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An effective estimate for the density of zeros of Hecke–Landau zeta-functions

by

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1. Let K be an algebraic number field on finite degree $n \geqslant 2$ and absolute value of discriminant d. Denote by $\mathfrak{f}$ a given nonzero integral ideal of the ring of algebraic integers R_K of K. Let $\chi(C)$ be a Dirichlet character of the abelian group $H^*(\mathfrak{f})$ of ideal classes $C(\bmod\, \mathfrak{f})$ in the "narrow" sense. For an integral ideal $\mathfrak{a}$ of R_K let $\chi(\mathfrak{a})$ be the usual extension of $\chi(C)$ (see [8], Def. LVI) and χ^* the primitive character mod $\mathfrak{f}_{\chi^*}$ induced by $\chi(\bmod\, \mathfrak{f})$, $\mathfrak{f}_{\chi^*}|\mathfrak{f}$. Let $A = \sqrt{d}\left(\frac{5\log d}{2(n-1)}\right)^{n-1} n^5 < d$ denote the constant appearing in Siegel's theorem on the fundamental system of units (see [10]).

Denote by $\zeta_K(s, \chi, \mathfrak{f})$, $s = \sigma+it$, the Hecke–Landau zeta-function defined for $\sigma > 1$ by the series

$$\zeta_K(s, \chi, \mathfrak{f}) = \sum_{\mathfrak{a}} \chi(\mathfrak{a}) N\mathfrak{a}^{-s},$$

where $\mathfrak{a}$ runs through integral ideals of K. Let $N(\alpha, T, \chi)$ denote the number of zeros of $\zeta_K(s, \chi, \mathfrak{f})$, $s = \sigma+it$, in the rectangle $\alpha \leqslant \sigma \leqslant 1$, $0 < t \leqslant T$. Basing on some effective estimates of the order of Hecke–Landau zeta-functions near the line $\sigma = 1$ (see [1], Th. 1) and using Halász–Turán ideas (see [4] and [5]) we shall prove the following theorem:

THEOREM. *There exist absolute positive constants $c_0 > 1$ and $c_1 < 1$ such that for all α and T with*

$$1-\min(c_1, e^{-20(A+1)}) \leqslant \alpha \leqslant 1, \qquad T > c_0, \tag{1.1}$$

the following inequality holds:

$$\sum_{\substack{\mathfrak{a} \\ N\mathfrak{a} < N\mathfrak{f}}} \sideset{}{^*}\sum_{\chi(\bmod\, \mathfrak{a})} N(\alpha, T, \chi) < \exp\left[250 M_1(\mathfrak{f}, T)(1-\alpha)^{3/2}\log^3\frac{1}{1-\alpha}\right], \tag{1.2}$$

where

$$M_1(\mathfrak{f}, T) = \max(\log^{3/2}(N\mathfrak{f}), A^{3/2}\log T). \tag{1.3}$$

The star in the inner sum indicates that χ runs through primitive characters mod $\mathfrak{a}$ *only.*

For the Riemann zeta-function, the estimate of the form (1.1)–(1.3) is due to G. Halász and P. Turán (see [4]) and for the Dirichlet L-functions to the second of the present authors (see [2]). Our estimate can be compared with the following result of W. Staś [11] for the Dedekind zeta-functions $\zeta_K(s)$:

For all α with

$$1-(3n\exp(-10^9))^{-1} \leqslant \alpha \leqslant 1 \tag{1.4}$$

and $T \geqslant e$ the following inequality holds:

$$N(\alpha,\, T,\, K) < \exp\exp(cn^{5600}d^{160})\, T^{0.1n^2(n(1-\alpha))^{3/2}} \log^2 \frac{1}{n(1-\alpha)}, \tag{1.5}$$

where c is a positive absolute constant.

Putting $\mathfrak{f} = R_K$ in (1.2) we obtain an estimate for $N(\alpha, T, K)$ which is better than (1.5) in respect of the dependence on the parameters of the field K, but our rectangle (1.1) is narrower than that in Staś's theorem. And *vice versa*, since $\prod_{\chi (\mathrm{mod}\, \mathfrak{f})} \zeta_K(s, \chi^*) = \zeta_L(s)$, $n_L = nh^*(\mathfrak{f})$, $d_L = d^{h^*(\mathfrak{f})} \prod_{\chi(\mathrm{mod}\,\mathfrak{f})} N\mathfrak{f}_{\chi^*}$, where L is the class-field of the group $H^*(\mathfrak{f})$, we can obtain an estimate similar to (1.2) from (1.5), but the dependence on the parameters will be much worse.

2. The proof of (1.1)–(1.3) will rest on the following lemmas:

Lemma 1 (see [7] and [4], p. 130). *Let $G(z)$ be regular for $|z| \leqslant R$, $G(0) \neq 0$ and $\left|\frac{G(z)}{G(0)}\right| \leqslant U$. Then if $0 < r < R$ and the zeros of $G(z)$ in the disc $|z| \leqslant r$ are $z_1, z_2, \ldots$ then for all non-negative integers μ we have*

$$\left|\sum_{|z_j| \leqslant r} \frac{1}{z^{\mu+1}}\right| < \frac{2(\mu+1)\log U}{r^{\mu+1}}\left(1+\frac{1}{\log(R/r)}\right)+\frac{1}{\mu!}\left|\left[\frac{G'}{G}(z)\right]_{z=0}^{(\mu)}\right|. \tag{2.1}$$

To obtain a lower bound we will use Turán's theorem.

Lemma 2 (Turán's second main theorem, see [13], p. 52). *For any $m > 0$, positive integer $n \leqslant N^*$ and complex numbers $w_1, w_2, \ldots, w_n$, there is an integer v_0 with $m \leqslant v_0 \leqslant m+N^*$ such that*

$$\Big|\sum_{j=1}^{n} w_j^{v_0}\Big| \geqslant \left(\frac{N^*}{8e(m+N^*)}\right)^{N^*} |w_k|^{v_0} \tag{2.2}$$

where w_k stands for any of the w_j's.

Lemma 3 (see [12], Lemma 6). *Denoting by $N(T, \chi)$ the number of roots of the Hecke–Landau zeta-function $\zeta_K(s, \chi, \mathfrak{f})$ in the region $|t| \leqslant T$, $-1 \leqslant \sigma \leqslant 1$, we have the estimate*

$$N(T+1,\, \chi)-N(T,\, \chi) < c_2 \log\big(dN\mathfrak{f}(|T|+3)^n\big), \tag{2.3}$$

where c_2 denotes an absolute constant.

We will also use the following estimate due to Landau (see [9]):

$$|H(x)| = \Big|\sum_{N\mathfrak{a} \leqslant x} 1\Big| \leqslant n^{c_3 n} d^{2/(n+1)} (\log d)^{2n} x^{1-2/(n+1)} + c_4^n\, x (\log d)^{n-1}, \tag{2.4}$$

and for $\chi \neq \chi_0$,

$$H(x,\, \chi) = \Big|\sum_{N\mathfrak{a} \leqslant x} \chi(\mathfrak{a})\Big| < n^{c_5 n} (dN\mathfrak{f})^{1/(n+1)} \big(\log(dN\mathfrak{f})\big)^n x^{1-2/(n+1)}, \tag{2.5}$$

where c_3, c_4 and c_5 are absolute constants. An effective version of (2.4) and (2.5) can be found in [1] (Lemma 8).

Lemma 4. *For $1-1/(n+1) \leqslant \sigma \leqslant 1$, $t \geqslant 1.1$,*

$$\begin{aligned} |\zeta_K(\sigma+it,\, \chi,\, \mathfrak{f})| \leqslant {} & e^{c_6 A} (N\mathfrak{f})^{1-\sigma}\, t^{14000n^{2.5}(n+2)(1-\sigma)^{3/2}} (\log t)^{2/3} \\ & + n^{c_7 n} \sqrt{d}\, (\log d)^{2n} (N\mathfrak{f})^{1-\sigma} \log(N\mathfrak{f}), \end{aligned} \tag{2.6}$$

and for $\sigma \geqslant 1$, $t \geqslant 1.1$ we have

$$|\zeta_K(s,\, \chi,\, \mathfrak{f})| \leqslant e^{c_6 A} (\log t)^{2/3} + n^{c_7 n} (\log d)^{2n} \log(N\mathfrak{f}), \tag{2.7}$$

where c_6 and c_7 are absolute constants.

Proof. Theorem 1 in [1] yields (2.6), (2.7) can be proved similarly: For $\sigma \geqslant 1-1/(n+1)$, $t > 1$ we have (see (4.2) in [1])

$$\begin{aligned} |\zeta_K(s,\, \chi,\, \mathfrak{f})| \leqslant {} & \Big|\sum_{1 \leqslant m \leqslant Y_1 \exp(n \log^{2/3} t)} F(m,\, \chi)\, m^{-s}\Big| \\ & + \Big|\sum_{Y_1 \exp(n\log^{2/3} t) < m < Y_2 t^{n+1}} F(m,\, \chi)\, m^{-s}\Big| + b_1^n (\log d)^{n-1} \\ = {} & |S_1| + |S_2| + b_1^n (\log d)^{n-1}, \end{aligned}$$

where

$$Y_1 = 2^{-n} n^{n(n+1)} d^{n/2} (N\mathfrak{f}), \quad Y_2 = n^{b_2 n^2} d^2 (\log d)^{2n(n+1)} \big(\log(1+N\mathfrak{f})\big)^{n(n+1)} (N\mathfrak{f})$$

and b_1 and b_2 are absolute constants.

For $\sigma > 1$, $t \geqslant 1.1$, we estimate $|S_1|$ trivially by partial summation using (2.5):

$$|S_1| \leqslant \sum_{m \leqslant Y_1 \exp(n \log^{2/3} t)} F(m)\, m^{-1} \leqslant n^{b_3 n} (\log d)^{2n} \big((\log t)^{2/3} + \log(N\mathfrak{f})\big)$$

since $Y_1 > d$. The second sum $|S_2|$ is estimated in the same way as $|S_2|$ in [1] (see (4.6)) and we obtain for $\sigma > 1$, $t \geqslant 1.1$

$$|S_2| \leqslant \exp(b_4 A)(N\mathfrak{f})^{1-\sigma}(\log t)^{2/3}.$$

Finally, we get (2.7). The constants b_3 and b_4 are absolute.

LEMMA 5 (see [1], Th. 2). *There exists a positive constant $c_8 > 1$, independent of K and χ, such that in the region*

$$\sigma \geqslant 1 - \left(c_8 \max\left(\log(N\mathfrak{f}),\ A(\log(|t|+3))^{2/3}(\log\log(|t|+3))^{1/3}\right)\right)^{-1}, \quad -\infty < t < \infty \tag{2.8}$$

the function $\zeta_K(\sigma+it, \chi)$ has no zeros except for the hypothetical real simple zero of $\zeta_K(s, \chi_1)$, χ_1 real.

3. Proof of the theorem. Let θ be such that

$$(0.07 c_8 M(\mathfrak{f}, T))^{-1} \leqslant 1-\theta \leqslant \min\left(c_1, \exp(-20(A+1))\right), \tag{3.1}$$

where c_1 is a sufficiently small absolute constant, c_8 is taken from Lemma 5 and

$$M(\mathfrak{f}, T) = \max\left(\log(N\mathfrak{f}),\ A(\log(|T|+3))^{2/3}(\log\log(|T|+3))^{1/3}\right).$$

Further, set

$$\lambda = (1-\theta)^{3/2}\left(\log\frac{1}{1-\theta}\right)^3. \tag{3.2}$$

It is easy to notice that

$$\frac{\lambda}{\log\frac{1}{1-\theta}} = (1-\theta)^{3/2}\log^2\frac{1}{1-\theta} \geqslant \frac{\gamma \log M_1(\mathfrak{f}, T)}{M_1(\mathfrak{f}, T)}, \tag{3.3}$$

where $M_1(\mathfrak{f}, T)$ is given by (1.3) and γ is an arbitrarily large absolute constant, provided $T > c_0$. Let I denote the segment

$$I\colon\ \sigma = \sigma_0 = 2-\theta, \quad T/2 \leqslant t \leqslant T.$$

LEMMA 6. *For a suitable set $\bar{H}^* < I$ of measure*

$$|\bar{H}^*| \ll n^{c_9 n} d^{4/(n+1)}(\log d)^{4n}(M_1(\mathfrak{f}, T))^{10.5}\exp(2\lambda M_1(\mathfrak{f}, T)) \tag{3.4}$$

the inequality

$$\left|\frac{\zeta_K'}{\zeta_K}(s, \chi^*, \mathfrak{a})^{(\nu)}\right| < \frac{\nu!\exp(-\lambda M_1(\mathfrak{f}, T))}{(1-\theta)^\nu} \tag{3.5}$$

holds for all $s \in I \setminus \bar{H}^*$ whenever $T > c_{10}$ (with a sufficiently large absolute constant c_{10}) for all $\zeta_K(s, \chi^*, \mathfrak{a})$ with characters $\chi^* (\mathrm{mod}\ \mathfrak{a})$, where $N\mathfrak{a} \leqslant N\mathfrak{f}$, and for all ν with

$$\frac{\lambda M_1(\mathfrak{f}, T)}{\log\frac{3}{2}}\left(1 + \frac{A}{\log(1/(1-\theta))}\right) \leqslant \nu \leqslant \frac{\lambda M_1(\mathfrak{f}, T)}{\log\frac{3}{2}}\left(1 + \frac{2A}{\log(1/(1-\theta))}\right). \tag{3.6}$$

Proof. For a fixed natural ν, consider the set $H = H(\nu, \mathfrak{f})$ of those $s \in I$ for which

$$\left|\frac{\zeta_K'}{\zeta_K}(s, \chi^*, \mathfrak{a})^{(\nu)}\right| \geqslant \frac{\nu!}{(\sigma_0-1)^\nu}\exp(-\lambda M_1(\mathfrak{f}, T))$$

for $\zeta_K(s, \chi^*, \mathfrak{a})$ attached to some primitive character $\chi^* (\mathrm{mod}\ \mathfrak{a})$, $N\mathfrak{a} \leqslant N\mathfrak{f}$ (not necessarily the same for all s in H).

Let τ_1 be the smallest t-value in H and $\tau_1, \ldots, \tau_l$ being defined, let τ_{l+1} be the smallest t-value in H satisfying $\tau_{l+1} \geqslant \tau_l + 6$ (if there is any). If $\tau_1, \ldots, \tau_P$ are all these points then $H \subset \bigcup_{l=1}^{P} [\tau_l, \tau_l+6]$ and hence $|H| \leqslant 6P$. Analogously to [4], pp. 347–348, we get

$$\frac{P^2 \nu!^2}{(\sigma_0-1)^{2\nu}} e^{-2\lambda M_1(\mathfrak{f}, T)} \leqslant \left(\sum_{\mathfrak{a} \neq R_k} \frac{1}{N\mathfrak{a}\log^2 N\mathfrak{a}}\right)\left(\sum_{\mathfrak{a}} \frac{\log^{2\nu+4} N\mathfrak{a}}{(N\mathfrak{a})^{2\sigma_0-1}}\left|\sum_{j=1}^{P} \eta_j \chi_j^*(\mathfrak{a}) N\mathfrak{a}^{-i\tau_j}\right|^2\right), \tag{3.7}$$

where $|\eta_j| = 1$ and $T/2 \geqslant |\tau_i - \tau_j| \geqslant 6$. The first factor on the right-hand side is estimated using Abel's formula and inequality (2.4):

$$\sum_{\mathfrak{a} \neq R_K} \frac{1}{N\mathfrak{a}(\log N\mathfrak{a})^2} = \int_2^\infty H(x)\left(1 + \frac{2}{\log x}\right)\frac{dx}{x^2(\log x)^2} \ll n^{c_3 n} d^{2/(n+1)}(\log d)^{2n}.$$

Hence we get

$$\begin{aligned}\frac{\nu!^2 P^2}{(1-\theta)^{2\nu}} e^{-2\lambda M_1(\mathfrak{f}, T)} &\ll n^{c_3 n} d^{2/(n+1)}\log^{2n} d \sum_{\mathfrak{a}} \frac{\log^{2\nu+4} N\mathfrak{a}}{(N\mathfrak{a})^{2\sigma_0-1}} \sum_{j_1, j_2=1}^{P} \frac{\eta_{j_1}\bar{\eta}_{j_2}\chi_{j_1}^*(\mathfrak{a})\overline{\chi_{j_2}^*(\mathfrak{a})}}{(N\mathfrak{a})^{i(\tau_{j_1}-\tau_{j_2})}} \\ &= n^{c_3 n} d^{2/(n+1)}\log^{2n} d \sum_{j_1, j_2=1}^{P} \eta_{j_1}\bar{\eta}_{j_2} \sum_{\mathfrak{a}} \frac{(\log N\mathfrak{a})^{2\nu+4}\chi_{j_1 j_2}(\mathfrak{a})}{(N\mathfrak{a})^{2\sigma_0-1+i(\tau_{j_1}-\tau_{j_2})}},\end{aligned} \tag{3.8}$$

where if $\mathfrak{a}_1$ is the modulus of $\chi_{j_1}^*$, $\mathfrak{a}_2$ is the modulus of $\chi_{j_2}^*$, then $\chi_{j_1 j_2}$ is a character modulo $\mathfrak{a}_1\mathfrak{a}_2$. Next, separating the terms on the right of (3.8) with

$j_1 = j_2$ and those with $j_1 \neq j_2$, we obtain

$$(3.9)\qquad \frac{Pv!^2}{(1-\theta)^{2v}} e^{-2\lambda M_1(\mathfrak{f},T)} \ll n^{c_3^{\cdot} n} d^{2/(n+1)} \log^{2n} d \Big(|\zeta_K(2\sigma_0-1)^{(2v+4)}| + P \max_{\substack{N\mathfrak{a} \leqslant N\mathfrak{f}^2 \\ 6 \leqslant t \leqslant T/2 \\ \chi \neq \chi_0}} |\zeta_K(2\sigma_0 - 1 + it, \chi, \mathfrak{a})^{(2v+4)}| \Big).$$

To estimate the derivatives of Dedekind zeta functions we apply Cauchy's coefficient estimation, first to the disc $|s-(2\sigma_0-1)| \leqslant \left(1-\frac{1}{v}\right)(2\sigma_0-2)$. This circle is situated on the right of the line $\sigma = 1$, thus Abel's formula and (2.4) give for $\sigma > 1$

$$\zeta_K(s) = s \int_1^{\infty} H(x) x^{-s} dx \ll n^{c_3 n} d^{2/(n+1)} (\log d)^{2n} \frac{|s|}{\sigma - 1},$$

and Cauchy's estimation yields

$$(3.10)\qquad |\zeta_K(2\sigma_0-1)^{(2v+4)}| \ll n^{c_3 n} d^{2/(n+1)} (\log d)^{2n} \frac{(2v+5)!}{(2(1-\theta))^{2v+5}}.$$

Applying the same reasoning to the disc $|s-(2\sigma_0-1+it)| < 2\sigma_0 - 1 - \theta = 3(1-\theta)$, but using (2.6) and (2.7) we get

$$(3.11)\qquad \max_{\substack{N\mathfrak{a} \leqslant N\mathfrak{f}^2 \\ 6 \leqslant t \leqslant T/2 \\ \chi \neq \chi_0}} |\zeta_K(2\sigma_0-1+it, \chi, \mathfrak{a})^{(2v+4)}| \ll \frac{(2v+4)!}{(3(1-\theta))^{2v+4}} \exp(c_6 A) (N\mathfrak{f})^{2(1-\theta)} M_1(\mathfrak{f}, T)\, T^{14000 n^{2.5}(n+2)(1-\theta)^{3/2}}.$$

Applying (3.10) and (3.11) in (3.9) we get

$$(3.12)\qquad Pe^{-2\lambda M_1(\mathfrak{f},T)} \ll n^{2c_3 n} d^{4/(n+1)} (\log d)^{4n} \frac{(2v+5)!}{2^{2v+5} v!^2 (1-\theta)^5} + P \frac{(2v+4)!}{3^{2v+4} v!^2 (1-\theta)^4} e^{c_6 A} (N\mathfrak{f})^{2(1-\theta)} M_1(\mathfrak{f}, T)\, T^{14000 n^{2.5}(n+2)(1-\theta)^{3/2}}.$$

Since $\frac{1}{1-\theta} \ll M_1(\mathfrak{f}, T)$ and, by the Stirling formula,

$$\frac{(2v+5)!}{2^{2v+5} v!^2} \ll v^{4.5} \quad \text{and} \quad \frac{(2v+4)!}{2^{2v+4} v!^2} \ll v^{3.5}$$

and $v \ll M_1(\mathfrak{f}, T)$, we obtain

$$(3.13)\qquad Pe^{-2\lambda M_1(\mathfrak{f},T)} \ll n^{2c_3 n} d^{4/(n+1)} (\log d)^{4n} M_1(\mathfrak{f}, T)^{9.5} + Pe^{c_6 A} (N\mathfrak{f})^{2(1-\theta)} (M_1(\mathfrak{f}, T))^{8.5}\, T^{14000 n^{2.5}(n+2)(1-\theta)^{3/2}} (2/3)^{2v}.$$

Now, we have

$$v \geqslant \frac{\lambda M_1(\mathfrak{f}, T)}{\log(3/2)} \left(1 + \frac{A}{\log(1/(1-\theta))}\right),$$

and since $1-\theta$ is bounded by an absolute constant, for $T > c_{10}$ the estimate (3.3) allows us to reduce any numerical factor of the second expression on the right of (3.13). Hence the measure of the set H is estimated as follows:

$$(3.14)\qquad |H| \ll n^{2c_3 n} d^{4/(n+1)} (\log d)^{4n} M_1(\mathfrak{f}, T)^{9.5} e^{2\lambda M_1(\mathfrak{f},T)}.$$

Let us denote by H^* the set of $s \in I$ for which the assumptions of the lemma hold. Its complement $\bar{H}^*$ in I is certainly covered by the union of the above $H = H(v, \mathfrak{f})$ sets. Hence owing to (3.6) and (3.14) the lemma is proved.

4. Let us consider the horizontal strips l_j defined by

$$(4.1)\qquad \frac{T}{2} + \frac{j}{M_1^3(\mathfrak{f}, T)} \leqslant t < \frac{T}{2} + \frac{j+1}{M_1^3(\mathfrak{f}, T)}; \quad j = 0, 1, \ldots, \left[\frac{T}{2} M_1^3(\mathfrak{f}, T)\right].$$

We call a strip l_j "good" if its intersection with I contains at least one point of the set H, otherwise we call it "bad". By (3.14) the number of "bad" strips is

$$(4.2)\qquad \ll n^{c_9 n} d^{4/(n+1)} (\log d)^{4n} M_1(\mathfrak{f}, T)^{13.5} e^{2\lambda M_1(\mathfrak{f},T)}.$$

In every "bad" strip l_j let us fix a point $z_j'' = \sigma_0 + it''$.

Lemma 7. *For all $\zeta_K(s, \chi, \mathfrak{a})$ functions, $N\mathfrak{a} \leqslant N\mathfrak{f}$, except at most*

$$(4.3)\qquad n^{c_{12} n} d^{4/(n+1)} (\log d)^{4n} M_1(\mathfrak{f}, T)^{9.5} e^{2\lambda M_1(\mathfrak{f},T)},$$

the inequality (3.5) holds at z_j for all v satisfying (3.6), provided T is sufficiently large.

Proof. Let

$$\chi_i^* (\mathrm{mod}\, \mathfrak{a}_i), \qquad N\mathfrak{a}_i \leqslant N\mathfrak{f}, \qquad i = 1, \ldots, N,$$

be all primitive characters for which

$$\left| \frac{\zeta_K'}{\zeta_K}(z_j'', \zeta^*, \mathfrak{a})^{(v)} \right| \geqslant \frac{v! \exp(-\lambda M_1(\mathfrak{f}, T))}{(1-\theta)^v}$$

for a fixed v from (3.6). Analogously to (3.9) we get

$$(4.4)\qquad \frac{Nv!^2 e^{-2\lambda M_1(\mathfrak{f},T)}}{(1-\theta)^{2v}} \ll n^{c_3 n} d^{2/(n+1)} (\log d)^{2n} \Big(\max_{N\mathfrak{a} \leqslant N\mathfrak{f}} |\zeta(2\sigma_0 - 1, \chi_0, \mathfrak{a})^{(2v+4)}| + N \max_{\substack{N\mathfrak{a} \leqslant N\mathfrak{f}^2 \\ \chi \neq \chi_0}} |\zeta(2\sigma_0 - 1, \chi, \mathfrak{a})^{(2v+4)}| \Big).$$

Now, the first expression on the right of (4.3) is estimated using (3.10), and the second using Cauchy's inequality for the circle $|s-(2\sigma_0-1)| \leqslant 2\sigma_0-1-\theta = 3(1-\theta)$. In this circle, if $\chi \neq \chi_0$ and $N\mathfrak{a} \leqslant N\mathfrak{f}^2$ we have

$$(4.5) \qquad |\zeta_K(s, \chi, \mathfrak{a})| \leqslant \frac{n^{c_{11}n} d^{2/(n+1)}}{1-\theta} (\log d)^{2n} \left(N\mathfrak{f} (\log N\mathfrak{f})^{n(n+1)/2}\right)^{1-\theta}.$$

To prove this we take $\sigma > 1$ and use Abel's formula. We obtain

$$\zeta_K(s, \chi, \mathfrak{a}) = s\int_1^x H(\xi, \chi)\xi^{-1-s}d\xi + s\int_x^\infty H(\xi, \chi)\xi^{-1-s}d\xi.$$

By (2.4), for $\sigma \geqslant \theta \geqslant 1-1/(n+1)$ the first integral is estimated as follows:

$$\Big|s\int_1^x H(\xi, \chi)\xi^{-1-s}d\xi\Big| \ll |s|\, n^{c_6 n} d^{2/(n+1)} (\log d)^{2n} \frac{x^{1-\theta}}{1-\theta}.$$

Similarly, (2.5) gives for the second integral (for $\sigma \geqslant \theta \geqslant 1-1/(n+1)$) the bound

$$\Big|s\int_x^\infty H(\xi, \chi)\xi^{-1-s}d\xi\Big| \ll |s|\, n^{c_5 n} (dN\mathfrak{f})^{1/(n+1)} (\log(dN\mathfrak{f}))^n x^{1-\theta-2/(n+1)}.$$

Putting $x = \sqrt{N\mathfrak{a}}(\log N\mathfrak{a})^{n(n+1)/2}$, where $N\mathfrak{a} \leqslant N\mathfrak{f}^2$, we get (4.5).

Hence, if $\chi \neq \chi_0$ and $N\mathfrak{a} \leqslant N\mathfrak{f}^2$, Cauchy's inequality yields by (4.5)

$$|\zeta_K(2\sigma_0-1, \chi, \mathfrak{a})^{(2\nu+4)}| \leqslant \frac{(2\nu+4)!\, n^{c_{11}n}}{3^{2\nu+4}(1-\theta)^{2\nu+5}} d^{2/(n+1)} \left(N\mathfrak{f}(\log N\mathfrak{f})^{(n^2+n)/2}\right)^{1-\theta}$$

and this shows that using (4.4) we get the inequality

$$Ne^{-2\lambda M_1(\mathfrak{f}, T)} \ll n^{c_{12}n} d^{4/(n+1)} (\log d)^{4n} \left(\frac{(2\nu+5)!}{\nu!^2\, 2^{2\nu+5}(1-\theta)^5} + N\frac{(2\nu+4)!}{3^{2\nu+4}(1-\theta)^5\nu!^2}(N\mathfrak{f})^{1-\theta}(\log N\mathfrak{f})^{n(n+1)(1-\theta)/2}\right)$$

analogous to (3.12). Thus for sufficiently large T we have for N the same estimate as before for P and the lemma is proved.

We call a zeta function "good" in a "bad" strip l_j if it satisfies inequality (3.5) at z_j'' for all ν from (3.6), and "bad" in the opposite case.

Owing to Lemma 2, from (4.2) and (4.3) the number of zeros of "bad" ζ-functions in all "bad" strips of the rectangle

$$1/0.07c_8 M(\mathfrak{f}, T) \leqslant 1-\theta \leqslant \min(c_1, e^{-20(A+1)}), \qquad T/2 \leqslant t \leqslant T,\ T > c_{13}$$

cannot exceed

$$(4.6) \qquad n^{c_{14}n} d^{8/(n+1)} \log^{8n} d M_1^{24}(\mathfrak{f}, T)\, e^{4\lambda M_1(\mathfrak{f}, T)}$$

5. Let $z^* = \sigma_0 + it^*$ be a point of H^* in any fixed "good" strip or the point z_j'' in any fixed "bad" strip. Hence for all ν from the interval (3.6) we have

$$(5.1) \qquad \left|\frac{\zeta'}{\zeta}(z^*, \chi^*, \mathfrak{a})^{(\nu)}\right| < \frac{\nu!\, e^{-\lambda M_1(\mathfrak{f}, T)}}{(1-\theta)^\nu},$$

where ζ is any ζ-function in the first case, and a "good" one in the second case. We shall apply Lemma 1 with $r = e(1-\theta)$, $R = e^2(1-\theta)$, $G(z) = \zeta(z+z^*, \chi^*, \mathfrak{a})$. We have

$$(5.2) \qquad \left|\sum_{|z^*-\varrho| \leqslant e(1-\theta)} \left(\frac{1-\theta}{z^*-\varrho}\right)^{\nu+1}\right| \leqslant \frac{4(\nu+1)\log U}{e^{\nu+1}} + \frac{(1-\theta)^{\nu+1}}{\nu!}\left|\frac{\zeta'}{\zeta}(z^*, \chi^*, \mathfrak{a})^{(\nu)}\right|,$$

where ϱ runs through the zeros of ζ in the disc $|z^*-\varrho| \leqslant e(1-\theta)$ and

$$U = \max_{|z| \leqslant e^2(1-\theta)} \left|\frac{\zeta(z+z^*, \chi^*, \mathfrak{a})}{\zeta(z^*, \chi^*, \mathfrak{a})}\right|.$$

Owing to (2.4) we get

$$\left|\frac{1}{\zeta(z^*, \chi^*, \mathfrak{a})}\right| = \left|\prod_{\mathfrak{p}}\left(1-\frac{\chi^*(\mathfrak{p})}{N\mathfrak{p}^{z^*}}\right)\right| < \zeta_K(\sigma_0) \leqslant n^{c_3 n} d^{2/(n+1)} \log^{2n} d \frac{\sigma_0}{\sigma_0-1} \ll n^{c_3 n} d^{2/(n+1)} \log^{2n} d M_1,$$

and using Lemma 4,

$$|\zeta(z+z^*, \chi^*, \mathfrak{a})| \leqslant e^{c_6 A} N\mathfrak{f}^{(e^2-1)(1-\theta)}\, T^{14\cdot 10^3 n^{2.5}(n+2)(e^2-1)^{3/2}(1-\theta)^{3/2}} M_1(\mathfrak{f}, T).$$

Hence

$$(5.3) \qquad U < e^{c_{15}A} M_1^2(\mathfrak{f}, T)(N\mathfrak{f})^{(e^2-1)(1-\theta)}\, T^{14\cdot 10^3 n^{2.5}(n+2)(e^2-1)^{3/2}(1-\theta)^{3/2}}$$

and $\log U \ll M_1(\mathfrak{f}, T)$.

So the first expression on the right of (5.2) is arbitrarily small and the second can be estimated using (5.1). Therefore for $T > c_{16}$ and for all ν permitted by (3.6) we have

$$(5.4) \qquad \left|\sum_{|z^*-\varrho| \leqslant e(1-\theta)} \left(\frac{1-\theta}{z^*-\varrho}\right)^{\nu+1}\right| < \tfrac{1}{2}e^{-\lambda M_1(\mathfrak{f}, T)}.$$

In order to estimate the sum in (5.4) from below we shall apply Turán's theorem (Lemma 2). To estimate the number of terms in (5.4) we apply Jensen's inequality, which gives for the number of zeros of the regular function $f(s)$ in the disc $|s-s_0| \leqslant \vartheta R$ $(0 < \vartheta < 1)$ the bound

$$\frac{1}{\log(1/\vartheta)} \max_{|s-s_0| \leqslant R} \log\left|\frac{f(s)}{f(s_0)}\right|.$$

This means that the sum (5.4) has at most $\log U$ terms. Owing to (5.3),

$$\log U \leqslant \frac{\lambda M_1}{\log\frac{3}{2}\log\frac{1}{1-\theta}}\left(\frac{c_{15} A \log\frac{3}{2}}{(1-\theta)^{3/2}\log^2\frac{1}{1-\theta}M_1} + \frac{2\log M_1}{(1-\theta)^{3/2}\log^2\frac{1}{1-\theta}M_1}\right.$$
$$\left. + \frac{(e^2-1)\log N\mathfrak{f}}{(1-\theta)^{1/2}\log^2\frac{1}{1-\theta}M_1} + \frac{14\cdot 10^3 n^{2.5}(n+2)(e^2-1)^{3/2}\log T \log\frac{3}{2}}{M_1\log^2\frac{1}{1-\theta}}\right)$$

It is easy to notice that the first three terms on the right-hand side are, owing to (3.3), arbitrarily small, provided T is sufficiently large. Similarly, the last term can be made arbitrarily small for sufficiently large T, provided $1-\theta$ is sufficiently small. Hence we can choose in Lemma 2

$$N^* = \frac{\lambda M_1 A}{\log\frac{3}{2}\log(1/(1-\theta))}$$

and according to (3.6)

$$m = \frac{\lambda M_1}{\log\frac{3}{2}}\left(1 + \frac{A}{\log(1/(1-\theta))}\right).$$

Moreover, in any strip l_j we have, owing to (3.6),

$$\left|\frac{1-\theta+1-\sigma_\varrho}{z^*-\varrho}\right|^{\nu+1} \geqslant \frac{1}{2},$$

provided T is sufficiently large. Hence by Lemma 2 there exists a ν_0 in the interval (3.6) such that

$$(5.5)\quad \left|\sum_{|z^*-\varrho| \leqslant e(1-\theta)} \left(\frac{1-\theta}{z^*-\varrho}\right)^{\nu_0+1}\right|$$
$$\geqslant \tfrac{1}{2}\exp\left(-N^*\log\left(8e\left(1+\frac{m}{N^*}\right)\right) - (\nu_0+1)\log\left(1+\frac{1-\sigma_{\varrho^*}}{1-\theta}\right)\right),$$

where $\varrho^* = \sigma_{\varrho^*} + it_{\varrho^*}$ denotes the zero with the greatest real part in the strip l_j.

Let

$$\beta = \frac{\log(1/(1-\theta))}{A}.$$

Comparing (5.4) with (5.5) we get

$$-\lambda M_1 > -\frac{\lambda M_1}{\beta\log\frac{3}{2}}\log(8e(1+\beta)) - \frac{\lambda M_1}{\log\frac{3}{2}}\left(1+\frac{1}{\beta}\right)\log\left(1+\frac{1-\sigma_{\varrho^*}}{1-\theta}\right)$$

and finally

$$\log\left(1+\frac{1-\sigma_{\varrho^*}}{1-\theta}\right) > \frac{\beta\log\frac{3}{2} - \log(8e(1+\beta))}{1+\beta}.$$

Putting $\beta \geqslant 20$ we get

$$\log\left(1+\frac{1-\sigma_{\varrho^*}}{1-\theta}\right) > 0.0945.$$

This proves that all zeros of our zeta-functions in "good" strips and also of "good" functions in "bad' strips satisfy the inequality

$$\sigma_\varrho < 1-0.09(1-\theta).$$

Putting

$$\alpha = 1-0.09(1-\theta)$$

and using (4.6) we have for $T > c_{17}$ the estimate

$$(5.6)\quad \sum_{N\mathfrak{a} \leqslant N\mathfrak{f}} \sum_{\chi(\mathrm{mod}\,\mathfrak{a})}^{*} (N(\alpha, T, \chi) - N(\alpha, T/2, \chi))$$
$$\leqslant n^{c_{14}^n} d^{8/(n+1)} \log^{8n} d M_1^{24} \exp(4\lambda M_1).$$

Replacing in (5.6) T by

$$T/2 \geqslant \ldots \geqslant T/2^i \geqslant c_{17} \geqslant T/2^{i+1}$$

we get after summation and application of (5.6)

$$(5.7)\quad \sum_{N\mathfrak{a} \leqslant N\mathfrak{f}} \sum_{\chi(\mathrm{mod}\,\mathfrak{a})}^{*} (N(\alpha, T, \chi) - N(\alpha, c_{17}, \chi))$$
$$< n^{c_{14}^n} d^{8/(n+1)} \log^{8n} d M_1^{25} \exp(4\lambda M_1)$$
$$\leqslant \exp\left(150 M_1(\mathfrak{f}, T)(1-\alpha)^{3/2}\log\frac{0.09}{1-\alpha}\right)$$

in the rectangle (1.1).

6. Now, as in the proof of (4.6) we shall show that in the rectangle

$$1-\min\bigl(c_1, \exp(-20(A+1))\bigr) \leqslant \alpha \leqslant 1, \quad 0 \leqslant t \leqslant c_{17}$$

the inequality

$$(6.1)\quad \sum_{N\mathfrak{a}\leqslant N\mathfrak{f}} \sum_{\chi(\mathrm{mod}\,\mathfrak{a})}^{*} N(\alpha, c_{17}, \chi)$$
$$\leqslant n^{c18^n} d^{4/(n+1)} \log^{4n} d \log^{14}(dN\mathfrak{f}) \exp\bigl(2\lambda \log^{3/2}(dN\mathfrak{f})\bigr)$$
$$\leqslant \exp\bigl((2+\varepsilon)\lambda \log^{3/2}(dN\mathfrak{f})\bigr)$$
$$\leqslant \exp\left(70(1-\alpha)^{3/2} \log^3 \frac{0.1}{1-\alpha} \log^{3/2}(dN\mathfrak{f})\right)$$

holds for an arbitrarily small ε.

Since there exists a numerical constant $c_{19} > 0$ such that in the region

$$(6.2)\quad \sigma \geqslant 1 - \frac{c_{19}}{n\log(dN\mathfrak{f})}, \quad |t| \leqslant c_{17}$$

the function $\Phi(s, \mathfrak{f}) = \prod_{N\mathfrak{a}\leqslant N\mathfrak{f}} \prod_{\chi(\mathrm{mod}\,\mathfrak{a})} \zeta_K(s, \chi, \mathfrak{a})$ has at most one zero (see [3]), let θ be such that

$$\frac{10c_{19}}{n\log(dN\mathfrak{f})} \leqslant 1-\theta \leqslant \min\bigl(c_1, e^{-20(A+1)}\bigr).$$

We obtain for the same λ as before

$$\frac{\lambda(\log(dN\mathfrak{f}))^{3/2}}{\log(1/(1-\theta))} \geqslant \gamma n d^{1/4} \log\bigl(n\log(dN\mathfrak{f})\bigr)$$

where γ is a sufficiently large constant, provided either n, d, or $N\mathfrak{f}$ is sufficiently large.

We divide the segment $\sigma_0 = 2-\theta$, $0 \leqslant t \leqslant c_{17}$ using the points

$$s_j = \sigma_0 + i\tau_j = 2-\theta + i\frac{jc_{19}^3 10^3}{(n\log(dN\mathfrak{f}))^3}$$

where $j = 0, 1, \ldots, (c_{19} 10 n \log(dN\mathfrak{f}))^3/c_{17}$.

Similarly to Lemma 7, we obtain

LEMMA 8. *For all $\zeta_K(s, \chi, \mathfrak{a})$ functions, $N\mathfrak{a} \leqslant N\mathfrak{f}$, except at most*

$$(6.3)\quad n^{c12^n} d^{4/(n+1)} (\log d)^{4n} (\log dN\mathfrak{f})^{11} \exp\bigl(2\lambda(\log dN\mathfrak{f})^{3/2}\bigr)$$

the inequality

$$\left|\frac{\zeta_K'}{\zeta_K}(s_j, \chi, \mathfrak{a})^{(\nu)}\right| < \frac{\nu! \exp\bigl(-\lambda(\log dN\mathfrak{f})^{3/2}\bigr)}{(1-\theta)^{\nu}}$$

holds for all ν with

$$\frac{\lambda(\log(dN\mathfrak{f}))^{3/2}}{\log\frac{3}{2}}\left(1+\frac{1}{\log(1/(1-\theta))}\right) \leqslant \nu \leqslant \frac{\lambda(\log(dN\mathfrak{f}))^{3/2}}{\log\frac{3}{2}}\left(1+\frac{2}{\log(1/(1-\theta))}\right),$$

provided either n, d or $N\mathfrak{f}$ is sufficiently large.

Using Landau's and Turán's theorems (Lemmas 1 and 2), we find that in the strip

$$|t-\tau_j| < 500\, c_{19}^3 (n\log(dN\mathfrak{f}))^{-3}$$

zeros $\varrho^* = \sigma_{\varrho^*} + it_{\varrho^*}$ of such zeta functions satisfy

$$1-\sigma_{\varrho^*} \geqslant 0.1(1-\theta).$$

Putting $\alpha = 1-0.1(1-\theta)$ we see that if $1-\theta \leqslant c_{20}$ and one of the parameters n, d or $N\mathfrak{f}$ is sufficiently large, all zeros of $\Phi(s, \mathfrak{f})$ for $\sigma \geqslant \alpha$ are zeros of "bad" $\zeta_K(s, \chi^*, \mathfrak{a})$-functions, which we estimate using (6.3) and Lemma 3 and finally get (6.1).

If n, d and $N\mathfrak{f}$ are all bounded we take $1-\alpha < c_{21}$ where c_{21} is a sufficiently small absolute constant such that the rectangle $0 \leqslant 1-\alpha < c_{21}$, $t \leqslant c_{17}$ is narrower than the rectangle (6.2). This means that there is at most one zero there.

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On representation of r-th powers by subset sums

by

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Let A be a set of x natural numbers

(1) $$A = \{a_1, \ldots, a_x\}, \quad 1 \leqslant a_1 < a_2 < \ldots < a_x \leqslant l, \quad |A| = x.$$

Let $\mathcal{M}$ be a given set of integers. Denote by $f(l, \mathcal{M})$ the maximum cardinality of a set A which contains no subset $B \subseteq A$ such that $\sum_{a_i \in B} a_i \in \mathcal{M}$.

Recently Erdős and Freud, and N. Alon proposed the following four similar problems:

1. Let $a_x \leqslant 3(x-1)$. Does there exist a subset $B \subseteq A$ such that $\sum_{a_i \in B} a_i$ is a power of two? ([Er].)

2. Let $a_x \leqslant 4(x-1)$. Does there exist a subset $B \subseteq A$ such that $\sum_{a_i \in B} a_i$ is a square-free number? ([Er].)

3. What is a maximal cardinality of set A which contains no subset $B \subseteq A$ such that $\sum_{a_i \in B} a_i$ is a square? In other words what is $f(l, \mathcal{M})$ if $\mathcal{M} = M_2$ is the set of all squares? ([Er].)

4. Let $f(l, m)$ denote for $m \geqslant 1$ the maximum cardinality of a set $A \subseteq \{1, \ldots, l\}$ which contains no subset $B \subseteq A$ such that $\sum_{a_i \in B} a_i = m$.

Conjecture of N. Alon is that if $l^{1.1} \leqslant m \leqslant l^{1.9}$, then

$$f(l, m) = (1+o(1))\frac{l}{\bar{m}} \quad \text{as } l \to \infty;$$

$\bar{m}$ denotes the smallest integer that does not divide m. ([Al])

G. Freiman stated a natural generalization of problem 3 of P. Erdős:

3′. What is $f(l, \mathcal{M})$ in the case when $\mathcal{M} = M_r$ is the set of all rth powers?

Problems 1 and 2 are considered in [Al] and [EF]. In [Al] it is shown

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