varepsilon^{-6}\frac{K^3}{\ln \Delta}\right\} \ll \varepsilon^3+\varepsilon^{-10}\frac{K^3}{\ln \Delta}$$

as required.

Now the proof of Theorem 1 is complete.

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Elementary estimates for the Chebyshev function $\psi(x)$ and for the Möbius function $M(x)$

by

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1. The general approach. We have shown in [4] that a technique first devised by Sylvester [11] to evaluate $\liminf \psi(x)/x$ and $\limsup \psi(x)/x$, could be transformed into an elementary method for estimating $\psi(x)$. In this way we established several elementary bounds for ψ and for the related function θ, including Rosser's result [9]:

$$\sup_{x>0}\frac{\psi(x)}{x} = \frac{\psi(113)}{113} < 1.038821, \tag{1.1}$$

and also

$$\sup_{x>0}\frac{\theta(x)}{x} < 1.01456 < \frac{69}{68} \tag{1.2}$$

and

$$\frac{\theta(x)}{x} > 0.985 > \frac{65}{66} \quad \text{for } x \geqslant 11927. \tag{1.3}$$

The present paper is devoted to a generalization and further refinement of these ideas, which allow us to obtain improved bounds for ψ and θ, as well as new estimates for the Möbius sum function.

Let f be a given function defined for all $x>0$ and vanishing identically for $0<x<1$. Assuming that the behaviour of f is sufficiently well known, we consider the problem of estimating its Möbius transform φ defined for $x>0$ by

$$\varphi(x) = \sum_{k\geqslant 1}\mu(k) f\left(\frac{x}{k}\right). \tag{1.4}$$

Let $(r_k)_{k\geqslant 1}$ be an increasing sequence of positive numbers which includes the positive integers. Extending $\mu(t)$ to all $t>0$ by letting $\mu(t)=0$ if t is not an integer, we may rewrite (1.4) in the form

$$\varphi(x) = \sum_{k \geq 1} \mu(r_k) f\left(\frac{x}{r_k}\right). \tag{1.5}$$

Then, if ν is a given function defined for $t > 0$, the new function σ defined for $x > 0$ by

$$\sigma(x) = \sum_{k \geq 1} \nu(r_k) f\left(\frac{x}{r_k}\right) \tag{1.6}$$

should be a good approximation to φ provided that ν is close enough to μ. To make things more precise we take an integer $m > 1$ and choose for ν an integer-valued function satisfying

$$\nu(t) = \mu(t) \quad \text{if } 0 < t < m. \tag{1.7}$$

We assume also that

$$R = \{t > 0: \nu(t) \neq 0\} \tag{1.8}$$

is a finite set of rational numbers and define $(r_k)_{k \geq 0}$ as the increasing sequence of all non-negative multiples of the members of R. Since $r_0 = 0$ and $1 \in R$, the set $\{r_k\}_{k \geq 1}$ includes all the positive integers; hence (1.5) holds and we take the function defined by (1.6) as an approximation to φ. In this case we have simply

$$\sigma(x) = \sum_{r_k \in R} \nu(r_k) f\left(\frac{x}{r_k}\right). \tag{1.9}$$

The following lemma relates the values of φ and σ.

Lemma 1. *If $F(x)$ is defined for all real x by*

$$F(x) = \sum_{r_k \in R} \nu(r_k) \left[\frac{x}{r_k}\right], \tag{1.10}$$

we have

$$\varphi(x) = \sigma(x) - \sum_{r_n \geq m} (F(r_n) - F(r_{n-1})) \varphi\left(\frac{x}{r_n}\right). \tag{1.11}$$

Proof. From (1.4), the Möbius formula gives

$$f(x) = \sum_{j \geq 1} \varphi\left(\frac{x}{j}\right)$$

and so

$$f\left(\frac{x}{r_k}\right) = \sum_{j \geq 1} \varphi\left(\frac{x}{jr_k}\right)$$

for each $k \geq 1$. Hence (1.6) yields

$$\sigma(x) = \sum_{k \geq 1} \nu(r_k) \sum_{j \geq 1} \varphi\left(\frac{x}{jr_k}\right) = \sum_{k, j \geq 1} \nu(r_k) \varphi\left(\frac{x}{jr_k}\right).$$

Since jr_k is a term of the sequence $(r_n)_{n \geq 1}$, letting $r_n = jr_k$ we obtain

$$\sigma(x) = \sum_{n \geq 1} \sum_{r_k | r_n} \nu(r_k) \varphi\left(\frac{x}{r_n}\right). \tag{1.12}$$

On the other hand, recalling again that $\{r_n\}_{n \geq 1}$ includes all the positive multiples of r_k we deduce that

$$\left[\frac{r_n}{r_k}\right] - \left[\frac{r_{n-1}}{r_k}\right] = \begin{cases} 1 & \text{if } r_k | r_n, \\ 0 & \text{if } r_k \nmid r_n \end{cases}$$

and so

$$\sum_{r_k | r_n} \nu(r_k) = \sum_{k \geq 1} \nu(r_k) \left(\left[\frac{r_n}{r_k}\right] - \left[\frac{r_{n-1}}{r_k}\right]\right).$$

In view of (1.10) this gives

$$\sum_{r_k | r_n} \nu(r_k) = F(r_n) - F(r_{n-1}), \tag{1.13}$$

and (1.12) is transformed into

$$\sigma(x) = \sum_{n \geq 1} (F(r_n) - F(r_{n-1})) \varphi\left(\frac{x}{r_n}\right).$$

Now from (1.7) and the identity

$$\sum_{k \geq 1} \mu(k) \left[\frac{x}{k}\right] = 1 \quad \text{if } x \geq 1,$$

we see that $F(r_n) = 1$ if $1 \leq r_n < m$. As $r_1 = 1$ and $F(r_0) = 0$, we obtain

$$\sigma(x) = \varphi(x) + \sum_{r_n \geq m} (F(r_n) - F(r_{n-1})) \varphi\left(\frac{x}{r_n}\right),$$

which is equivalent to (1.11).

As the identity (1.11) expresses $\varphi(x)$ in terms of $\varphi(x/r_k)$, with $r_k \geq m \geq 2$, and the "known" function $\sigma(x)$, it suggests the possibility of estimating $\varphi(x)$ by some kind of recursive method, provided that we have enough information about $F(x)$. To develop this idea it is convenient to restrict further the admissible functions ν. In view of the identity

$$\sum_{k \geq 1} \frac{\mu(r_k)}{r_k} = \sum_{k \geq 1} \frac{\mu(k)}{k} = 0,$$

we impose on ν the additional condition

$$\sum_{r_k \in R} \frac{\nu(r_k)}{r_k} = 0. \tag{1.14}$$

Since

$$F(x) = \sum_{r_k \in R} \nu(r_k) \frac{x}{r_k} - \sum_{r_k \in R} \nu(r_k) \left\{\frac{x}{r_k}\right\}$$

we have then

$$F(x) = -\sum_{r_k \in R} \nu(r_k) \left\{\frac{x}{r_k}\right\}, \tag{1.15}$$

which shows, in particular, that F is bounded. To estimate $F(x)$ efficiently we need the following lemma.

LEMMA 2. *Let S be a subset of R, τ a common multiple of the members of S and d the l.c.m. of the denominators of the members of S. Define $G(x)$ for all real x by*

$$G(x) = -\sum_{r_k \in S} \nu(r_k) \left\{\frac{x}{r_k}\right\}, \tag{1.16}$$

then G has period τ and we have, for every integer n,

$$G\left(\frac{n}{d}\right) + G\left(\tau - \frac{n+1}{d}\right) = -\sum_{r_k \in S} \nu(r_k) + \frac{1}{d} \sum_{r_k \in S} \frac{\nu(r_k)}{r_k}. \tag{1.17}$$

Proof. As τ/r_k is an integer for all $r_k \in S$, (1.16) immediately gives $G(x+\tau) = G(x)$, which shows that G has period τ. Then we have

$$G\left(\frac{n}{d}\right) + G\left(\tau - \frac{n+1}{d}\right) = G\left(\frac{n}{d}\right) + G\left(-\frac{n+1}{d}\right)$$
$$= \sum_{r_k \in S} -\nu(r_k) \left(\left\{\frac{n}{dr_k}\right\} + \left\{-\frac{n+1}{dr_k}\right\}\right).$$

Since dr_k is an integer for every $r_k \in S$, we have

$$\left\{\frac{n}{dr_k}\right\} + \left\{-\frac{n+1}{dr_k}\right\} = 1 - \frac{1}{dr_k},$$

and (1.17) follows.

Taking $S = R$ in this lemma, we conclude that F has a period τ equal to the least common multiple of the members of R. Also, if d is the l.c.m. of the denominators of the members of R, (1.17) gives

$$F\left(\frac{n}{d}\right) + F\left(\tau - \frac{n+1}{d}\right) = -\sum_{r_k \in R} \nu(r_k). \tag{1.18}$$

From (1.10) we see that F can only change at x if x is a multiple of some $r_k \in R$ and this shows that F is constant in each interval $[r_n, r_{n+1})$. Then, letting $\tau = q/d$, the mean value of F is given by

$$w = \frac{1}{q} \sum_{n=0}^{q-1} F\left(\frac{n}{d}\right) = \frac{1}{2q} \left(\sum_{n=0}^{q-1} F\left(\frac{n}{d}\right) + \sum_{n=0}^{q-1} F\left(\tau - \frac{n+1}{d}\right)\right),$$

and (1.18) yields

$$w = -\tfrac{1}{2} \sum_{r_k \in R} \nu(r_k). \tag{1.19}$$

Representing the maximum and the minimum of F respectively by s^+ and s^-, we also deduce from (1.18) that

$$s^+ + s^- = -\sum_{r_k \in R} \nu(r_k),$$

or

$$s^+ + s^- = 2w. \tag{1.20}$$

With this information about $F(x)$ we now return to the identity (1.11) and show that the difference $\varphi(x) - \sigma(x)$ can be expressed, in several ways, as a finite sum of alternating series.

For each integer s satisfying $s^- < s \leqslant s^+$, let $p_0(s)$ be the first term r_n such that $F(r_n) < s$. We define an increasing sequence $(p_i(s))_{i \geqslant 0}$ in the set $\{r_n\}_{n \geqslant 1}$ by the condition of $[p_{2k-1}(s), p_{2k}(s))$ being a maximal interval where $F(x) \geqslant s$ holds. Since $F(x) = 1$ for $1 \leqslant r_k < m$, we have

$$p_0(s) = 1 \quad \text{if } 1 < s \leqslant s^+;$$

and

$$p_0(s) \geqslant m \quad \text{if } s^- < s \leqslant 1.$$

On the other hand, each term $r_n \geqslant m$ has the form $p_{2k+1}(s)$ iff $F(r_n) \geqslant s > F(r_{n-1})$ and the form $p_{2k}(s)$ iff $F(r_{n-1}) \geqslant s > F(r_n)$. From this we obtain the identity

$$\sum_{r_n \geqslant m} (F(r_n) - F(r_{n-1})) \varphi\left(\frac{x}{r_n}\right) \tag{1.21}$$
$$= -\sum_{s^- < s \leqslant 1} \varphi\left(\frac{x}{p_0(s)}\right) + \sum_{s^- < s \leqslant s^+} \sum_{k \geqslant 1} \left(\varphi\left(\frac{x}{p_{2k-1}(s)}\right) - \varphi\left(\frac{x}{p_{2k}(s)}\right)\right),$$

since the coefficient of $\varphi(x/r_n)$ in the second member is exactly $F(r_n) - F(r_{n-1})$.

For each integer s in the range $s^- < s \leqslant s^+$ we define also an increasing sequence $(q_i(s))_{i \geqslant 0}$ by choosing $q_0(s)$ as the first term r_n such that $F(r_n) \geqslant s$ and by the condition of $[q_{2k-1}(s), q_{2k}(s))$ being a maximal interval where $F(x) < s$ holds. We have

$$q_k(s) = p_{k+1}(s) \quad \text{if } 1 < s \leqslant s^+;$$

and

$$q_{k+1}(s) = p_k(s) \quad \text{if } s^- < s \leqslant 1,$$

$$(1.22) \quad \sum_{r_n \geqslant m} (F(r_n) - F(r_{n-1}))\,\varphi\left(\frac{x}{r_n}\right) = \sum_{1 < s \leqslant s^+} \varphi\left(\frac{x}{q_0(s)}\right) - \sum_{s^- < s \leqslant s^+} \sum_{k \geqslant 1} \left(\varphi\left(\frac{x}{q_{2k-1}(s)}\right) - \varphi\left(\frac{x}{q_{2k}(s)}\right)\right).$$

Finally, from (1.11), (1.21) and (1.22) we obtain

$$(1.23) \quad \varphi(x) = \sigma(x) + \sum_{s^- < s \leqslant 1} \varphi\left(\frac{x}{p_0(s)}\right) - \sum_{s^- < s \leqslant s^+} \sum_{k \geqslant 1} \left(\varphi\left(\frac{x}{p_{2k-1}(s)}\right) - \varphi\left(\frac{x}{p_{2k}(s)}\right)\right),$$

$$(1.24) \quad \varphi(x) = \sigma(x) - \sum_{1 < s \leqslant s^+} \varphi\left(\frac{x}{q_0(s)}\right) + \sum_{s^- < s \leqslant s^+} \sum_{k \geqslant 1} \left(\varphi\left(\frac{x}{q_{2k-1}(s)}\right) - \varphi\left(\frac{x}{q_{2k}(s)}\right)\right)$$

and also the more symmetrical identity

$$(1.25) \quad \varphi(x) = \sigma(x) + \sum_{s^- < s \leqslant 1} \sum_{k \geqslant 0} \left(\varphi\left(\frac{x}{p_{2k}(s)}\right) - \varphi\left(\frac{x}{p_{2k+1}(s)}\right)\right) - \sum_{1 < s \leqslant s^+} \sum_{k \geqslant 0} \left(\varphi\left(\frac{x}{q_{2k}(s)}\right) - \varphi\left(\frac{x}{q_{2k+1}(s)}\right)\right).$$

The best way of dealing with these identities depends on the nature of the function φ to be estimated. We shall consider separately the cases of the Chebyshev function

$$\psi(x) = \sum_{\substack{k,p \\ p^k \leqslant x}} \log p; \qquad p \text{ prime}$$

and of the Möbius sum function

$$M(x) = \sum_{k \leqslant x} \mu(k).$$

2. Estimates for $\psi(x)$. From the well-known identity

$$\log [x]! = \sum_{k \geqslant 1} \psi\left(\frac{x}{k}\right),$$

the Möbius inversion formula gives

$$\psi(x) = \sum_{k \geqslant 1} \mu(k) \log \left[\frac{x}{k}\right]!,$$

which is a particular case of (1.4) with $\varphi(x) = \psi(x)$ and $f(x) = \log [x]!$. Choosing a suitable function ν we then take

$$(2.1) \quad \sigma(x) = \sum_{r_k \in R} \nu(r_k) \log \left[\frac{x}{r_k}\right]!$$

as a first approximation to $\psi(x)$.

To evaluate $\sigma(x)$ we observe that

$$\log \left[\frac{x}{r_k}\right]! = \left[\frac{x}{r_k}\right] \log \left[\frac{x}{r_k}\right] - \left[\frac{x}{r_k}\right] + O\left(\log \left[\frac{x}{r_k}\right]\right)$$

and

$$\left[\frac{x}{r_k}\right] = \frac{x}{r_k} + O(1)$$

when x increases to infinity. This gives

$$\sigma(x) = (x \log x - x) \sum_{r_k \in R} \frac{\nu(r_k)}{r_k} - \sum_{r_k \in R} \nu(r_k) \frac{x}{r_k} \log r_k + O(\log x)$$

$$= - \sum_{r_k \in R} \nu(r_k) \frac{x}{r_k} \log r_k + O(\log x),$$

where the last identity follows from (1.14). Introducing the constant

$$(2.2) \quad \omega = - \sum_{r_k \in R} \frac{\nu(r_k)}{r_k} \log r_k,$$

we have then

$$(2.3) \quad \sigma(x) = \omega x + O(\log x)$$

which shows, in particular, that

$$(2.4) \quad \lim_{x \to +\infty} \frac{\sigma(x)}{x} = \omega.$$

For our purposes, however, we also need a more precise version of (2.3) and this is given in the following lemma.

LEMMA 3. *We have*

(2.5) $$|\sigma(x)-\omega x| < (s^+ - w)\log x + h + \gamma/x \quad \textit{for } x > \max R,$$

where s^+ and w are, respectively, the maximum and the mean value of the function F, and h, γ are constants defined by

(2.6) $$h = \tfrac{1}{2}\sum_{r_k\in R} |\nu(r_k)|\log r_k + |w|\log 2\pi,$$

(2.7) $$\gamma = \sum_{r_k\in R} |\nu(r_k)|\, r_k.$$

Proof. For each $u > 1$, Stirling's formula gives

(2.8) $$\log[u]! = ([u]+\tfrac{1}{2})\log[u] - [u] + \tfrac{1}{2}\log 2\pi + \varrho(u)$$

$$\text{with } 0 < \varrho(u) < 1/(4[u]).$$

On the other hand, from the inequality

$$\log(1+t) \leqslant t \quad \text{if } t > -1,$$

we obtain easily

$$\{u\}/u \leqslant \log u - \log[u] \leqslant \{u\}/[u].$$

This gives

$$([u]+\tfrac{1}{2})(\log u - \log[u]) \leqslant \{u\} + \{u\}/[u]$$

and

$$([u]+\tfrac{1}{2})(\log u - \log[u]) \geqslant (u-\tfrac{1}{2})(\log u - \log[u]) \geqslant \{u\} - \{u\}/2u.$$

Hence, from (2.8) we deduce

(2.9) $$\log[u]! = ([u]+\tfrac{1}{2})\log u - u + \tfrac{1}{2}\log 2\pi + \varepsilon(u)$$

with

$$-\frac{\{u\}}{2[u]} < \varepsilon(u) < \frac{1}{4[u]} + \frac{\{u\}}{2u}.$$

Observing that $\{u\} < 1$ and $u = [u]+\{u\} < [u]+1 \leqslant 2[u]$, we have simply

(2.10) $$|\varepsilon(u)| < 1/u.$$

Now taking $u = x/r_k > 1$ for each $r_k \in R$, we obtain from (1.10), (1.14), (1.19), (2.1) and (2.2)

$$\sigma(x) = \omega x + (F(x)-w)\log x + \sum_{r_k\in R} \nu(r_k)(\{x/r_k\}-\tfrac{1}{2})\log r_k -$$
$$- w\log 2\pi + \sum_{r_k\in R} \nu(r_k)\,\varepsilon(x/r_k).$$

Using (2.10) and introducing the constants h and γ this gives

$$|\sigma(x)-\omega x| < |F(x)-w|\log x + h + \gamma/x.$$

However, if s^- is the minimum of F, (1.20) shows that $s^+ - w = w - s^-$. Hence

$$|F(x)-w| \leqslant s^+ - w,$$

and (2.5) holds.

To obtain estimates for $\psi(x)$ from our estimates of $\sigma(x)$ we use identities (1.23) and (1.24). Since ψ is a monotonic increasing function, we now have

$$\psi\left(\frac{x}{p_{2k-1}(s)}\right) - \psi\left(\frac{x}{p_{2k}(s)}\right) \geqslant 0$$

and

$$\psi\left(\frac{x}{q_{2k-1}(s)}\right) - \psi\left(\frac{x}{q_{2k}(s)}\right) \geqslant 0$$

in the second members of (1.23) and (1.24) respectively. Hence, denoting by I_s and J_s two finite sets of indices, we obtain

$$\psi(x) \leqslant \sigma(x) + \sum_{s^- < s \leqslant 1} \psi\left(\frac{x}{p_0(s)}\right) - \sum_{s^- < s \leqslant s^+} \sum_{k\in I_s} \left(\psi\left(\frac{x}{p_{2k-1}(s)}\right) - \psi\left(\frac{x}{p_{2k}(s)}\right)\right)$$

and

$$\psi(x) \geqslant \sigma(x) - \sum_{1 < s \leqslant s^+} \psi\left(\frac{x}{q_0(s)}\right) + \sum_{s^- < s \leqslant s^+} \sum_{k\in J_s} \left(\psi\left(\frac{x}{q_{2k-1}(s)}\right) - \psi\left(\frac{x}{q_{2k}(s)}\right)\right).$$

Replacing, if necessary, some of the $p_{2k}(s)$ or $q_{2k}(s)$ by suitable lower bounds, we then arrive at two inequalities of the form

(2.11) $$\psi(x) \leqslant \sigma(x) + \sum_{s^- < s \leqslant 1} \psi\left(\frac{x}{a_s}\right) - \sum_{s^- < s \leqslant s^+} \sum_{k=1}^{\alpha_s} \left(\psi\left(\frac{x}{b'_k(s)}\right) - \psi\left(\frac{x}{c'_k(s)}\right)\right)$$

and

(2.12) $$\psi(x) \geqslant \sigma(x) - \sum_{1 < s \leqslant s^+} \psi\left(\frac{x}{a_s}\right) + \sum_{s^- < s \leqslant s^+} \sum_{k=1}^{\beta_s} \left(\psi\left(\frac{x}{b''_k(s)}\right) - \psi\left(\frac{x}{c''_k(s)}\right)\right).$$

Here the constants a_s verify

$$a_s \leqslant q_0(s) \quad \text{if } s^- < s \leqslant 1; \quad a_s \leqslant p_0(s) \quad \text{if } 1 < s \leqslant s^+$$

and $[b'_k(s), c'_k(s))$, $[b''_k(s), c''_k(s))$ are intervals, not necessarily maximal, where $F(x) \geqslant s$ and $F(x) < s$ respectively.

Simplifying the notation we may rewrite (2.11) and (2.12) as

$$\psi(x) \leqslant \sigma(x) + \sum_{s^- < s \leqslant 1} \psi\left(\frac{x}{a_s}\right) - \sum_{k \in I}\left(\psi\left(\frac{x}{b_k}\right) - \psi\left(\frac{x}{c_k}\right)\right) \tag{2.13}$$

and

$$\psi(x) \geqslant \sigma(x) - \sum_{1 < s \leqslant s^+} \psi\left(\frac{x}{a_s}\right) + \sum_{k \in J}\left(\psi\left(\frac{x}{b_k}\right) - \psi\left(\frac{x}{c_k}\right)\right), \tag{2.14}$$

where I, J are finite sets of indices and $b_k < c_k$ for each $k \in I \cup J$.

Now let

$$\lambda^+ = \limsup \frac{\psi(x)}{x}; \quad \lambda^- = \liminf \frac{\psi(x)}{x}.$$

Dividing both members of (2.13) and (2.14) by x and using (2.4) we obtain

$$\lambda^+ \leqslant \omega + \sum_{s^- < s \leqslant 1} \frac{\lambda^+}{a_s} - \sum_{k \in I}\left(\frac{\lambda^-}{b_k} - \frac{\lambda^+}{c_k}\right) \tag{2.15}$$

and

$$\lambda^- \geqslant \omega - \sum_{1 < s \leqslant s^+} \frac{\lambda^+}{a_s} + \sum_{k \in J}\left(\frac{\lambda^-}{b_k} - \frac{\lambda^+}{c_k}\right). \tag{2.16}$$

Introducing the constants

$$A = 1 - \sum_{s^- < s \leqslant 1} \frac{1}{a_s} - \sum_{k \in I} \frac{1}{c_k}; \quad B = \sum_{k \in I} \frac{1}{b_k} \tag{2.17}$$

and

$$C = \sum_{1 < s \leqslant s^+} \frac{1}{a_s} + \sum_{k \in J} \frac{1}{c_k}; \quad D = 1 - \sum_{k \in J} \frac{1}{b_k}, \tag{2.18}$$

we can rewrite (2.15) and (2.16) in the form

$$A\lambda^+ + B\lambda^- \leqslant \omega; \quad C\lambda^+ + D\lambda^- \geqslant \omega. \tag{2.19}$$

On the other hand we have

$$\sum_{k \in I \cup J}\left(\frac{1}{b_k} - \frac{1}{c_k}\right) < \sum_{s^- < s \leqslant s^+} \sum_{k \geqslant 1}\left(\frac{1}{p_{2k-1}(s)} - \frac{1}{p_{2k}(s)} + \frac{1}{q_{2k-1}(s)} - \frac{1}{q_{2k}(s)}\right)$$
$$< \sum_{s^- < s \leqslant 1} \frac{1}{p_0(s)} + \sum_{1 < s \leqslant s^+} \frac{1}{q_0(s)} \leqslant \sum_{s^- < s \leqslant s^+} \frac{1}{a_s},$$

which shows that $A - C < D - B$. Assuming $A > C$ we then have $D > B$ and this implies that A, B, C, D are all positive. Hence, from (2.19) we deduce

$$(AD - BC)\lambda^+ \leqslant \omega(D - B); \quad (AD - BC)\lambda^- \geqslant \omega(A - C).$$

As $AD - BC > 0$ we conclude the following

THEOREM 1. *If $A > C$ we have*

$$\limsup \frac{\lambda(x)}{x} \leqslant \frac{\omega(D - B)}{AD - BC} \tag{2.20}$$

and

$$\liminf \frac{\lambda(x)}{x} \geqslant \frac{\omega(A - C)}{AD - BC}. \tag{2.21}$$

The quality of the bounds given by (2.20) and (2.21) is measured by the ratio

$$\varrho = (D - B)/(A - C) \tag{2.22}$$

which should be as near to unity as possible. Introducing one more term of the form $\psi(x/b_k) - \psi(x/c_k)$ in the right side of (2.13) or (2.14), we obtain new constants A', B', C', D', and Theorem 1 gives new bounds for $\psi(x)/x$. Since the new ratio is

$$\varrho' = \frac{D' - B'}{A' - C'} = \frac{D - B - 1/b_k}{A - C - 1/c_k},$$

and the condition $\varrho' < \varrho$ is equivalent to

$$\varrho < c_k/b_k \tag{2.23}$$

we see that these new bounds are sharper if and only if (2.23) holds.

To give a simple example, take $m = 6$ in (1.7). Following Chebyshev [3] we define $\nu(30) = 1$ and $\nu(t) = 0$ if $t \geqslant 6$, $t \neq 30$. In this case we have $r_k = k$ for each $k \geqslant 0$ and the identity

$$1 + \tfrac{1}{30} - \tfrac{1}{2} - \tfrac{1}{3} - \tfrac{1}{5} = 0 \tag{2.24}$$

shows that condition (1.14) is satisfied. The function F is now given by

$$F(x) = [x] + \left[\frac{x}{30}\right] - \left[\frac{x}{2}\right] - \left[\frac{x}{3}\right] - \left[\frac{x}{5}\right]$$

and has period $\tau = 30$. By direct computation along a period we can verify that $s^- = 0$, $s^+ = 1$ and we also see that

$$(p_k(1))_{k \geqslant 0} = (6, 7, 10, 13, 15, 17, 18, 19, 20, 23, 24, 29, 30, 31, 36, \ldots).$$

Hence, taking $I = J = \emptyset$ in (2.13) and (2.14), we obtain

(2.25) $$\psi(x) \leqslant \sigma(x) + \psi(x/6); \quad \psi(x) \geqslant \sigma(x),$$

and (2.22) gives $\varrho = 6/5$. However, we also have

(2.26) $$\psi(x) \leqslant \sigma(x) + \psi(x/6) - \psi(x/7) + \psi(x/10)$$

and

(2.27) $$\psi(x) \geqslant \sigma(x) + \psi(x/24) - \psi(x/29).$$

Since $10/7 > \varrho$, $29/24 > \varrho$, these last inequalities yield better bounds than (2.25). At this point (2.22) gives $\varrho = 1.16668\ldots > 7/6$ and we easily verify that no further improvement is possible. Actually, since

$$\omega = \frac{\log 2}{2} + \frac{\log 3}{3} + \frac{\log 5}{5} - \frac{\log 30}{30} = 0.92129202\ldots,$$

(2.25) leads to Chebyshev's classical estimates

$$\limsup \frac{\psi(x)}{x} < 1.105556; \quad \liminf \frac{\psi(x)}{x} > 0.921292,$$

while (2.26) and (2.27) yield

$$\limsup \frac{\psi(x)}{x} < 1.076578; \quad \liminf \frac{\psi(x)}{x} > 0.922610.$$

We note that these last bounds have a smaller ratio than those obtained recently in [6] with a more complicated method. They could easily be improved by taking higher values of m in (1.7) but these results are superseded by the prime number theorem which shows that $\lim \psi(x)/x = 1$.

Our next theorem gives finite estimates for ψ of the type

$$L^- < \psi(x)/x < L^+ \quad \text{if } x \geqslant N.$$

THEOREM 2. *Let $N > 1$ and choose $\lambda^+(t)$, $\lambda^-(t)$, ω^+, ω^- such that*

(2.28) $$\lambda^-(t) \leqslant \psi(x)/x \leqslant \lambda^+(t) \quad \textit{if } t \leqslant x < N$$

and

(2.29) $$\omega^- < \sigma(x)/x < \omega^+ \quad \textit{if } x \geqslant N.$$

Assuming (2.13) *and* (2.14), *we define L^+, L^- by*

(2.30) $$L^+ = \omega^+ + \sum_{s^- < s \leqslant 1} \frac{1}{a_s} \lambda^+\left(\frac{N}{a_s}\right) - \sum_{k \in I} \left(\frac{1}{b_k} \lambda^-\left(\frac{N}{b_k}\right) - \frac{1}{c_k} \lambda^+\left(\frac{N}{c_k}\right)\right)$$

and

(2.31) $$L^- = \omega^- - \sum_{1 < s \leqslant s^+} \frac{1}{a_s} \lambda^+\left(\frac{N}{a_s}\right) + \sum_{k \in J} \left(\frac{1}{b_k} \lambda^-\left(\frac{N}{b_k}\right) - \frac{1}{c_k} \lambda^+\left(\frac{N}{c_k}\right)\right).$$

Then if

(2.32) $$L^+ \leqslant \lambda^+(N/a_s) \quad \textit{for } s^- < s \leqslant s^+; \quad L^+ \leqslant \lambda^+(N/c_k) \quad \textit{for } k \in I \cup J$$

and

(2.33) $$L^- \geqslant \lambda^-(N/b_k) \quad \textit{for } k \in I \cup J,$$

we have

(2.34) $$L^- < \psi(x)/x < L^+ \quad \textit{for } x \geqslant N.$$

Proof. From (2.13), (2.14) and (2.29), we obtain for $x \geqslant N$

(2.35) $$\frac{\psi(x)}{x} < \omega^+ + \sum_{s^- < s \leqslant 1} \frac{1}{a_s} \psi\left(\frac{x}{a_s}\right) \Big/ \frac{x}{a_s} - \sum_{k \in I} \left(\frac{1}{b_k} \psi\left(\frac{x}{b_k}\right) \Big/ \frac{x}{b_k} - \frac{1}{c_k} \psi\left(\frac{x}{c_k}\right) \Big/ \frac{x}{c_k}\right)$$

and

(2.36) $$\frac{\psi(x)}{x} > \omega^- - \sum_{1 < s \leqslant s^+} \frac{1}{a_s} \psi\left(\frac{x}{a_s}\right) \Big/ \frac{x}{a_s} + \sum_{k \in J} \left(\frac{1}{b_k} \psi\left(\frac{x}{b_k}\right) \Big/ \frac{x}{b_k} - \frac{1}{c_k} \psi\left(\frac{x}{c_k}\right) \Big/ \frac{x}{c_k}\right).$$

Take an integer $n \geqslant N$ and assume that (2.34) holds for $N \leqslant x < n$. If $n \leqslant x < n+1$ we have

$$\frac{N}{a_s} \leqslant \frac{x}{a_s} < \frac{n+1}{a_s} \leqslant \frac{n+1}{m} < n \quad \text{for } s^- < s \leqslant s^+.$$

From the inductive hypothesis and (2.32) we deduce then

$$\psi\left(\frac{x}{a_s}\right) \leqslant L^+ \frac{x}{a_s} \leqslant \lambda^+\left(\frac{N}{a_s}\right) \frac{x}{a_s} \quad \text{for } N \leqslant \frac{x}{a_s} < n.$$

On the other hand, from the definition of $\lambda^+(t)$ we get

$$\psi\left(\frac{x}{a_s}\right) \leqslant \lambda^+\left(\frac{N}{a_s}\right) \frac{x}{a_s} \quad \text{for } \frac{N}{a_s} \leqslant \frac{x}{a_s} < N.$$

It follows that

$$\psi\left(\frac{x}{a_s}\right) \leqslant \lambda^+\left(\frac{N}{a_s}\right)\frac{x}{a_s} \quad \text{for } n \leqslant x < n+1,\ s^- < s \leqslant s^+.$$

In the same way we obtain

$$\psi\left(\frac{x}{b_k}\right) \geqslant \lambda^-\left(\frac{N}{b_k}\right)\frac{x}{b_k}; \quad \psi\left(\frac{x}{c_k}\right) \leqslant \lambda^+\left(\frac{N}{c_k}\right)\frac{x}{c_k}$$

for $n \leqslant x < n+1$ and all $k \in I \cup J$. Then, from (2.35), (2.36), (2.30), (2.31) we conclude that (2.34) holds for $n \leqslant x < n+1$ and the theorem is proved by induction.

When applying this theorem, if (2.32) or (2.33) is not true for some of the $\lambda^+(t)$, $\lambda^-(t)$, we replace them by the new bounds L^+, L^- in (2.30), (2.31) and we determine a new pair L^+, L^-. This process converges quickly and we soon arrive at a pair L^+, L^- satisfying the conditions (2.32) and (2.33). However, to deduce Rosser's inequality (1.1) we only need the following special case of the theorem.

COROLLARY 1. *Assuming* (2.13), (2.14) *and* (2.29) *let* A, B, C, D *be given by* (2.17), (2.18). *If* $A > C$ *define*

$$L^+ = \frac{D\omega^+ - B\omega^-}{AD-BC}; \quad L^- = \frac{A\omega^- - C\omega^+}{AD-BC}. \tag{2.37}$$

Take also N_0, N_0' *such that* $a_s \leqslant N_0$ *for* $s^- < s \leqslant s^+$, $c_k \leqslant N_0$ *for* $k \in I \cup J$ *and* $b_k \leqslant N_0'$ *for* $k \in I \cup J$. *Then if*

$$\frac{\psi(x)}{x} \leqslant L^+ \quad \text{for } \frac{N}{N_0} \leqslant x < N; \quad \frac{\psi(x)}{x} \geqslant L^- \quad \text{for } \frac{N}{N_0'} \leqslant x < N, \tag{2.38}$$

we have

$$L^- < \psi(x)/x < L^+ \quad \text{for } x \geqslant N. \tag{2.39}$$

Proof. (2.38) shows that we may take

$$\lambda^+(t) = L^+ \quad \text{for } N/N_0 \leqslant t < N; \quad \lambda^-(t) = L^- \quad \text{for } N/N_0' \leqslant t < N$$

in (2.28). Then (2.30) and (2.31) are replaced by

$$L^+ = \omega^+ + \sum_{s^- < s \leqslant 1} \frac{L^+}{a_s} - \sum_{k \in I}\left(\frac{L^-}{b_k} - \frac{L^+}{c_k}\right)$$

and

$$L^- = \omega^- - \sum_{1 < s \leqslant s^+} \frac{L^+}{a_s} + \sum_{k \in J}\left(\frac{L^-}{b_k} - \frac{L^+}{c_k}\right),$$

respectively.

As these conditions are equivalent to (2.37), the estimate (2.39) follows from the previous theorem.

To deduce (1.1) we take $m = 17$ in (1.7). From (2.24) we obtain

$$1 + \frac{1}{15} - \frac{1}{2} - \frac{1}{3} - \frac{1}{5} - \frac{1}{30} = 0.$$

Adding the identities

$$\frac{1}{6} - \frac{1}{7} - \frac{1}{42} = \frac{1}{10} - \frac{1}{11} - \frac{1}{110} = \frac{1}{14} + \frac{1}{182} - \frac{1}{13} = 0,$$

we deduce that

$$\sum_{k=1}^{16} \frac{\mu(k)}{k} + \frac{1}{182} - \frac{1}{30} - \frac{1}{42} - \frac{1}{110} = 0.$$

Thus, condition (1.14) is satisfied if we define $\nu(t)$ by $\nu(t) = \mu(t)$ for $1 \leqslant t < 17$ and

$$\nu(182) = 1; \quad \nu(30) = \nu(42) = \nu(110) = -1;$$
$$\nu(t) = 0 \quad \text{otherwise}.$$

In this case $(r_k)_{k \geqslant 1}$ is again the sequence of the positive integers and the function F is given by

$$F(x) = \sum_{k \geqslant 1} \nu(k)\left[\frac{x}{k}\right].$$

To estimate $F(x)$ we use the decomposition

$$F(x) = F_0(x) + F_1(x) + F_2(x) + F_3(x)$$

where

$$F_0(x) = [x] + \left[\frac{x}{15}\right] - \left[\frac{x}{2}\right] - \left[\frac{x}{3}\right] - \left[\frac{x}{5}\right],$$

$$F_1(x) = \left[\frac{x}{6}\right] - \left[\frac{x}{7}\right] - \left[\frac{x}{42}\right],$$

$$F_2(x) = \left[\frac{x}{10}\right] - \left[\frac{x}{11}\right] - \left[\frac{x}{110}\right]$$

and

$$F_3(x) = \left[\frac{x}{14}\right] + \left[\frac{x}{182}\right] - \left[\frac{x}{13}\right].$$

Since

$$F_0(x) = \left\{\frac{x}{2}\right\} + \left\{\frac{x}{3}\right\} + \left\{\frac{x}{5}\right\} - \{x\} - \left\{\frac{x}{15}\right\} < 3$$

and $F_0(x)$ is an integer, we see that $F_0(x) \leqslant 2$; in the same way we obtain $F_1(x) \leqslant 1$, $F_2(x) \leqslant 1$, $F_3(x) \leqslant 0$ and this gives $F(x) \leqslant 4$. From (1.19) we obtain $w = 1/2$, and (1.20) shows that

$$-1 \leqslant F(x) \leqslant 4.$$

As F has period $\tau = 30030$, the terms $p_k(s)$, $q_k(s)$ would be completely determined by computing $F(x)$ for the integral values of x up to 15014 and using (1.18). To obtain (1.1), however, we need only to evaluate $F(x)$ at the integers n such that $0 < n \leqslant 876$. The most efficient way of achieving this is to observe that (1.13) now reduces to

$$\sum_{k|n} \nu(k) = F(n) - F(n-1). \tag{2.40}$$

Hence we determine $\sum_{k|n} \nu(k)$ in the range $0 < n \leqslant 876$ by an obvious sieve method and we compute $F(n)$ recursively starting with $F(16) = 1$. These calculations can be performed easily since the labour required is comparable to that needed to construct a table of primes up to 876. We obtain

$$p_0(1) = 66, \quad p_0(0) > 876; \quad q_0(2) = 17, \quad q_0(3) = 19, \quad q_0(4) = 439.$$

Now, in the inequalities (2.11), (2.12) we take

$$a_1 = 66, \quad a_0 = 877; \quad a_2 = 17, \quad a_3 = 19, \quad a_4 = 439,$$

and we choose the following intervals $[a'_k(s), b'_k(s))$, $[a''_k(s), b''_k(s))$.

	$[a'_k(s), b'_k(s))$	$[a''_k(s), b''_k(s))$
$s = 1$	[67,126) [157,176) [179,220) [223,275) [277,330) [359,429)	———
$s = 2$	[17,22) [23,26) [29,35) [47,52) [59,65) [71,78) [79,88) [191,210)	[26,29) [65,71) [117,139)
$s = 3$	[19,21)	[21,31) [33,61) [63,73) [84,103) [110,193) [208,229) [242,271) [294,323) [325,373) [440,493)
$s = 4$	———	[440,877)

Then (2.17) and (2.18) give

$$A = 0.732878785\ldots, \qquad B = 0.297381533\ldots,$$

$$C = 0.263737183\ldots, \qquad D = 0.802852410\ldots$$

Taking $N = 10^5$ we may apply Lemma 3 with

$$\omega = 1.04652442\ldots; \quad h = 21.15419476\ldots; \quad \gamma = 451.$$

We obtain the estimate

$$\omega^- < \sigma(x)/x < \omega^+ \quad \text{for } x > N,$$

where

$$\omega^- = 1.046025; \quad \omega^+ = 1.047024$$

and (2.37) yields

$$L^+ = 1.038383\ldots < 27/26; \quad L^- = 0.961776\ldots > 25/26.$$

On the other hand, from a table of primes up to 10^5 we can verify that

$$\psi(x)/x < 27/26 \quad \text{if } 114 \leqslant x < 10^5$$

and

$$\psi(x)/x > 25/26 \quad \text{if } 227 \leqslant x < 10^5.$$

Applying Corollary 1 with $N_0 = 877$, $N'_0 = 440$, and noticing that $N/N_0 > 114$, $N/N'_0 > 227$, we conclude that

$$\psi(x)/x < 27/26 \text{ if } x \geqslant 114; \quad \psi(x)/x > 25/26 \text{ if } x \geqslant 227. \tag{2.41}$$

But now Rosser's inequality (1.1) follows simply from

$$\psi(x)/x < \psi(113)/113 = 1.0388205\ldots \quad \text{if } 0 < x < 113.$$

To improve (2.41) substantially we must choose higher values of m in (1.7) and we need also sharp estimates of the type (2.28) holding for wide intervals. With a computer we have obtained several bounds $\lambda^+(t)$, $\lambda^-(t)$ in the range $t \leqslant x < 10^8$ and this enables us to use Theorem 2 with $N = 10^8$. These values of $\lambda^+(t)$ and $\lambda^-(t)$ are listed in Table I which is a more detailed version of a similar table given in [5].

For high values of m in (1.7), the task of choosing a suitable function ν satisfying (1.14) is much simplified if we allow the set R to include non-integer elements. To keep the sequence $(r_k)_{k \geqslant 1}$ as simple as possible within the zone where $F(r_n)$ is to be computed, we search for identities of the form

$$\sum_{k<m} \frac{\mu(k)}{k} - \frac{1}{j} + \frac{1}{r} = 0 \tag{2.42}$$

where j is an integer, $m \leqslant |j| < |r|$ and $5 \cdot 10^7 < |r| < 10^8$.

Testing several functions ν obtained by this method shows that the value $m = 8021$ is nearly optimal. In this case (2.42) holds with $j = 34502$, $r = 92516105.9109\ldots$ and we consequently define $\nu(t)$ by

$$\nu(t) = \mu(t) \quad \text{if } 0 < t < 8021;\ \nu(34502) = -1;\ \nu(r) = 1; \qquad \nu(t) = 0 \text{ otherwise.} \tag{2.43}$$

Now (1.15) gives

$$F(x) = -\sum_{k<m} \mu(k)\left\{\frac{x}{k}\right\} + \left\{\frac{x}{j}\right\} - \left\{\frac{x}{r}\right\}. \tag{2.44}$$

Counting the non-negative terms in the right side of this identity we see that $F(x) < 2440$ but a much better bound for $F(x)$ can be obtained from the following lemma.

LEMMA 4. *Take a positive integer n_0 and let Q be the set of the square-free integers, prime to n_0, in the range $1 \leqslant q < m$. For each $q \in Q$ denote by S_q the set of the positive integers d such that $d \mid n_0$ and $qd < m$. Defining*

$$G_q(x) = -\mu(q) \sum_{d \in S_q} \mu(d) \left\{\frac{x}{d}\right\}$$

and

$$u_q = -\mu(q) \sum_{d \in S_q} \frac{\mu(d)}{d},$$

we have

$$F(x) \leqslant \Big[\sum_{q \in Q} (m_q + \varepsilon_q)\Big] + 1, \tag{2.45}$$

with

$$\varepsilon_q = \begin{cases} 0 & \text{if } u_q \leqslant 0, \\ \dfrac{q-1}{q} u_q & \text{if } u_q > 0 \end{cases}$$

and

$$m_q = \max_k G_q(k).$$

Proof. With $n = [x]$ we easily obtain from (2.44)

$$F(x) = -\sum_{k<m} \mu(k)\left\{\frac{n}{k}\right\} + \left\{\frac{n}{j}\right\} + \frac{\{x\}}{r} - \left\{\frac{x}{r}\right\}.$$

As $0 < j < r$ we have

$$\left\{\frac{n}{j}\right\} + \frac{\{x\}}{r} < 1 - \frac{1}{j} + \frac{1}{r} < 1,$$

and so

$$F(x) < -\sum_{k<m} \mu(k)\left\{\frac{n}{k}\right\} + 1.$$

Since $F(x)$ is an integer, we obtain

$$F(x) \leqslant \left[-\sum_{k<m} \mu(k)\left\{\frac{n}{k}\right\}\right] + 1. \tag{2.46}$$

On the other hand, observing that every square-free integer k in the range $1 \leqslant k < m$ has just one representation of the form $k = qd$ with $q \in Q$ and $d \in S_q$, we see that

$$-\sum_{k<m} \mu(k)\left\{\frac{n}{k}\right\} = \sum_{q \in Q} \sum_{d \in S_q} -\mu(qd)\left\{\frac{n}{qd}\right\}. \tag{2.47}$$

We have now

$$\begin{aligned} \sum_{d \in S_q} -\mu(qd)\left\{\frac{n}{qd}\right\} &= -\mu(q) \sum_{d \in S_q} \mu(d) \left\{\frac{n/q}{d}\right\} \\ &= -\mu(q) \sum_{d \in S_q} \mu(d) \left\{\frac{[n/q]}{d}\right\} + \left\{\frac{n}{q}\right\} u_q \\ &= G_q\left(\left[\frac{n}{q}\right]\right) + \left\{\frac{n}{q}\right\} u_q. \end{aligned}$$

This gives

$$\sum_{d \in S_q} -\mu(qd)\left\{\frac{n}{qd}\right\} \leqslant m_q + \varepsilon_q,$$

and the lemma follows directly from (2.46) and (2.47).

From Lemma 2 we see that each function G_q has a period τ_q such that $\tau_q \mid n_0$ and (1.17) shows further that we can determine m_q by evaluating $G_q(n)$ along a half-period. Hence, choosing $n_0 = 30030$, the calculations required by Lemma 4 can be efficiently performed by a computer and (2.45) yields $F(x) \leqslant 1182$.

On the other hand (1.19) gives $w = 1/2$ and we conclude

$$-1181 \leqslant F(x) \leqslant 1182. \tag{2.48}$$

From (2.2), (2.6) and (2.7) we obtain

$$\omega = 0.9999376831\ldots; \qquad h = 19495.43942\ldots; \qquad \gamma = 112106728.91\ldots$$

Now Lemma 3 shows that (2.29) holds for $x \geqslant 10^8$ with

$$\omega^+ = 1.0003503; \qquad \omega^- = 0.9995250.$$

With our choice of v all the r_k less than 10^8 are integers, except for the term corresponding to r, and this enables us to evaluate $F(r_n)$ for $1 \leqslant r_n < 10^8$ by a sieve method using a trivial modification of (2.40). In Table II we list the initial terms $p_0(s)$ and $q_0(s)$ for each value s of $F(r_n)$ in this region, together with the number of intervals $[b'_k(s),\ c'_k(s))$, $[b''_k(s),\ c''_k(s))$ selected for the second members of (2.11) and (2.12) respectively. These are maximal intervals of the type previously described, truncated at 10^7, such that the corresponding terms in (2.30) or (2.31) are positive.

Taking

$$a_s = p_0(s) \quad \text{if } -73 \leqslant s \leqslant 1; \qquad a_s = 10^8 \quad \text{if } -1180 \leqslant s \leqslant -74;$$

$$a_s = q_0(s) \quad \text{if } 2 \leqslant s \leqslant 78; \qquad a_s = 10^8 \quad \text{if } 79 \leqslant s \leqslant 1182,$$

we obtain from (2.30) and (2.31)

$$L^+ < 1.0018823 < 532/531; \quad L^- > 0.998118 > 530/531.$$

With these values of L^+ and L^-, conditions (2.32) and (2.33) are satisfied and Theorem 2 yields

$$\left|\frac{\psi(x)}{x} - 1\right| < \frac{1}{531} \quad \text{if } x \geqslant 10^8.$$

Hence, from Table I we conclude

$$\frac{\psi(x)}{x} < \frac{532}{531} \quad \text{if } x \geqslant 60299 \tag{2.49}$$

and

$$\frac{\psi(x)}{x} > \frac{530}{531} \quad \text{if } x \geqslant 70841. \tag{2.50}$$

We note that L. Schoenfeld [10] gives

$$\left|\frac{\psi(x)}{x} - 1\right| < 0.00119721 \quad \text{if } x \geqslant 10^8,$$

but this estimate requires much deeper analytical methods as well as the result of very extensive computations on the zeros of Riemann's zeta function.

As an application of (2.49) and (2.50) we prove

THEOREM 3. *We have*

$$\frac{\theta(x)}{x} < \frac{532}{531} \quad \textit{if } x > 0 \tag{2.51}$$

and

$$\frac{\theta(x)}{x} > \frac{499}{500} \quad \textit{if } x \geqslant 487381. \tag{2.52}$$

Proof. (2.51) follows directly from (2.49) together with the inequalities $\theta(x) \leqslant \psi(x)$ for $x > 0$ and $\theta(x) < x$ for $x < 60299$.

(2.52) can be verified directly for $487381 \leqslant x < 839973$. In the interval $839973 \leqslant x < 10^8$ it follows from the estimate

$$\frac{x - \theta(x)}{\sqrt{x}} < 1.833 \quad \text{if } 839973 \leqslant x < 10^8,$$

taken from Appel and Rosser [1]. To prove it for $x \geqslant 10^8$ we use the inequality

$$\theta(x) \geqslant \psi(x) - \psi(x^{1/2}) - \psi(x^{1/3}) - \psi(x^{1/5}) \quad \text{if } x > 0,$$

which is established in [5]. From (2.50) and Table I we obtain

$$\frac{\theta(x)}{x} > \frac{530}{531} - \frac{1.051616}{10^4} - \frac{1.021163}{10^{16/3}} - \frac{1.038821}{10^{32/5}} > \frac{499}{500} \quad \text{if } x \geqslant 10^8.$$

An immediate consequence of this theorem is that each interval $[x,\ 258x/257)$ contains a prime if $x \geqslant 485492$, a result that still holds when $x \geqslant 8469$.

Taking $N > 10^8$ in Theorem 2, we can obtain better results with the same function v, provided that we have sharp estimates for ψ in the range $10^8 \leqslant x < N$. Actually, with $N = 10^{11}$, Theorem 2 shows that

$$\left|\frac{\psi(x)}{x} - 1\right| < \frac{1}{2976}$$

holds for every $x \geqslant 10^{11}$ if it holds for $10^8 \leqslant x < 10^{11}$ and this last condition follows from R. Brent's estimates of $\pi(x)$ [2]. However, our choice of $m = 8021$ in Theorem 2 is not optimal for $N = 10^{11}$ and these results could still be improved.

3. Estimates for $M(x)$. From the definition of $M(x)$ we have

$$M(x) = \sum_{k \geqslant 1} \mu(k)\,\delta(x/k)$$

where δ is defined by $\delta(t) = 1$ if $t \geqslant 1$ and $\delta(t) = 0$ if $0 < t < 1$. In this case (1.6) and (1.19) give simply

$$\sigma(x) = -2w \quad \text{if } x \geqslant \max R,$$

and we obtain from (1.25)

$$(3.1)\quad |M(x)| \leqslant 2|w| + \sum_{s^- < s \leqslant 1} \sum_{k \geqslant 0} \left| M\left(\frac{x}{p_{2k}(s)}\right) - M\left(\frac{x}{p_{2k+1}(s)}\right) \right| + \sum_{1 < s \leqslant s^+} \sum_{k \geqslant 0} \left| M\left(\frac{x}{q_{2k}(s)}\right) - M\left(\frac{x}{q_{2k+1}(s)}\right) \right|$$

if $x \geqslant \max R$.

Let v^+ be the maximum of $|F(x)-1|$. For each integer v such that $1 \leqslant v \leqslant v^+$, the region where $|F(x)-1| \geqslant v$ is the union of all intervals

$$[q_{2k}(v+1),\ q_{2k+1}(v+1)) \quad \text{and} \quad [p_{2k}(2-v),\ p_{2k+1}(2-v)).$$

As these are non-overlapping intervals, then if $[a_k(s), b_k(s))$ denotes a maximum interval where $|F(x)-1| \geqslant v$ and $F(x)-1$ has constant sign, (3.1) gives

$$(3.2)\quad |M(x)| \leqslant 2|w| + \sum_{v=1}^{v^+} \sum_{k \geqslant 0} \left| M\left(\frac{x}{a_k(v)}\right) - M\left(\frac{x}{b_k(v)}\right) \right| \quad \text{if } x \geqslant \max R.$$

If $Q(t)$ denotes the number of square-free integers in the interval $[1, t]$ we clearly have

$$(3.3)\quad \left| M\left(\frac{x}{a_k(v)}\right) - M\left(\frac{x}{b_k(v)}\right) \right| \leqslant Q\left(\frac{x}{a_k(v)}\right) - Q\left(\frac{x}{b_k(v)}\right)$$

and also

$$(3.4)\quad \left| M\left(\frac{x}{a_k(v)}\right) - M\left(\frac{x}{b_k(v)}\right) \right| \leqslant \left| M\left(\frac{x}{a_k(v)}\right) \right| + \left| M\left(\frac{x}{b_k(v)}\right) \right|.$$

Hence, if I_v $(1 \leqslant v \leqslant v^+)$ are finite sets of indices, defining

$$(3.5)\quad H(x) = \sum_{v=1}^{v^+} \sum_{k \in I_v} \left(\left| M\left(\frac{x}{a_k(v)}\right) \right| + \left| M\left(\frac{x}{b_k(v)}\right) \right| \right),$$

$$(3.6)\quad S(x) = \sum_{v=1}^{v^+} \sum_{k \notin I_v} \left(Q\left(\frac{x}{a_k(v)}\right) - Q\left(\frac{x}{b_k(v)}\right) \right)$$

we have

$$(3.7)\quad |M(x)| \leqslant 2|w| + H(x) + S(x) \quad \text{if } x \geqslant \max R.$$

We refer to the intervals $[a_k(v), b_k(v))$ such that $k \in I_v$ as the "selected intervals" of the estimate (3.7).

THEOREM 4. *Assuming* (3.7) *we define* α *and* β *by*

$$(3.8)\quad \alpha = \sum_{v=1}^{v^+} \sum_{k \in I_v} \left(\frac{1}{a_k(v)} + \frac{1}{b_k(v)} \right),$$

$$(3.9)\quad \beta = \sum_{v=1}^{v^+} \sum_{k \notin I_v} \left(\frac{1}{a_k(v)} - \frac{1}{b_k(v)} \right).$$

Then, if $\alpha < 1$, *we have*

$$(3.10)\quad \limsup \left| \frac{M(x)}{x} \right| \leqslant \frac{6\beta}{\pi^2(1-\alpha)}.$$

Proof. Let

$$\lambda = \limsup |M(x)/x|.$$

Taking an integer n such that $n > \max_{1 \leqslant v \leqslant v^+} I_v$, we have from (3.7)

$$|M(x)| \leqslant 2|w| + H(x) + \sum_{v=1}^{v^+} \sum_{\substack{k \notin I_v \\ k < n}} \left(Q\left(\frac{x}{a_k(v)}\right) - Q\left(\frac{x}{b_k(v)}\right) \right) + \sum_{v=1}^{v^+} Q\left(\frac{x}{a_n(v)}\right).$$

Dividing both members by x and using the well-known relation

$$\lim_{t \to +\infty} \frac{Q(t)}{t} = \frac{6}{\pi^2},$$

we obtain

$$\lambda \leqslant \lambda\alpha + \frac{6}{\pi^2} \sum_{v=1}^{v^+} \sum_{\substack{k \notin I_v \\ k < n}} \left(\frac{1}{a_k(v)} - \frac{1}{b_k(v)} \right) + \frac{6}{\pi^2} \sum_{v=1}^{v^+} \frac{1}{a_n(v)}.$$

Taking the limit when n increases to infinity we obtain

$$\lambda \leqslant \lambda\alpha + 6\beta/\pi^2,$$

and (3.10) holds.

If a new interval $[a_k(v), b_k(v))$ is selected in (3.7), (3.10) gives a new estimate

$$\limsup \left| \frac{M(x)}{x} \right| \leqslant \frac{6(\beta - 1/a_k(v) + 1/b_k(v))}{\pi^2(1 - \alpha - 1/a_k(v) - 1/b_k(v))},$$

which is sharper if and only if

$$(3.11)\quad \frac{b_k(v)}{a_k(v)} > \frac{1-\alpha+\beta}{1-\alpha-\beta}.$$

As an application we define

$$(3.12)\quad \nu(t) = \mu(t) \quad \text{if } 0 < t < 7; \quad \nu(7.5) = -1; \quad \nu(t) = 0 \quad \text{otherwise},$$

following Diamond and McCurley [7]. In this case the function F has period $\tau = 30$. By a simple calculation we obtain $s^+ = 1$ and we see that the intervals $[a_k(s), b_k(s))$ are

$$[7, 7.5),\quad [10, 11),\quad [13, 15),\quad [15, 17),\quad [19, 20),\quad [22.5, 23),\quad [29, 30),\quad [30, 31),\quad [37, 37.5),\quad \ldots$$

Taking $I_1 = \emptyset$, we obtain from (3.9)

$$\beta = \sum_{k\geq 0}\left(\frac{1}{30k+7} - \frac{1}{30k+7.5} + \ldots + \frac{1}{30k+29} - \frac{1}{30k+31}\right).$$

Truncating this series at $k = 99$ and evaluating the tail by

$$\sum_{k>n}\left(\frac{1}{ak+b} - \frac{1}{ak+b+\varepsilon}\right) < \frac{\varepsilon}{na^2+ab+a\varepsilon}; \quad b>0,\ 0<\varepsilon<a,$$

we get

$$\beta < 0.0513365.$$

Then (3.10) yields

$$\limsup |M(x)/x| < 0.03121 < 1/32,$$

which is the result obtained in [7]. However, if we transfer from $S(x)$ to $H(x)$ the terms corresponding to the intervals

$$[10, 11),\quad [13, 15),\quad [15, 17),$$

(3.10) gives

$$\limsup |M(x)/x| < 0.0271829 < 1/36.$$

At this point (3.11) shows that no further improvement is possible with this function ν by selecting other intervals in (3.7).

Take now a fixed number $N \geq \max R$. To obtain a finite estimate of the type

$$|M(x)/x| < L \quad \text{if } x \geq N,$$

we need two lemmas.

LEMMA 5. *If $y \geq x > 0$ we have*

$$Q(x) \leq \tfrac{2}{3}x + \tfrac{4}{3}, \tag{3.13}$$

and

$$Q(y) - Q(x) < \tfrac{2}{3}(y-x) + \tfrac{8}{3}. \tag{3.14}$$

Proof. Denoting by $f(t)$ the number of integers in the interval $[1, t]$ which are not divisible by 4 or 9, we have

$$f(t) = [t] - \left[\frac{t}{4}\right] - \left[\frac{t}{9}\right] + \left[\frac{t}{36}\right],$$

and this gives

$$\tfrac{2}{3}t - \tfrac{4}{3} < f(t) \leq \tfrac{2}{3}t + \tfrac{4}{3}.$$

Now the lemma follows from the inequalities $Q(x) \leq f(x)$ and $Q(y) - Q(x) \leq f(y) - f(x)$.

LEMMA 6. *Let $(c_k)_{k\geq 0}$ be an increasing sequence of positive numbers. If $x \geq N$ we have*

$$\frac{1}{x}\sum_{k=0}^{n-1}\left(Q\left(\frac{x}{c_{2k}}\right) - Q\left(\frac{x}{c_{2k+1}}\right)\right) + \frac{1}{x}Q\left(\frac{x}{c_{2n}}\right) \leq \frac{2}{3c_0} + \frac{4}{3N} - \frac{2}{3}\sum_{k=1}^{n}\delta_k\left(\frac{1}{c_{2k-1}} - \frac{1}{c_{2k}} - \frac{4}{N}\right), \tag{3.15}$$

where

$$\delta_k = 1 \quad \textit{if } \frac{1}{c_{2k-1}} - \frac{1}{c_{2k}} > \frac{4}{N}; \qquad \delta_k = 0 \quad \textit{otherwise}.$$

Proof. If $(d_k)_{k\geq 0}$ is an increasing sequence of positive numbers, Lemma 5 gives

$$\frac{1}{x}\sum_{k=0}^{m-1}\left(Q\left(\frac{x}{d_{2k}}\right) - Q\left(\frac{x}{d_{2k+1}}\right)\right) + \frac{1}{x}Q\left(\frac{x}{d_{2m}}\right) \leq \frac{2}{3}\sum_{k=0}^{m-1}\left(\frac{1}{d_{2k}} - \frac{1}{d_{2k+1}} - \frac{4}{N}\right) + \frac{2}{3d_{2m}} + \frac{4}{3N},$$

and this can be rewritten in the form

$$\frac{1}{x}Q\left(\frac{x}{d_0}\right) - \frac{1}{x}\sum_{k=1}^{m}\left(Q\left(\frac{x}{d_{2k-1}}\right) - Q\left(\frac{x}{d_{2k}}\right)\right) \leq \frac{2}{3d_0} + \frac{4}{3N} - \frac{2}{3}\sum_{k=1}^{m}\left(\frac{1}{d_{2k-1}} - \frac{1}{d_{2k}} - \frac{4}{N}\right). \tag{3.16}$$

Returning now to the sequence $(c_k)_{k\geq 0}$ and denoting by I the subset of $\{1, \ldots, n\}$ such that $\delta_k = 1$ iff $k \in I$, we have

$$\frac{1}{x}\sum_{k=0}^{n-1}\left(Q\left(\frac{x}{c_{2k}}\right) - Q\left(\frac{x}{c_{2k+1}}\right)\right) + \frac{1}{x}Q\left(\frac{x}{c_{2n}}\right)$$
$$= \frac{1}{x}Q\left(\frac{x}{c_0}\right) - \frac{1}{x}\sum_{k=1}^{n}\left(Q\left(\frac{x}{c_{2k-1}}\right) - Q\left(\frac{x}{c_{2k}}\right)\right)$$
$$\leq \frac{1}{x}Q\left(\frac{x}{c_0}\right) - \frac{1}{x}\sum_{k\in I}\left(Q\left(\frac{x}{c_{2k-1}}\right) - Q\left(\frac{x}{c_{2k}}\right)\right).$$

Using (3.16) we see that this last expression does not exceed

$$\frac{2}{3c_0}+\frac{4}{3N}-\frac{2}{3}\sum_{k\in I}\left(\frac{1}{c_{2k-1}}-\frac{1}{c_{2k}}-\frac{4}{N}\right),$$

which proves (3.15).

If $(c_k(v))_{k\geq 0}$ denotes the increasing sequence formed by the end points of the intervals $[a_k(v), b_k(v))$ such that $k\notin I_v$, then (3.6) is transformed into

$$S(x)=\sum_{v=1}^{v^+}\sum_{k\geq 0}\left(Q\left(\frac{x}{c_{2k}(v)}\right)-Q\left(\frac{x}{c_{2k+1}(v)}\right)\right).$$

Taking the limit in Lemma 6 and changing the notation we obtain

$$\frac{S(x)}{x}\leqslant\sum_{v=1}^{v^+}\left(\frac{2}{3c_0(v)}+\frac{4}{3N}-\frac{2}{3}\sum_{k\geq 1}\delta_k(v)\left(\frac{1}{c_{2k-1}(v)}-\frac{1}{c_{2k}(v)}-\frac{4}{N}\right)\right)$$

if $x\geqslant N$. Hence, (3.7) gives

$$\left|\frac{M(x)}{x}\right|\leqslant\frac{2|w|}{N}+\frac{4v^+}{3N}+\frac{H(x)}{x}+s\quad\text{if } x\geqslant N \tag{3.17}$$

with

$$s=\frac{2}{3}\sum_{v=1}^{v^+}\left(\frac{1}{c_0(v)}-\sum_{k\geq 1}\delta_k(v)\left(\frac{1}{c_{2k-1}(v)}-\frac{1}{c_{2k}(v)}-\frac{4}{N}\right)\right). \tag{3.18}$$

Now, from (3.17), we obtain the following analogues to Theorem 2 and Corollary 1; the proofs are similar and we omit them.

THEOREM 5. *Choose $\lambda(t)$ such that*

$$|M(x)/x|\leqslant\lambda(t)\quad \textit{if } t\leqslant x<N \tag{3.19}$$

and let

$$L=\frac{2|w|}{N}+\frac{4v^+}{3N}+\sum_{v=1}^{v^+}\sum_{k\in I_v}\left(\frac{1}{a_k(v)}\lambda\left(\frac{N}{a_k(v)}\right)+\frac{1}{b_k(v)}\lambda\left(\frac{N}{b_k(v)}\right)\right)+s. \tag{3.20}$$

Then, if

$$L\leqslant\lambda\left(\frac{N}{a_k(v)}\right);\quad L\leqslant\lambda\left(\frac{N}{b_k(v)}\right)\quad \textit{for } k\in I_v,\ 1\leqslant v\leqslant v^+, \tag{3.21}$$

we have

$$|M(x)/x|\leqslant L\quad \textit{for } x\geqslant N.$$

COROLLARY 2. *Take N_0 such that $b_k(v)\leqslant N_0$ for each $k\in I_v$, $1\leqslant v\leqslant v^+$ and let α be given by* (3.8). *Assuming $\alpha<1$, define*

$$L=\frac{2|w|/N+4v^+/3N+s}{1-\alpha}. \tag{3.22}$$

Then, if

$$|M(x)/x|\leqslant L\quad \textit{for } N/N_0\leqslant x<N, \tag{3.23}$$

we have

$$|M(x)/x|\leqslant L\quad \textit{for } x\geqslant N.$$

Consider again the function v defined by (3.12) and take $N=10^8$. Selecting the intervals [10, 11), [13, 15), [15, 17) we may apply Corollary 2 with $N_0=17$ to obtain

$$\alpha<0.45998904;\quad s<0.0161882.$$

Since $w=1$, (3.22) yields

$$L<0.0299777<3/100.$$

On the other hand Neubauer [8] has shown that

$$|M(x)|<\sqrt{x}/2\quad\text{for } 201\leqslant x<10^8,$$

which implies

$$|M(x)/x|<3/100\quad\text{if } 278\leqslant x<10^8.$$

Thus condition (3.23) is satisfied and Corollary 2 gives

$$|M(x)/x|<3/100\quad\text{if } x\geqslant 10^8.$$

As this inequality holds also for $202\leqslant x<278$, we conclude

$$|M(x)/x|<3/100\quad\text{if } x\geqslant 202.$$

To improve this result we have applied Theorem 5 with $N=10^8$ and the bounds $\lambda(t)$ given in Table III. For the function v defined by (2.43), the intervals $[a_k(v), b_k(v))$ appearing in (3.2) were determined in the zone $x\leqslant 10^7$ and truncated at 10^7. The initial terms $a_0(v)$ can be obtained directly from Table II up to 10^8 and this gives $a_0(v)$ for $1\leqslant v\leqslant 77$. As (2.48) shows that $s^+\leqslant 1182$, we have replaced $a_0(v)$ by 10^8 for $78\leqslant v\leqslant 1182$. To evaluate L we have taken the intervals $[a_k(v), b_k(v))$ sequentially for each v, comparing the effect on L of selecting or not a new interval in (3.7). With this method a total of 418 intervals were selected and the highest of $b_k(v)$ with $k\in I_v$ is 85845. Now (3.20) yields

$$L<0.0009647<1/1036,$$

and the conditions (3.21) are satisfied. Hence, Theorem 5 gives

$$|M(x)/x|<1/1036\quad\text{if } x\geqslant 10^8.$$

From Table III we see that this inequality still holds for $120865\leqslant x<10^8$ and it can be also checked in the range $120727\leqslant x<120865$. We conclude

$$|M(x)/x|<1/1036\quad\text{if } x\geqslant 120727.$$

Table I

$\lambda^-(t)\,x < \psi(x) < \lambda^+(t)\,x \quad \text{if } t \leqslant x \leqslant 10^8$

t	$\lambda^+(t)$	t	$\lambda^-(t)$
1	1.0388206	23	0.8658345
114	1.0359089	41	0.9060283
201	1.0272755	59	0.9223773
294	1.0211630	101	0.9484202
469	1.0180185	227	0.9676432
664	1.0138597	347	0.9749497
684	1.0136306	569	0.9787068
1630	1.0119595	1429	0.9870819
1670	1.0098987	1447	0.9875864
2868	1.0074348	2657	0.9882855
2974	1.0066131	3299	0.9900226
3948	1.0064876	3461	0.9922755
6380	1.0063952	3511	0.9923791
6404	1.0054256	5387	0.9933098
7045	1.0051616	7451	0.9934334
10359	1.0045711	7477	0.9934380
24271	1.0036023	11801	0.9948622
24297	1.0029673	19379	0.9954703
43068	1.0029036	19423	0.9964387
59851	1.0023695	32059	0.9970335
60299	1.0018667	32321	0.9977007
60977	1.0015609	69997	0.9978769
96021	1.0015220	70843	0.9981698
102688	1.0014363	88807	0.9985121
155941	1.0012069	175939	0.9987077
230569	1.0011420	303287	0.9989190
356185	1.0008886	312229	0.9990620
359810	1.0007863	463447	0.9991479
445208	1.0007157	467867	0.9993803
618740	1.0005627	643847	0.9993884
1198547	1.0004825	1092893	0.9996014
1520824	1.0003639	1790479	0.9996740
3459604	1.0002563	3597037	0.9997171
4996151	1.0002148	4420517	0.9997916
7118551	1.0001838	5880041	0.9998085
12898940	1.0001288	10393637	0.9998633
30980127	1.0000960	16577753	0.9998764
33896936	1.0000905	36999173	0.9999030
40886484	1.0000754	38113423	0.9999254

Table II

(see text)

s	$p_0(s)$	α_s	β_s	s	$q_0(s)$	α_s	β_s
1	8021	15	19	2	9161	15	17
0	8022	17	18	3	9219	14	17
−1	8023	17	18	4	9221	14	18
−2	8026	18	18	5	9283	13	18
−3	8027	19	17	6	9285	15	20
−4	8031	21	17	7	9294	12	21
−5	8033	22	13	8	9403	12	21
−6	8034	24	12	9	9417	11	19
−7	8035	24	11	10	9418	9	18
−8	8038	24	11	11	9419	8	19
−9	8049	24	7	12	9421	8	20
−10	8051	26	7	13	9422	8	21
−11	8057	24	7	14	9426	8	21
−12	8058	22	8	15	9433	8	21
−13	8065	22	7	16	9434	8	20
−14	8071	25	7	17	9439	8	21
−15	8141	23	6	18	9465	6	22
−16	8142	23	5	19	9474	5	23
−17	8143	22	5	20	9478	4	24
−18	8146	25	4	21	9479	4	25
−19	8151	27	4	22	9483	4	28
−20	8153	28	4	23	9485	4	28
−21	8158	24	4	24	9499	3	27
−22	8159	25	3	25	9551	3	28
−23	8201	25	2	26	9698	3	26
−24	8413	23	2	27	9699	3	26
−25	8418	20	2	28	9710	3	24
−26	8489	17	2	29	9715	1	24
−27	8490	16	1	30	9717	1	23
−28	8491	17	0	31	9718	1	24
−29	8503	18	0	32	9719	1	23
−30	8506	17	0	33	9721	1	21
−31	8507	16	0	34	9726	1	19
−32	8509	15	0	35	9741	1	18
−33	8510	14	0	36	9749	1	16
−34	8511	15	0	37	9822	0	17
−35	11759	15	0	38	9823	0	15
−36	11761	16	0	39	9830	0	15
−37	11762	16	0	40	9831	0	13
−38	11769	16	0	41	9833	0	13
−39	11770	12	0	42	9857	0	14
−40	11771	14	0	43	9861	0	10
−41	11773	13	0	44	92967	0	8
−42	69971	11	0	45	157579	0	7
−43	69973	10	0	46	180064	0	4
−44	69986	10	0	47	180071	0	6
−45	70561	10	0	48	180077	0	5
−46	70569	9	0	49	180079	0	5

Table II (continued)

s	$p_0(s)$	α_s	β_s	s	$q_0(s)$	α_s	β_s
−47	70570	10	0	50	180082	0	4
−48	70596	8	0	51	180259	0	4
−49	70597	6	0	52	180263	0	5
−50	70601	7	0	53	180271	0	5
−51	70603	6	0	54	180386	0	3
−52	302741	5	0	55	180390	0	3
−53	302742	3	0	56	180390	0	4
−54	1523273	2	0	57	180391	0	3
−55	2942825	1	0	58	180444	0	3
−56	2942829	1	0	59	180463	0	3
−57	2942875	1	0	60	180466	0	3
−58	2942877	1	0	61	180467	0	3
−59	2942879	1	0	62	1675007	0	2
−60	2942879	1	0	63	1675070	0	2
−61	2942880	1	0	64	1675073	0	1
−62	2942887	1	0	65	1675080	0	1
−63	2942957	1	0	66	1675087	0	1
−64	2943035	1	0	67	27413461	0	0
−65	2943042	1	0	68	27423461	0	0
−66	2943050	1	0	69	34043952	0	0
−67	2943053	1	0	70	34043957	0	0
−68	2943055	1	0	71	34043967	0	0
−69	2943076	1	0	72	36690681	0	0
−70	2943082	1	0	73	36690681	0	0
−71	2943083	1	0	74	36690681	0	0
−72	2943085	1	0	75	36690682	0	0
−73	2943108	1	0	76	36690683	0	0
				77	36690704	0	0
				78	36690705	0	0

Table III

$|M(x)| < \lambda(t)\, x \quad \text{if } t \leqslant x \leqslant 10^8$

t	$\lambda(t)$	t	$\lambda(t)$
10	0.2307693	11821	0.0029771
14	0.1578948	24522	0.0022820
21	0.1290323	32018	0.0020484
34	0.0697675	42982	0.0019822
46	0.0638298	48517	0.0018968
74	0.0526316	60982	0.0013748
118	0.0402011	97077	0.0011042
203	0.0279721	120865	0.0009446
298	0.0203161	142278	0.0007986
445	0.0180452	300914	0.0007253
689	0.0135257	359891	0.0005269
1137	0.0097740	464551	0.0004610
1641	0.0089191	604362	0.0004324
2867	0.0069803	618062	0.0004050
3422	0.0048793	1079317	0.0003066
4262	0.0043607	1802578	0.0002459
9959	0.0035675	2159549	0.0002367

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