Conspectus materiae tomi LIII, fasciculi 2

La revue est consacrée à la Théorie des Nombres
The journal publishes papers on the Theory of Numbers
Die Zeitschrift veröffentlicht Arbeiten aus der Zahlentheorie
Журнал посвящен теории чисел

L'adresse de la Rédaction et de l'échange

Address of the Editorial Board and of the exchange

Die Adresse der Schriftleitung und des Austausches

Адрес редакции и книгообмена

ACTA ARITHMETICA
ul. Śniadeckich 8, 00-950 Warszawa

Les auteurs sont priés d'envoyer leurs manuscrits en deux exemplaires à l'adresse ci-dessus
The authors are requested to submit papers in two copies to the above address
Die Autoren sind gebeten um Zusendung von 2 Exemplaren jeder Arbeit an die obige Adresse
Рукописи статей редакция просит предлагать в двух экземплярах на вышеуказанный адрес

ISBN 83-01-09179-7 ISSN 0065-1036

PRINTED IN POLAND

W R O C Ł A W S K A D R U K A R N I A N A U K O W A

ACTA ARITHMETICA
LIII (1989)

Linear forms in p-adic logarithms

by

KUNRUI YU (Beijing, China)

Dedicated to the Memory of Professor V. G. Sprindžuk

Contents

Introduction and results. The p-adic theory of transcendental numbers was initiated by Mahler in the 1930s. Mahler [20], [21] obtained in 1932 and 1935 the p-adic analogues of both the Hermite–Lindemann and the Gelfond–Schneider theorems; and during the course of the work he founded the p-adic theory of analytic functions.

In 1939, Gelfond [14] proved a quantitative result on linear forms in two p-adic logarithms in analogy with his classic work on Hilbert's seventh problem relating to two complex logarithms. In 1967, Schinzel [28] improved Gelfond's result and computed all the constants explicitly.

In the 1960s Baker published his first series of papers [3], [4] on linear forms in $n \geqslant 2$ logarithms of algebraic numbers. His method has subsequently been employed to the investigation on linear forms in $n \geqslant 2$ p-adic logarithms of algebraic numbers. To begin with, in 1967, Brumer [9] proved that if $\alpha_1, \ldots, \alpha_n$ are multiplicatively independent p-adic units then any nontrivial linear form in p-adic logarithms

$$\beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n$$

does not vanish. Later, Coates [12] proved a quantitative p-adic analogue following Baker's result [4]; Sprindžuk [30], [31] proved p-adic analogues of Baker's results [3], [4]; Kaufman [18] proved a p-adic analogue of Feldman's result [13]. Further, in 1975, Baker and Coates [7] established in the case $n=2$ a p-adic analogue of a sharpened inequality of Baker [5]. In 1977, van der Poorten [26] published a paper, containing four theorems on linear forms in p-adic logarithms, with much more generality than the previous work and essentially with the same degree of precision as Baker's result [6]. In order to state van der Poorten's results, we introduce some notation. Denote by $\alpha_1, \ldots, \alpha_n$ $(n \geqslant 2)$ non-zero algebraic numbers in an algebraic number field K of degree D over $\boldsymbol{Q}$, and of heights not exceeding $A_1, \ldots, A_n$ respectively (with $A_j \geqslant e^e$, $1 \leqslant j \leqslant n$). Write

$$\Omega' = \log A_1 \ldots \log A_{n-1}, \qquad \Omega = \Omega' \log A_n.$$

Denote by $b_1, \ldots, b_n$ $(b_n \neq 0)$ rational integers with absolute values not exceeding B. Denote by $\mathfrak{p}$ a prime ideal in the ring of algebraic integers O_K in K, lying above the rational prime p; write $e_\mathfrak{p}$ for the ramification index of $\mathfrak{p}$ and $f_\mathfrak{p}$ for its residue class degree, so $N\mathfrak{p} = N_{K/\boldsymbol{Q}}\mathfrak{p} = p^{f_\mathfrak{p}}$. Let

$$g_\mathfrak{p} = [\tfrac{1}{2} + e_\mathfrak{p}/(p-1)], \qquad G_\mathfrak{p} = N\mathfrak{p}^{g_\mathfrak{p}}(N\mathfrak{p}-1).$$

For $\alpha \in K$, $\alpha \neq 0$ denote by $\mathrm{ord}_\mathfrak{p}\,\alpha$ the order to which $\mathfrak{p}$ divides the fractional ideal (α) and put $\mathrm{ord}_\mathfrak{p}\, 0 = \infty$. Then van der Poorten's [26] Theorem 1 (the main theorem) and Theorem 2 are as follows.

THEOREM 1 VdP. *The inequalities*

$$\infty > \mathrm{ord}_\mathfrak{p}(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) > (16(n+1)D)^{12(n+1)} G_\mathfrak{p} \Omega \log \Omega' \log B$$

have no solutions in rational integers $b_1, \ldots, b_n$; $b_n \not\equiv 0 \pmod p$, *with absolute values at most* B.

THEOREM 2 VdP. *The inequalities*

$$\infty > \mathrm{ord}_\mathfrak{p}(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) > (16(n+1)D)^{12(n+1)} (G_\mathfrak{p}/\log p)\, \Omega (\log B)^2$$

have no solutions in rational integers $b_1, \ldots, b_n$ *with absolute values at most* B.

Unfortunately, the proof in van der Poorten [26] involves several errors and inaccuracies, which we should like to remark upon at the end of § 3.4 and in the Appendix, so that a complete revision is necessary.

In the present paper we prove two theorems, which imply the results we reported on in the Proceedings of the Durham Symposium on Transcendental Number Theory, July 1986. (See Yu [36].) Take now

$$K = \boldsymbol{Q}(\alpha_1, \ldots, \alpha_n)$$

and keep the notations D, $\mathfrak{p}$, p, $e_\mathfrak{p}$, $f_\mathfrak{p}$, $N\mathfrak{p} = N_{K/\boldsymbol{Q}}\mathfrak{p}$ and $\mathrm{ord}_\mathfrak{p}$ introduced above. Denote by $K_\mathfrak{p}$ the completion of K with respect to the (additive) valuation $\mathrm{ord}_\mathfrak{p}$, and the completion of $\mathrm{ord}_\mathfrak{p}$ will be denoted again by $\mathrm{ord}_\mathfrak{p}$. Now let Σ be an algebraic closure of $\boldsymbol{Q}_p$. Write $\boldsymbol{C}_p$ for the completion of Σ with respect to its valuation, which is the unique extension of the valuation $|\ |_p$ of $\boldsymbol{Q}_p$. Denote by ord_p the additive form of the valuation of $\boldsymbol{C}_p$. According to Hasse [17], pp. 298–302, we can embed $K_\mathfrak{p}$ into $\boldsymbol{C}_p$: there exists a $\boldsymbol{Q}$-isomorphism σ from K into Σ such that $K_\mathfrak{p}$ is value-isomorphic to $\boldsymbol{Q}_p(\sigma(K))$, whence we can identify $K_\mathfrak{p}$ with $\boldsymbol{Q}_p(\sigma(K))$. Obviously,

$$\mathrm{ord}_\mathfrak{p}\,\beta = e_\mathfrak{p}\,\mathrm{ord}_p\,\beta \qquad \text{for all } \beta \in K_\mathfrak{p}.$$

Further, for an algebraic number α, write $h(\alpha)$ for its logarithmic absolute height (see Chapter 2). Let $b_1, \ldots, b_n$ be rational integers and q a rational prime such that

$$q \nmid p(p^{f_\mathfrak{p}} - 1). \tag{0.1}$$

Let $V_1, \ldots, V_n, V_{n-1}^+, B_0, B_n, B', B, W$ be real numbers satisfying the following conditions

$$\begin{gathered} V_j \geqslant \max\left(h(\alpha_j), \frac{f_\mathfrak{p}\log p}{D}\right) \quad (1 \leqslant j \leqslant n), \\ V_1 \leqslant \ldots \leqslant V_{n-1}, \quad V_{n-1}^+ = \max(1, V_{n-1}), \end{gathered} \tag{0.2}$$

$$\begin{gathered} B_0 \geqslant \min_{1 \leqslant j \leqslant n, b_j \neq 0} |b_j|, \quad B_n \geqslant |b_n|, \\ B' \geqslant \max_{1 \leqslant j < n} |b_j|, \quad B \geqslant \max\{|b_1|, \ldots, |b_n|, 2\}, \end{gathered} \tag{0.3}$$

$$W \geqslant \begin{cases} \max\left\{\log\left(1 + \dfrac{3}{8n}\dfrac{f_\mathfrak{p}\log p}{D}\left(\dfrac{B_n}{V_1} + \dfrac{B'}{V_n}\right)\right), \log B_0, \dfrac{f_\mathfrak{p}\log p}{D}\right\}, \\ \qquad\qquad \text{if } \min_{1 \leqslant j \leqslant n} \mathrm{ord}_p b_j > 0, \\ \max\left\{\log\left(1 + \dfrac{3}{8n}\dfrac{f_\mathfrak{p}\log p}{D}\left(\dfrac{B_n}{V_1} + \dfrac{B'}{V_n}\right)\right), \dfrac{f_\mathfrak{p}\log p}{D}\right\}, \\ \qquad\qquad \text{if } \min_{1 \leqslant j \leqslant n} \mathrm{ord}_p b_j = 0. \end{cases} \tag{0.4}$$

(It is easy to see, by (0.2), that (0.4) is implied by

$$W \geqslant \begin{cases} \max\left\{\log\left(1 + \dfrac{3}{4n}B\right), \log B_0, \dfrac{f_\mathfrak{p}\log p}{D}\right\}, & \text{if } \min_{1 \leqslant j \leqslant n} \mathrm{ord}_p b_j > 0, \\ \max\left\{\log\left(1 + \dfrac{3}{4n}B\right), \dfrac{f_\mathfrak{p}\log p}{D}\right\}, & \text{if } \min_{1 \leqslant j \leqslant n} \mathrm{ord}_p b_j = 0.) \end{cases}$$

Then we have

THEOREM 1. *Suppose that*

$$\mathrm{ord}_\mathfrak{p}\,\alpha_j = 0 \quad (1 \leqslant j \leqslant n), \tag{0.5}$$

$$[K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K] = q^n, \tag{0.6}$$

$$\mathrm{ord}_p b_n \leqslant \mathrm{ord}_p b_j \quad (1 \leqslant j \leqslant n-1) \tag{0.7}$$

and

$$\alpha_1^{b_1} \dots \alpha_n^{b_n} \neq 1. \tag{0.8}$$

Then

$$\operatorname{ord}_p(\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1)$$
$$< C_1(p, n)\, a_1^n\, n^{n+5/2}\, q^{2n}(q-1)\log^2(nq)(p^{f_p}-1)\frac{(2+1/(p-1))^n}{(f_p \log p)^{n+2}}$$
$$\times D^{n+2} V_1 \dots V_n \left(\frac{W}{6n} + \log(4D)\right)\left(\log(4DV_{n-1}^{+}) + \frac{f_p \log p}{8n}\right),$$

where

$$a_1 = \begin{cases} \frac{56}{15}e, & 2 \leqslant n \leqslant 7, \\ \frac{8}{3}e, & n \geqslant 8 \end{cases}$$

and $C_1(p, n)$ *is given by the following table with*

$$C_1(p, n) = C_1'(p, n)\left(2 + \frac{1}{p-1}\right)^2 \quad \textit{for } p \geqslant 5.$$

n	2	3	4	5	6	7	$n \geqslant 8$
$C_1(2, n)$	768523	476217	373024	318871	284931	261379	2770008
$C_1(3, n)$	167881	104028	81486	69657	62243	57098	116055
$C_1'(p, n)$	87055	53944	42255	36121	32276	24584	311077

Remark. By a little computation it is easy to verify that

$$C_1(2, n)\, a_1^n \leqslant 2770008\,(\tfrac{8}{3}e)^n \quad \text{for all } n \geqslant 2$$

and

$$C_1(p, n)\, a_1^n \leqslant 311077\left(2 + \frac{1}{p-1}\right)^2 (\tfrac{8}{3}e)^n \leqslant 2770008\,(\tfrac{8}{3}e)^n$$
$$\text{for all } p \geqslant 3,\ n \geqslant 2.$$

Thus

$$C_1(p, n)\, a_1^n \leqslant 2770008\,(\tfrac{8}{3}e)^n \quad \text{for all } p \text{ and } n \geqslant 2.$$

Therefore Theorem 1 implies Theorem 1 in Yu [36].

In the following Theorem 2, we assume, instead of (0.4),

$$W \geqslant \begin{cases} \max\left\{\log\left(1 + \frac{2}{5n}\frac{f_p \log p}{D}\left(\frac{B_n}{V_1} + \frac{B'}{V_n}\right)\right),\ \log B_0,\ \frac{f_p \log p}{D}\right\}, \\ \qquad \text{if } \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j > 0, \\ \max\left\{\log\left(1 + \frac{2}{5n}\frac{f_p \log p}{D}\left(\frac{B_n}{V_1} + \frac{B'}{V_n}\right)\right),\ \frac{f_p \log p}{D}\right\}, \\ \qquad \text{if } \min_{1 \leqslant j \leqslant n} \operatorname{ord}_p b_j = 0. \end{cases} \tag{0.9}$$

THEOREM 2. *Suppose that* (0.5)–(0.8) *hold. Then*

$$\operatorname{ord}_p(\alpha_1^{b_1} \dots \alpha_n^{b_n} - 1)$$
$$< C_2(p, n)\, a_2^n n^{n+7/2}\, q^{2n}(q-1)\log^2(nq)\, e_p (p^{f_p}-1)\frac{(2+1/(p-1))^n}{(f_p \log p)^{n+2}}$$
$$\times D^{n+2} V_1 \dots V_n \left(\frac{W}{6n} + \log(4D)\right)^2,$$

where $a_2 = a_2(p, n)$ *and* $C_2(p, n)$ *are given as follows:*

$$a_2(2, n) = \begin{cases} \frac{8}{3}e, & 2 \leqslant n \leqslant 17, \\ \frac{5}{2}e, & n \geqslant 18, \end{cases} \qquad a_2(3, n) = \begin{cases} \frac{8}{3}e, & 2 \leqslant n \leqslant 7, \\ \frac{5}{2}e, & n \geqslant 8, \end{cases}$$

$$a_2(p, n) = \begin{cases} \frac{8}{3}e, & 2 \leqslant n \leqslant 16, \\ \frac{5}{2}e, & n \geqslant 17, \end{cases} \qquad (p \geqslant 5);$$

n	2	3	4	5	6	7	$8 \leqslant n \leqslant 17$	$n \geqslant 18$
$C_2(2, n)$	338071	244589	202601	178202	161998	150321	141430	441432

n	2	3	4	5	6	7	$n \geqslant 8$
$C_2(3, n)$	61716	44650	36985	32531	29573	27442	24871

$$C_2(p, n) = C_2'(p, n)\left(2 + \frac{1}{p-1}\right)^3, \quad p \geqslant 5$$

n	2	3	4	5	6	7	$8 \leqslant n \leqslant 16$	$n \geqslant 17$
$C_2'(p, n)$	14491	10484	8685	7639	6944	6444	6063	17401

Remark. It is easy to verify, by a little computation, that Theorem 2 implies Theorem 2 in Yu [36].

COROLLARY OF THEOREM 2. *One may remove in Theorem 2 the hypothesis* (0.7), *provided* (0.9) *is replaced by*

$$(0.10)\quad W \geqslant \begin{cases} \max\left\{\log\left(1+\frac{4B}{5n}\right), \log B_0, \frac{f_p \log p}{D}\right\}, & \text{if } \min_{1\leqslant j\leqslant n} \operatorname{ord}_p b_j > 0, \\ \max\left\{\log\left(1+\frac{4B}{5n}\right), \frac{f_p \log p}{D}\right\}, & \text{if } \min_{1\leqslant j\leqslant n} \operatorname{ord}_p b_j = 0. \end{cases}$$

To check this it is sufficient to reorder $\alpha_1, \ldots, \alpha_n; b_1, \ldots, b_n$ as $\alpha_{i_1}, \ldots, \alpha_{i_n}$; $b_{i_1}, \ldots, b_{i_n}$ so that $\operatorname{ord}_p b_{i_n} = \min_{1\leqslant j\leqslant n} \operatorname{ord}_p b_j$ and $V_{i_1} \leqslant \ldots \leqslant V_{i_{n-1}}$ and then to apply Theorem 2.

Studies with G. Wüstholz are in progress so as to remove the Kummer condition (0.6) and the appearance of V_{n-1}^+ in the bounds of Theorem 1. This can now be achieved by the recent work of Wüstholz concerning multiplicity estimates in connexion with Baker's theory of linear forms in logarithms of algebraic numbers. (See Wüstholz [35].) Furthermore it seems certain that a combination of Kummer theory with multiplicity estimates will yield very sharp effective bounds.

The research of this paper was done when I was enjoying the hospitality of the Max-Planck-Institut für Mathematik, Bonn, with the support of an Alexander von Humboldt-Fellowship. I am very grateful to Professor F. Hirzebruch, the director of the MPI, for his constant care and encouragement. I wish to thank Professor G. Wüstholz for suggesting the topic of the title and for helpful advice and also to thank Professors A. Baker, D. W. Masser, R. Tijdeman and M. Waldschmidt for their encouragement.

I am greatly indebted to the referee of this paper for helpful proposals to improve the exposition and to shorten some proofs.

Chapter 1. p-Adic analysis

In this chapter we work in $\boldsymbol{C}_p$ introduced in the Introduction. Thus $\boldsymbol{C}_p$ is a complete non-archimedean valued field of characteristic zero with residue class field of characteristic p, and $\operatorname{ord}_p z$ $(z \in \boldsymbol{C}_p)$ is the additive valuation of $\boldsymbol{C}_p$ such that

$$\operatorname{ord}_p p = 1.$$

Throughout this chapter, the variable z takes values from $\boldsymbol{C}_p$. If $\operatorname{ord}_p z \geqslant 0$, we say that z is integral.

1.1. p-Adic exponential and logarithmic functions in $\boldsymbol{C}_p$. We record the following facts, which can be found in Hasse [17], pp. 262–274.

(a) The exponential series

$$\exp(z) = \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

has the region of convergence $\operatorname{ord}_p z > 1/(p-1)$, where

$$\exp(z_1+z_2) = \exp(z_1)\exp(z_2) \quad \text{and} \quad \operatorname{ord}_p(\exp(z)-1) = \operatorname{ord}_p z.$$

(b) The logarithmic series

$$\log(1+z) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} z^n}{n}$$

has the region of convergence $\operatorname{ord}_p z > 0$, where

$$\log((1+z_1)(1+z_2)) = \log(1+z_1)+\log(1+z_2).$$

In the subregion $\operatorname{ord}_p z > 1/(p-1)$,

$$\operatorname{ord}_p \log(1+z) = \operatorname{ord}_p z.$$

(c) For $\operatorname{ord}_p z > 1/(p-1)$, we have

$$\log\exp(z) = z \quad \text{and} \quad \exp(\log(1+z)) = 1+z.$$

(d) For $\operatorname{ord}_p x > 1/(p-1)$ and integral z, we define

$$(1+x)^z = \exp(z\log(1+x)).$$

(Note that, for $z \in \boldsymbol{Z}$, this definition coincides with the usual powers.) Thus, by (c), we have

$$\log(1+x)^z = z\log(1+x).$$

Furthermore for integral z, z' and x, x' with $\operatorname{ord}_p x > 1/(p-1)$, $\operatorname{ord}_p x' > 1/(p-1)$, we have

$$(1+x)^{z+z'} = (1+x)^z(1+x)^{z'}, \quad (1+x)^{zz'} = ((1+x)^z)^{z'},$$
$$(1+x)^z(1+x')^z = ((1+x)(1+x'))^z.$$

Note that for $\beta_1, \ldots, \beta_m \in \boldsymbol{C}_p$ with

$$(1.1)\qquad \operatorname{ord}_p(\beta_j-1) > 1/(p-1) \quad (1 \leqslant j \leqslant m)$$

and integral $z_1, \ldots, z_m \in \boldsymbol{C}_p$, we have

$$\operatorname{ord}_p(\log\beta_j) = \operatorname{ord}_p(\beta_j-1) > 1/(p-1) \quad (1 \leqslant j \leqslant m),$$

hence

$$\operatorname{ord}_p(z_1\log\beta_1 + \ldots + z_m\log\beta_m) > 1/(p-1).$$

Thus, by (d) and (a),

$$(1.2)\qquad \operatorname{ord}_p(\beta_1^{z_1}\ldots\beta_m^{z_m}-1) = \operatorname{ord}_p(\exp(z_1\log\beta_1 + \ldots + z_m\log\beta_m)-1)$$
$$= \operatorname{ord}_p(z_1\log\beta_1 + \ldots + z_m\log\beta_m).$$

1.2. Normal series and functions. For the p-adic analytic parts of the proofs of our theorems, instead of using Schnirelman integral [29] (see also Adams [1]), which yields a p-adic analogue of the Cauchy integral formula, we introduce a kind of Hermite interpolation formula (see the Appendix, Theorem A); then we give, based on Mahler's [21] concept on normal functions, and similarly to the work of Schinzel [28] and van der Poorten [26], a lemma for the extrapolation procedure (see Section 4 of this chapter).

The following concepts of normal series and functions are due to Mahler [21]. A p-adic power series

$$f(z) = \sum_{h=0}^{\infty} f_h (z-z_0)^h, \quad f_h \in C_p \quad (h = 0, 1, \ldots),$$

where z_0 is an integral element of C_p, is called a *normal series*, if

$$\mathrm{ord}_p f_h \geqslant 0 \quad (h = 0, 1, \ldots)$$

and

$$\mathrm{ord}_p f_h \to \infty \quad (h \to \infty).$$

Clearly $f(z)$ converges for every integral z.

Let z_1 be an arbitrary integral element in C_p. By the p-adic analogue of Taylor's theorem, we have

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(z_1)}{k!}(z-z_1)^k$$

where

$$f^{(k)}(z_1) = k! \sum_{h=k}^{\infty} \binom{h}{k} f_h (z_1 - z_0)^{h-k} \quad (k = 0, 1, \ldots)$$

denotes the derivative at z_1 of order k. Obviously

$$\mathrm{ord}_p \frac{f^{(k)}(z_1)}{k!} \geqslant 0 \quad (k = 0, 1, \ldots)$$

and

$$\mathrm{ord}_p \frac{f^{(k)}(z_1)}{k!} \to \infty \quad (k \to \infty).$$

Thus, if a p-adic function is representable by a normal series in a neighborhood of an integral point in C_p, then so is it in a neighborhood of every integral point in C_p. Therefore we may call a p-adic function, which is definable by a normal series in a neighborhood of an integral point in C_p, a *normal function*.

The following lemma is fundamental.

LEMMA 1.1 (Mahler [21]). *If a normal function $f(z)$ has zeros at the distinct integral points $\beta_1, \ldots, \beta_h$ in C_p of multiplicities at least $m_1, \ldots, m_h$, respectively, then*

$$f(z) = g(z) \prod_{j=1}^{h} (z-\beta_j)^{m_j},$$

where $g(z)$ is a normal function.

Remark. If $\delta \in C_p$ satisfies $\mathrm{ord}_p \delta > 1/(p-1)$, then the p-adic series

$$\exp(\delta z) = \sum_{k=0}^{\infty} \frac{\delta^k}{k!} z^k$$

is a normal series, because of the well-known fact that $\mathrm{ord}_p k! \leqslant k/(p-1)$.

1.3. Supernormality. For $\theta = c/d$, where c, d are positive rational integers with $(c, d) = 1$, we define

$$p^\theta = \varrho^c,$$

where ϱ is a fixed root of $x^d - p = 0$ in C_p. Thus

$$\mathrm{ord}_p p^\theta = \theta.$$

If $\delta \in C_p$ satisfies

$$\mathrm{ord}_p \delta > \theta + \frac{1}{p-1},$$

then $\exp(\delta z)$ has supernormality in the sense that

$$\exp(\delta p^{-\theta} z) = \sum_{k=0}^{\infty} \frac{(\delta p^{-\theta})^k}{k!} z^k$$

is a normal function.

The following lemma shows that there exists a nonnegative integer $\varkappa$ bounded in terms of p and e_p such that for every $\beta \in C_p$ satisfying $\mathrm{ord}_p(\beta - 1) \geqslant 1/e_p$ the p-adic function

$$(\beta^{p^\varkappa})^z = \exp(z \log \beta^{p^\varkappa})$$

has supernormality required for our p-adic analytic part of the proofs of our theorems.

LEMMA 1.2. *Let $\varkappa$ be the rational integer satisfying*

$$p^{\varkappa-1}(p-1) \leqslant \left(1 + \frac{p-1}{p}\right) e_p < p^\varkappa (p-1) \tag{1.3}$$

and put

$$\theta = \begin{cases} 1, & \text{if } \varkappa \geqslant 1 \text{ and } p^{\varkappa-1}(p-1) > e_p; \\ \dfrac{p^\varkappa}{(2+1/(p-1))e_p}, & \text{otherwise.} \end{cases} \tag{1.4}$$

If $\beta \in C_p$ satisfies

$$\mathrm{ord}_p(\beta - 1) \geqslant 1/e_p,$$

then

$$\mathrm{ord}_p(\beta^{p^\varkappa} - 1) > \theta + \frac{1}{p-1}.$$

Remark. By the remark at the end of the last section, $(\beta^{p^\varkappa})^{p^{-\theta}z} = \exp(p^{-\theta}z\log\beta^{p^\varkappa})$ is a normal function.

Proof. (For details see Lemma 2 in Yu [36].) By considering $(\gamma-1)^p$ it is easy to verify that for integral $\gamma\in C_p$ we have

$$\operatorname{ord}_p(\gamma^p-1)\geqslant\min(p\operatorname{ord}_p(\gamma-1),1+\operatorname{ord}_p(\gamma-1)). \tag{1.5}$$

The lemma is evidently true if $\varkappa=0$. If $\varkappa\geqslant1$, then we obtain by inductive use of (1.5) that

$$\operatorname{ord}_p(\beta^{p^j}-1)\geqslant p^j/e_{\mathfrak p}\quad\text{for } j=0,1,\ldots,\varkappa-1. \tag{1.6}$$

On combining (1.6) for $j=\varkappa-1$ with (1.5) we obtain the required conclusion

$$\operatorname{ord}_p(\beta^{p^\varkappa}-1)\geqslant\min\left(\frac{p^\varkappa}{e_{\mathfrak p}},\frac{p^{\varkappa-1}}{e_{\mathfrak p}}+1\right)>\theta+\frac{1}{p-1}.$$

For later references, note that by (1.3) and (1.4) we have

$$\theta\leqslant1\quad\text{and}\quad\frac{p^\varkappa}{e_{\mathfrak p}}\leqslant\frac{p^\varkappa}{e_{\mathfrak p}\theta}\leqslant2+\frac{1}{p-1}. \tag{1.7}$$

Let

$$G=N_{K/Q}\mathfrak p-1=p^{f_{\mathfrak p}}-1.$$

It is well known (see Hasse [17], p. 220) that if m is a positive rational integer with $(p,m)=1$, then $K_{\mathfrak p}$ contains the mth roots of unity if and only if $m|G$. In particular, $K_{\mathfrak p}$ contains the Gth roots of unity. In the remaining part of this paper, let ζ be a fixed Gth primitive root of unity in $K_{\mathfrak p}$.

For any integral elements α, β in $K_{\mathfrak p}$ we write

$$\alpha\equiv\beta\pmod{\mathfrak p},$$

if $\operatorname{ord}_{\mathfrak p}(\alpha-\beta)\geqslant1$. Obviously, this defines an equivalence relation on

$$O_{\mathfrak p}=\{\alpha\in K_{\mathfrak p}|\ \operatorname{ord}_{\mathfrak p}\alpha\geqslant0\}.$$

LEMMA 1.3. *For any $\alpha\in K_{\mathfrak p}$ with $\operatorname{ord}_{\mathfrak p}\alpha=0$, there exists $r\in\mathbf Z$ with $0\leqslant r<G$ such that*

$$e_{\mathfrak p}\operatorname{ord}_p(\alpha\zeta^r-1)=\operatorname{ord}_{\mathfrak p}(\alpha\zeta^r-1)\geqslant1.$$

Proof. By Hasse [17], p. 153, 155, 220, we see that the set

$$\{0,1,\zeta,\zeta^2,\ldots,\zeta^{G-1}\}$$

is a complete residue system of $O_{\mathfrak p}$ mod $\mathfrak p$. Since $\operatorname{ord}_{\mathfrak p}\alpha=0$, there exists $r'\in\mathbf Z$ with $0\leqslant r'<G$ such that

$$\alpha\equiv\zeta^{r'}\pmod{\mathfrak p}.$$

Let $r\in\mathbf Z$ satisfy $r\equiv-r'\pmod G$ and $0\leqslant r<G$. We get then

$$\alpha\zeta^r\equiv1\pmod{\mathfrak p},$$

and the lemma follows at once.

1.4. A lemma for extrapolation.

LEMMA 1.4. *Suppose that $\theta>0$ is a rational number, $q>0$ is a rational prime with $q\neq p$, and $M>0$, $R>0$ are rational integers with $q|R$. Suppose further that $F(z)$ is a p-adic normal function and*

$$\min_{\substack{1\leqslant s\leqslant R,(s,q)=1\\ t=0,\ldots,M-1}}\left(\operatorname{ord}_p\frac{F^{(t)}(sp^\theta)}{t!}+t\theta\right)\geqslant\left(1-\frac1q\right)RM\theta+M\operatorname{ord}_pR!+(M-1)\frac{\log R}{\log p}. \tag{1.8}$$

Then for all $l\in\mathbf Z$, we have

$$\operatorname{ord}_pF\left(\frac lq p^\theta\right)\geqslant\left(1-\frac1q\right)RM\theta.$$

Remark. Here $\log R$ and $\log p$ denote the usual logarithms for positive real numbers.

Proof. By Theorem A of the Appendix, the unique polynomial $Q(z)$ of degree at most $\left(1-\frac1q\right)RM-1$ satisfying

$$Q^{(t-1)}(sp^\theta)=F^{(t-1)}(sp^\theta),\quad1\leqslant s\leqslant R,\ (s,q)=1,\ 1\leqslant t\leqslant M$$

is given by the formula

$$Q(z)=\sum_{\substack{s=1\\(s,q)=1}}^{R}\sum_{t=1}^{M}\frac{F^{(t-1)}(sp^\theta)}{(t-1)!}(-1)^{M-t}(z-sp^\theta)^{t-1}\left\{\prod_{\substack{j=1\\(j,q)=1\\j\neq s}}^{R}\left(\frac{z-jp^\theta}{(s-j)p^\theta}\right)^M\right\} \tag{1.9}$$
$$\times\sum_{h=1}^{M-t}(-1)^{h-1}\sum_{\substack{\lambda_1+\ldots+\lambda_{M-t}=M-t\\ \lambda_i=0\,(i<h)\\ \lambda_i\geqslant1\,(i\geqslant h)}}\prod_{i=1}^{M-t}\frac{1}{\lambda_i!}\left(\frac{\partial}{\partial\eta}\right)^{\lambda_i}\left\{(z-\eta)\prod_{\substack{k=1\\(k,q)=1\\k\neq s}}^{R}\left(\frac{\eta-kp^\theta}{(s-k)p^\theta}\right)^M\right\}_{\eta=sp^\theta}$$

where the second line of (1.9) reads as 1 when $t=M$. Let

$$E_s(z)=\prod_{\substack{k=1\\(k,q)=1,k\neq s}}^{R}\frac{z-kp^\theta}{(s-k)p^\theta},$$

$$A_{s,t}(z)=(z-sp^\theta)^{t-1}(E_s(z))^M,$$

$$B_{s,\lambda}(z)=\frac{1}{\lambda!}\left(\frac{\partial}{\partial\eta}\right)^\lambda\{(z-\eta)(E_s(\eta))^M\}_{\eta=sp^\theta}.$$

Then (1.9) can be written as

$$(1.10)\qquad Q(z)=\sum_{\substack{s=1\\(s,q)=1}}^{R}\sum_{t=1}^{M}(-1)^{M-t}\frac{F^{(t-1)}(sp^{\theta})}{(t-1)!}A_{s,t}(z)$$
$$\times\sum_{h=1}^{M-t}(-1)^{h-1}\sum_{\substack{\lambda_1+\ldots+\lambda_{M-t}=M-t\\ \lambda_i=0(i<h)\\ \lambda_i\geqslant 1(i\geqslant h)}}\prod_{i=1}^{M-t}B_{s,\lambda_i}(z).$$

We first show that for every $l\in\boldsymbol{Z}$,

$$(1.11)\qquad \operatorname{ord}_pQ\left(\frac{l}{q}p^{\theta}\right)$$
$$\geqslant\min_{\substack{1\leqslant s\leqslant R,(s,q)=1\\ t=0,\ldots,M-1}}\left(\operatorname{ord}_p\frac{F^{(t)}(sp^{\theta})}{t!}+t\theta\right)-M\operatorname{ord}_pR!-(M-1)\frac{\log R}{\log p}.$$

Note that for every s with $1\leqslant s\leqslant R$, $(s,q)=1$, we have, by $(q,p)=1$,

$$\operatorname{ord}_pE_s\left(\frac{l}{q}p^{\theta}\right)=\operatorname{ord}_p\prod_{\substack{k=1\\(k,q)=1,k\neq s}}^{R}\frac{l/q-k}{s-k}\geqslant-\operatorname{ord}_p\prod_{\substack{k=1\\k\neq s}}^{R}(s-k)$$
$$\geqslant-\operatorname{ord}_p(R-1)!\geqslant-\operatorname{ord}_pR!.$$

Thus we get, by $(q,p)=1$,

$$(1.12)\qquad \operatorname{ord}_pA_{s,t}\left(\frac{l}{q}p^{\theta}\right)\geqslant(t-1)\theta-M\operatorname{ord}_pR!$$
$$\text{for } l\in\boldsymbol{Z},\ 1\leqslant s\leqslant R,\ (s,q)=1,\ 1\leqslant t\leqslant M.$$

On noting that

$$(1.13)\qquad E_s(sp^{\theta})=1$$

and for every $\mu\in\boldsymbol{Z}$ with $1\leqslant\mu\leqslant\left(1-\frac{1}{q}\right)R-1$

$$(1.14)\qquad \frac{1}{\mu!}\left(\frac{d}{d\eta}\right)^{\mu}E_s(\eta)=E_s(\eta)\sum_{\substack{1\leqslant k_1<\ldots<k_\mu\leqslant R\\(k_j,q)=1,k_j\neq s\\(1\leqslant j\leqslant\mu)}}\frac{1}{(\eta-k_1p^{\theta})\ldots(\eta-k_\mu p^{\theta})},$$

we obtain

$$\frac{1}{\mu!}\left\{\left(\frac{d}{d\eta}\right)^{\mu}E_s(\eta)\right\}_{\eta=sp^{\theta}}=\sum_{\substack{1\leqslant k_1<\ldots<k_\mu\leqslant R\\(k_j,q)=1,k_j\neq s\\(1\leqslant j\leqslant\mu)}}\frac{1}{(s-k_1)\ldots(s-k_\mu)p^{\mu\theta}}.$$

Observing that

$$\operatorname{ord}_p(s-k_j)\leqslant\left[\frac{\log(R-1)}{\log p}\right]<\frac{\log R}{\log p},$$

we get

$$(1.15)\qquad \operatorname{ord}_p\frac{1}{\mu!}\left\{\left(\frac{d}{d\eta}\right)^{\mu}E_s(\eta)\right\}_{\eta=sp^{\theta}}\geqslant-\mu\left(\theta+\frac{\log R}{\log p}\right)$$
$$\text{for } 1\leqslant\mu\leqslant\left(1-\frac{1}{q}\right)R-1,\ 1\leqslant s\leqslant R,\ (s,q)=1.$$

Note that (1.15) is also true for $\mu=0$ and $\mu>\left(1-\frac{1}{q}\right)R-1$, because of (1.13) and the fact that $E_s(z)$ is a polynomial in z of degree $\left(1-\frac{1}{q}\right)R-1$. Now for positive $\lambda\in\boldsymbol{Z}$

$$(1.16)\qquad \frac{1}{\lambda!}\left(\frac{d}{d\eta}\right)^{\lambda}(E_s(\eta))^M=\sum_{\substack{\mu_1+\ldots+\mu_M=\lambda\\ \mu_j\geqslant 0(1\leqslant j\leqslant M)}}\prod_{j=1}^{M}\frac{1}{\mu_j!}\left(\frac{d}{d\eta}\right)^{\mu_j}E_s(\eta).$$

On combining (1.15) and (1.16), we get

$$(1.17)\qquad \operatorname{ord}_p\frac{1}{\lambda!}\left\{\left(\frac{d}{d\eta}\right)^{\lambda}(E_s(\eta))^M\right\}_{\eta=sp^{\theta}}\geqslant-\lambda\left(\theta+\frac{\log R}{\log p}\right)$$
$$\text{for } \lambda\geqslant 1,\ 1\leqslant s\leqslant R,\ (s,q)=1.$$

Note that (1.17) is also true for $\lambda=0$, by (1.13). Now we estimate $\operatorname{ord}_pB_{s,\lambda_i}\left(\frac{l}{q}p^{\theta}\right)$. By the definition of $B_{s,\lambda}(z)$ we obtain for $\lambda\geqslant 1$

$$(1.18)\qquad B_{s,\lambda}(z)$$
$$=(z-sp^{\theta})\frac{1}{\lambda!}\left\{\left(\frac{d}{d\eta}\right)^{\lambda}(E_s(\eta))^M\right\}_{\eta=sp^{\theta}}-\frac{1}{(\lambda-1)!}\left\{\left(\frac{d}{d\eta}\right)^{\lambda-1}(E_s(\eta))^M\right\}_{\eta=sp^{\theta}}.$$

So by (1.18) and (1.17) we get

$$(1.19)\qquad \operatorname{ord}_pB_{s,\lambda}\left(\frac{l}{q}p^{\theta}\right)\geqslant\min\left\{\theta-\lambda\left(\theta+\frac{\log R}{\log p}\right),-(\lambda-1)\left(\theta+\frac{\log R}{\log p}\right)\right\}$$
$$=-(\lambda-1)\theta-\lambda\frac{\log R}{\log p}$$
$$\text{for } \lambda\geqslant 1,\ 1\leqslant s\leqslant R,\ (s,q)=1.$$

Note that (1.19) is also true for $\lambda = 0$. By (1.19) we see that

$$\text{(1.20)}\quad \mathrm{ord}_p \sum_{h=1}^{M-t} (-1)^{h-1} \sum_{\substack{\lambda_1+\ldots+\lambda_{M-t}=M-t \\ \lambda_i=0(i<h) \\ \lambda_i\geqslant 1(i\geqslant h)}} \prod_{i=1}^{M-t} B_{s,\lambda_i}\left(\frac{l}{q}p^\theta\right) \geqslant -(M-t)\frac{\log R}{\log p}$$

$$\text{for } l\in \mathbf{Z},\ 1\leqslant s\leqslant R,\ (s,q)=1,\ 1\leqslant t\leqslant M-1.$$

Note that (1.20) is also valid for $t = M$. On combining (1.10), (1.12) and (1.20), we conclude

$$\mathrm{ord}_p Q\left(\frac{l}{q}p^\theta\right) \geqslant \min_{\substack{1\leqslant s\leqslant R,(s,q)=1 \\ t=1,\ldots,M}} \left\{\mathrm{ord}_p \frac{F^{(t-1)}(sp^\theta)}{(t-1)!} + (t-1)\theta - M\,\mathrm{ord}_p R! - (M-t)\frac{\log R}{\log p}\right\},$$

which implies (1.11).

Now we proceed to prove that $Q(z)$ is a p-adic normal function, that is, to show that

$$\text{(1.21)}\qquad \mathrm{ord}_p \frac{Q^{(m)}(0)}{m!} \geqslant 0 \quad \text{for} \quad 0\leqslant m\leqslant \left(1-\frac{1}{q}\right)RM-1.$$

By (1.11) with $l = 0$ and (1.8), we see that (1.21) is true for $m = 0$. So we may assume $m \geqslant 1$ in the sequel. We assert that

$$\text{(1.22)}\quad \mathrm{ord}_p \frac{E_s^{(\mu)}(0)}{\mu!} \geqslant -\mu\theta - \mathrm{ord}_p R!$$

$$\text{for } \mu\geqslant 0 \text{ and } 1\leqslant s\leqslant R,\ (s,q)=1,$$

for by the definition of $E_s(z)$, (1.22) is true for $\mu = 0$; it is obvious for $\mu > \left(1-\frac{1}{q}\right)R-1$; and for $1 \leqslant \mu \leqslant \left(1-\frac{1}{q}\right)R-1$, it follows from (1.14) at once. Further (1.22) and (1.16) imply that

$$\text{(1.23)}\quad \mathrm{ord}_p \frac{1}{\lambda!}\left\{\left(\frac{d}{dz}\right)^\lambda (E_s(z))^M\right\}_{z=0} \geqslant -\lambda\theta - M\,\mathrm{ord}_p R!$$

$$\text{for } \lambda\geqslant 0,\ 1\leqslant s\leqslant R,\ (s,q)=1.$$

Now we show that

$$\text{(1.24)}\quad \mathrm{ord}_p \frac{1}{\mu!}\left\{\left(\frac{d}{dz}\right)^\mu A_{s,t}(z)\right\}_{z=0} \geqslant (t-1-\mu)\theta - M\,\mathrm{ord}_p R!$$

$$\text{for } \mu\geqslant 0,\ 1\leqslant s\leqslant R,\ (s,q)=1,\ 1\leqslant t\leqslant M.$$

By the definition of $A_{s,t}(z)$ and (1.23) with $\lambda = 0$, we see that (1.24) is true for $\mu = 0$. Assume $\mu \geqslant 1$. Then

$$\frac{1}{\mu!}\left(\frac{d}{dz}\right)^\mu A_{s,t}(z) = \sum_{\substack{\lambda=0 \\ \lambda\geqslant \mu-t+1}}^{\mu} \left\{\frac{1}{\lambda!}\left(\frac{d}{dz}\right)^\lambda (E_s(z))^M\right\}\binom{t-1}{\mu-\lambda}(z-sp^\theta)^{t-1-(\mu-\lambda)}.$$

This and (1.23) imply (1.24) at once. Now we prove that if $1 \leqslant t \leqslant M-1$ and $\lambda_1, \ldots, \lambda_{M-t}$ are non-negative integers satisfying $\lambda_1 + \ldots + \lambda_{M-t} = M-t$, then

$$\text{(1.25)}\qquad \mathrm{ord}_p \frac{1}{(m-\mu)!}\left\{\left(\frac{d}{dz}\right)^{m-\mu}\left(\prod_{i=1}^{M-t} B_{s,\lambda_i}(z)\right)\right\}_{z=0} \geqslant -(m-\mu)\theta - (M-t)\frac{\log R}{\log p}$$

$$\text{for } 1\leqslant s\leqslant R,\ (s,q)=1,\ 0\leqslant \mu\leqslant m.$$

By (1.17) and (1.18), we have

$$\text{(1.26)}\quad B_{s,\lambda}(z) = a_{s,\lambda}(z-sp^\theta)+b_{s,\lambda} \quad \text{for } \lambda\geqslant 0,\ 1\leqslant s\leqslant R,\ (s,q)=1,$$

where $a_{s,\lambda},\ b_{s,\lambda} \in \mathbf{C}_p$ $(b_{s,0} = 0)$ satisfy

$$\text{(1.27)}\qquad \begin{aligned} \mathrm{ord}_p a_{s,\lambda} &\geqslant -\lambda\left(\theta + \frac{\log R}{\log p}\right), \\ \mathrm{ord}_p b_{s,\lambda} &\geqslant -(\lambda-1)\left(\theta + \frac{\log R}{\log p}\right). \end{aligned}$$

(1.25) is obvious for μ with $m-\mu > M-t$, by (1.26). It is also true for $\mu = m$ by (1.19) with $l = 0$ and the fact that $\lambda_1 + \ldots + \lambda_{M-t} = M-t$. So we may assume $1 \leqslant m-\mu \leqslant M-t$.

Now

$$\frac{1}{(m-\mu)!}\left(\frac{d}{dz}\right)^{m-\mu}\left(\prod_{i=1}^{M-t} B_{s,\lambda_i}(z)\right) = \sum_{1\leqslant i_1<\ldots<i_{m-\mu}\leqslant M-t}\left(\prod_{j=1}^{m-\mu} a_{s,\lambda_{i_j}}\right)\prod_{\substack{1\leqslant i\leqslant M-t \\ i\neq i_j(1\leqslant j\leqslant m-\mu)}} B_{s,\lambda_i}(z).$$

This together with (1.27), (1.19) with $l = 0$ and the fact that $\lambda_1 + \ldots + \lambda_{M-t} = M-t$ yields (1.25). Observing (1.10), (1.24) and (1.25), we obtain for $m = 0, 1, \ldots, \left(1-\frac{1}{q}\right)RM-1$

$$\mathrm{ord}_p \frac{Q^{(m)}(0)}{m!} \geqslant \min_{\substack{1\leqslant s\leqslant R,(s,q)=1 \\ t=1,\ldots,M \\ \mu=0,\ldots,m}} \left\{\mathrm{ord}_p \frac{F^{(t-1)}(sp^\theta)}{(t-1)!} + (t-1-\mu)\theta - (m-\mu)\theta - (M-t)\frac{\log R}{\log p}\right\} - M\,\mathrm{ord}_p R!$$

$$\geqslant \min_{\substack{1\leqslant s\leqslant R,(s,q)=1\\ t=0,\dots,M-1}} \left\{\operatorname{ord}_p \frac{F^{(t)}(sp^\theta)}{t!}+t\theta\right\}$$
$$-\left(\left(1-\frac{1}{q}\right)RM-1\right)\theta - M\operatorname{ord}_p R! - (M-1)\frac{\log R}{\log p}$$
$$\geqslant \theta,$$

where the last inequality follows from (1.8). This proves (1.21), i.e., $Q(z)$ is a normal function.

The normal function

$$F(z)-Q(z)$$

has zeros at

$$sp^\theta, \quad 1\leqslant s\leqslant R, \quad (s,q)=1$$

of multiplicities at least M. By Lemma 1.1, there exists a normal function $g(z)$ such that

$$F(z) = Q(z)+g(z)\prod_{\substack{s=1\\(s,q)=1}}^{R}(z-sp^\theta)^M.$$

Note that $\operatorname{ord}_p g\left(\frac{l}{q}p^\theta\right)\geqslant 0$, because $g(z)$ is normal and $(q, p)=1$, whence $\operatorname{ord}_p\left(\frac{l}{q}p^\theta\right)\geqslant \theta > 0$. Thus for every $l\in \mathbf{Z}$, we have

$$\operatorname{ord}_p F\left(\frac{l}{q}p^\theta\right)\geqslant \min\left(\operatorname{ord}_p Q\left(\frac{l}{q}p^\theta\right), \left(1-\frac{1}{q}\right)RM\theta\right).$$

This together with (1.11) and (1.8) implies

$$\operatorname{ord}_p F\left(\frac{l}{q}p^\theta\right)\geqslant \left(1-\frac{1}{q}\right)RM\theta.$$

The proof of the lemma is thus complete.

Chapter 2. Arithmetic tools and estimates

We first introduce briefly the concept of logarithmic absolute height of an algebraic number α. Let α be of degree d, $a_0>0$ be the leading coefficient of its minimal polynomial f over $\mathbf{Z}$, $H_0(\alpha)$ be its usual height, i.e., the maximum of the absolute values of the coefficients of f, $\alpha_1,\dots,\alpha_d$ be its conjugates over $\mathbf{Q}$. Write

$$M(\alpha) = a_0\prod_{i=1}^{d}\max(1,|\alpha_i|).$$

Let E be a number field containing α. Write

(2.1) $$H_E(\alpha) = \prod_v \max(1,|\alpha|_v),$$

where v runs over all valuations of E normalized in the usual way to satisfy the product formula $\prod_v |\alpha|_v = 1$ for $\alpha\neq 0$. More precisely, for each embedding σ of E into $\mathbf{C}$ there is an archimedean valuation v defined by $|\alpha|_v = |\sigma(\alpha)|$; and for each prime ideal $\mathfrak{P}$ of O_E (the ring of algebraic integers in E) with absolute norm $N\mathfrak{P} = N_{E/\mathbf{Q}}\mathfrak{P}$ there is a non-archimedean valuation defined by

$$|\alpha|_v = (N\mathfrak{P})^{-\operatorname{ord}_{\mathfrak{P}}\alpha},$$

where $\mathfrak{P}^{\operatorname{ord}_{\mathfrak{P}}\alpha}$ is the exact power of $\mathfrak{P}$ in the fractional principal ideal of E generated by α. The numbers

$$H(\alpha) = (H_E(\alpha))^{\frac{1}{[E:\mathbf{Q}]}}$$

and

$$h(\alpha) = \log H(\alpha)$$

are independent of E. We call $H(\alpha)$ and $h(\alpha)$ the *absolute height* and the *logarithmic absolute height* of α, respectively. The relation

$$H_{\mathbf{Q}(\alpha)}(\alpha) = M(\alpha)$$

(see, for example, Bertrand [8], Lemma 11) shows that

$$h(\alpha) = \frac{1}{d}\log M(\alpha).$$

For any algebraic numbers $\alpha, \beta, \alpha_1, \dots, \alpha_n$ and any $0\neq m\in\mathbf{Z}$, we have

(2.2) $$h(\alpha\beta)\leqslant h(\alpha)+h(\beta),$$

(2.3) $$h(\alpha^m) = |m|\,h(\alpha),$$

(2.4) $$h(\alpha_1+\dots+\alpha_n)\leqslant h(\alpha_1)+\dots+h(\alpha_n)+\log n.$$

From the inequality

$$M(\alpha)\leqslant (d+1)^{1/2}H_0(\alpha)$$

(see Mahler [22]) it follows that

$$h(\alpha)\leqslant \frac{1}{d}(\log H_0(\alpha)+\log d),$$

since $h(\alpha) = \log H_0(\alpha)$ for $\alpha\in\mathbf{Q}$ and $x+1\leqslant x^2$ for $x\geqslant 2$. By (2.1) and the product formula, we have

(2.5) $$H_E(\beta) = H_E(1/\beta)\quad \text{for } \beta\in E,\ \beta\neq 0.$$

Now we give a p-adic analogue of the Liouville inequality. For every prime ideal $\mathfrak{P}$ of O_E, let $e_{\mathfrak{P}}$ be its ramification index, $f_{\mathfrak{P}}$ its residue class degree, p the unique rational prime contained in $\mathfrak{P}$. Write

$$\operatorname{ord}_p = \frac{1}{e_{\mathfrak{P}}} \operatorname{ord}_{\mathfrak{P}}.$$

Denote by $|\ |_v$ the non-archimedean valuation determined by $\mathfrak{P}$. Then for every $\beta \in E$, we have

$$p^{-f_{\mathfrak{P}} \operatorname{ord}_{\mathfrak{P}} \beta} = (N\mathfrak{P})^{-\operatorname{ord}_{\mathfrak{P}} \beta} = |\beta|_v \leqslant H_E(\beta).$$

If $\beta \neq 0$, we can apply the above inequality to $1/\beta$ and obtain, by (2.5),

$$p^{f_{\mathfrak{P}} \operatorname{ord}_{\mathfrak{P}} \beta} \leqslant H_E(\beta),$$

whence

$$\operatorname{ord}_p \beta \leqslant \frac{\log H_E(\beta)}{e_{\mathfrak{P}} f_{\mathfrak{P}} \log p} = \frac{[E:Q]}{e_{\mathfrak{P}} f_{\mathfrak{P}} \log p} h(\beta). \tag{2.6}$$

For a polynomial P denote by $L(P)$ its length, i.e., the sum of the absolute values of its coefficients.

LEMMA 2.1. *Suppose* $P(x_1, \ldots, x_m) \in \mathbf{Z}[x_1, \ldots, x_m]$ *satisfies*

$$\deg_{x_k} P \leqslant N_k \quad (\geqslant 1), \qquad 1 \leqslant k \leqslant m.$$

If $\beta_1, \ldots, \beta_m \in E$ *and* $P(\beta_1, \ldots, \beta_m) \neq 0$, *then*

$$\operatorname{ord}_p P(\beta_1, \ldots, \beta_m) \leqslant \frac{[E:Q]}{e_{\mathfrak{P}} f_{\mathfrak{P}} \log p} \{\log L(P) + \sum_{k=1}^{m} N_k h(\beta_k)\}.$$

Proof. For each valuation v of E we have

$$\max(1, |P(\beta_1, \ldots, \beta_m)|_v) \leqslant C_v \prod_{k=1}^{m} (\max(1, |\beta_k|_v))^{N_k}, \tag{2.7}$$

where $C_v = L(P)$ if v is archimedean and $C_v = 1$ otherwise. On multiplying (2.7) for all v and taking $[E:Q]$-th root we obtain

$$H(P(\beta_1, \ldots, \beta_m)) \leqslant L(P) \prod_{k=1}^{m} (H(\beta_k))^{N_k},$$

whence

$$h(P(\beta_1, \ldots, \beta_m)) \leqslant \log L(P) + \sum_{k=1}^{m} N_k h(\beta_k).$$

This together with (2.6) proves the lemma.

We will deduce a version of Siegel's lemma (Lemma 2.2 below) from the following

LEMMA (Anderson and Masser [2]). *Let E be an algebraic number field of degree D. For each valuation v of E let μ_v be an element of E and let M_v be*

a non-negative real number such that $M_v = 1$ except for finitely many v. Put $M = \prod_v M_v$. Then there are at most $(2M^{1/D}+1)^D$ elements ξ of E such that

$$|\xi - \mu_v|_v \leqslant M_v$$

for all v.

LEMMA 2.2. *Let $\beta_1, \ldots, \beta_r$ be algebraic numbers in an algebraic number field E of degree D. Suppose that*

$$P_{i,j} \in \mathbf{Z}[x_1, \ldots, x_r] \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m)\ (\textit{not all zero})$$

satisfy

$$\deg_{x_k} P_{i,j} \leqslant N_{j,k} \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m,\ 1 \leqslant k \leqslant r).$$

Write

$$X = \max_{1 \leqslant j \leqslant m} \{(\sum_{i=1}^{n} L(P_{i,j})) \exp(\sum_{k=1}^{r} N_{j,k} h(\beta_k))\}$$

and

$$\gamma_{i,j} = P_{i,j}(\beta_1, \ldots, \beta_r) \quad (1 \leqslant i \leqslant n,\ 1 \leqslant j \leqslant m).$$

If $n > mD$, then there exist rational integers $y_1, \ldots, y_n$ with

$$0 < \max_{1 \leqslant i \leqslant n} |y_i| \leqslant X^{mD/(n-mD)}$$

such that

$$\sum_{i=1}^{n} \gamma_{i,j} y_i = 0 \quad (1 \leqslant j \leqslant m).$$

Remark. This is a slight refinement of Lemma 4 in Mignotte and Waldschmidt [24].

Proof. Let

$$A = [X^{mD/(n-mD)}]. \tag{2.8}$$

For each $y = (y_1, \ldots, y_n) \in \mathbf{Z}^n$ with

$$0 \leqslant y_i \leqslant A \quad (1 \leqslant i \leqslant n)$$

we set $\lambda = (\lambda_1, \ldots, \lambda_m)$ by

$$\lambda_j = \sum_{i=1}^{n} \gamma_{i,j} y_i \in E \quad (1 \leqslant j \leqslant m). \tag{2.9}$$

Further for each j with $1 \leqslant j \leqslant m$ and each valuation v of E, let

$$\mu_{v,j} = \begin{cases} \sum_{i=1}^{n} \gamma_{i,j} \cdot \frac{1}{2} A, & \text{if } v \text{ is archimedean}, \\ 0, & \text{otherwise}, \end{cases}$$

and

$$M_{v,j} = A_{v,j} \prod_{k=1}^{r} (\max(1, |\beta_k|_v))^{N_{j,k}}$$

where

$$A_{v,j} = \begin{cases} \frac{1}{2} A \sum_{i=1}^{n} L(P_{i,j}), & \text{if } v \text{ is archimedean,} \\ 1, & \text{otherwise.} \end{cases}$$

Note that

(2.10) $$M_j = \prod_v M_{v,j} = \left\{\tfrac{1}{2} A \left(\sum_{i=1}^{n} L(P_{i,j})\right) \prod_{k=1}^{r} (H(\beta_k))^{N_{j,k}}\right\}^D \leqslant (\tfrac{1}{2} AX)^D.$$

Evidently $\mu_{v,j} \in E$ and for each j, $M_{v,j} = 1$ except for finitely many v. By (2.9), we have for archimedean v

$$|\lambda_j - \mu_{v,j}|_v = |\sum_{i=1}^{n} \gamma_{i,j}(y_i - \tfrac{1}{2}A)|_v \leqslant \tfrac{1}{2} A \sum_{i=1}^{n} |\gamma_{i,j}|_v$$

$$\leqslant \tfrac{1}{2} A \left(\sum_{i=1}^{n} L(P_{i,j})\right) \prod_{k=1}^{r} (\max(1, |\beta_k|_v))^{N_{j,k}} = M_{v,j} \quad (1 \leqslant j \leqslant m),$$

and for non-archimedean v

$$|\lambda_j - \mu_{v,j}|_v = |\sum_{i=1}^{n} \gamma_{i,j} y_i|_v \leqslant \max_{1 \leqslant i \leqslant n} |\gamma_{i,j}|_v$$

$$\leqslant \prod_{k=1}^{r} (\max(1, |\beta_k|_v))^{N_{j,k}} = M_{v,j} \quad (1 \leqslant j \leqslant m).$$

Thus all the $(A+1)^n$ $\lambda = (\lambda_1, \ldots, \lambda_m)$, which correspond by (2.9) to the $(A+1)^n$ $y = (y_1, \ldots, y_n) \in \mathbf{Z}^n$ with $0 \leqslant y_i \leqslant A$ $(1 \leqslant i \leqslant n)$, satisfy

$$|\lambda_j - \mu_{v,j}|_v \leqslant M_{v,j} \quad \text{for all } v \quad (1 \leqslant j \leqslant m).$$

On the other hand, by the Lemma of Anderson and Masser, and by (2.10), there exist at most

$$\prod_{j=1}^{m} (2M_j^{1/D} + 1)^D \leqslant (AX+1)^{mD}$$

$\xi = (\xi_1, \ldots, \xi_m) \in E^m$ satisfying

$$|\xi_j - \mu_{v,j}|_v \leqslant M_{v,j} \quad \text{for all } v \quad (1 \leqslant j \leqslant m).$$

Now (2.8) and the fact that $X \geqslant 1$ imply

$$(AX+1)^{mD} \leqslant (X(A+1))^{mD} < (A+1)^n.$$

Thus by the box-principle, there exist two distinct integral points $y' = (y'_1, \ldots, y'_n)$ and $y'' = (y''_1, \ldots, y''_n)$ with

$$0 \leqslant y'_i \leqslant A, \quad 0 \leqslant y''_i \leqslant A \quad (1 \leqslant i \leqslant n)$$

such that

$$\sum_{i=1}^{n} \gamma_{i,j} y'_i = \sum_{i=1}^{n} \gamma_{i,j} y''_i \quad (1 \leqslant j \leqslant m).$$

Hence $y = (y_1, \ldots, y_n) = (y'_1 - y''_1, \ldots, y'_n - y''_n) \in \mathbf{Z}^n$ satisfies

$$\sum_{i=1}^{n} \gamma_{i,j} y_i = 0 \quad (1 \leqslant j \leqslant m)$$

and

$$0 < \max_{1 \leqslant i \leqslant n} |y_i| \leqslant A \leqslant X^{mD/(n-mD)}.$$

This completes the proof of the lemma.

For every positive integer k, let $\nu(k)$ be the least common multiple of $1, 2, \ldots, k$. Define for $z \in \mathbf{C}$

(2.11) $\Delta(z; k) = (z+1) \ldots (z+k)/k!$ $\quad (k \in \mathbf{Z}, k \geqslant 1)$ $\quad$ and $\quad \Delta(z; 0) = 1$,

and for l, m non-negative integers

(2.12) $$\Delta(z; k, l, m) = \frac{1}{m!} \left\{\frac{d^m}{dy^m} (\Delta(y; k))^l\right\}_{y=z}.$$

LEMMA 2.3. *For any $z \in \mathbf{C}$ and any integers $k \geqslant 1$, $l \geqslant 1$, $m \geqslant 0$, we have*

(2.13) $$|\Delta(z; k, l, m)| \leqslant (2e)^{kl} \left(\frac{|z|+k}{k}\right)^{kl}.$$

Let q be a positive integer, and let x be a rational number such that qx is a positive integer. Then

(2.14) $$q^{2kl} (\nu(k))^m \Delta(x; k, l, m) \in \mathbf{Z},$$

and we have

$$\nu(k) \leqslant 3^k.$$

Finally, for any positive integers k, R and L with $k \geqslant R$, the polynomials $(\Delta(z+r; k))^l$ $(r = 0, 1, \ldots, R-1;\ l = 1, \ldots, L)$ are linearly independent.

Proof. Inequality (2.13) is a slight improvement of Lemma 2.4 of Waldschmidt [33] and is proved below. Formula (2.14) is just Lemma T 1 of Tijdeman [32]. His upper bound 4^k of $\nu(k)$ can be replaced by 3^k by using inequality (3.35) of Rosser and Schoenfeld [27] (see also Hanson [16] for a simple and alternative proof). The last assertion of Lemma 2.3 is just Lemma 4 of Cijsouw and Waldschmidt [11].

To prove (2.13), we may assume $m \leqslant kl$. Then

$$\Delta(y;k,l,m) = (\Delta(y;k))^l \sum ((y+j_1)\dots(y+j_m))^{-1}, \tag{2.15}$$

where the summation is over all selections $j_1, \dots, j_m$ of m integers from the set $1, \dots, k$ repeated l times. Hence

$$|\Delta(z;k,l,m)| \leqslant \binom{kl}{m} (\Delta(|z|;k))^l \leqslant 2^{kl} (\Delta(|z|;k))^l.$$

This together with the fact that

$$|\Delta(z;k)| \leqslant \Delta(|z|;k) \leqslant \frac{(|z|+k)^k}{k!} \leqslant \left(\frac{|z|+k}{k}\right)^k e^k \tag{2.16}$$

implies (2.13) at once.

Let B', B_n be positive real numbers, $L_1, \dots, L_n$ $(n \geqslant 2)$, T be positive integers. Put $L = \max_{1 \leqslant j \leqslant n-1} L_j$.

Lemma 2.4. *Suppose that $b_1, \dots, b_n, \lambda_1, \dots, \lambda_n, \tau_1, \dots, \tau_{n-1}$ are rational integers satisfying*

$$|b_j| \leqslant B' \quad (1 \leqslant j \leqslant n-1), \qquad |b_n| \leqslant B_n,$$
$$0 \leqslant \lambda_j \leqslant L_j \quad (1 \leqslant j \leqslant n),$$
$$\tau_j \geqslant 0 \quad (1 \leqslant j \leqslant n-1), \qquad \tau_1 + \dots + \tau_{n-1} \leqslant T.$$

Then

$$\prod_{j=1}^{n-1} |\Delta(b_n\lambda_j - b_j\lambda_n; \tau_j)| \leqslant e^T \left(1 + \frac{(n-1)(B_n L + B' L_n)}{T}\right)^T. \tag{2.17}$$

Remark. This is essentially an estimate in Loxton, Mignotte, van der Poorten and Waldschmidt [19], but we have modified their estimate

$$\prod_{j=1}^{n-1} |\Delta(b_n\lambda_j - b_j\lambda_n; \tau_j)| \leqslant \left\{2e\left(1 + (n-1)\frac{B_n L + B' L_n + 1}{T}\right)\right\}^T$$

by (2.17).

Proof. Without loss of generality, we may assume $\tau_1 > 0, \dots, \tau_{n-1} > 0$. By (2.16), we have

$$|\Delta(b_n\lambda_j - b_j\lambda_n; \tau_j)| \leqslant e^{\tau_j} \left(\frac{B_n L + B' L_n + \tau_j}{\tau_j}\right)^{\tau_j}. \tag{2.18}$$

From the convexity of the function $f(x) = x \log x$, we see that for any $a_i > 0$ and $x_i > 0$ $(i = 1, \dots, m)$

$$\sum_{i=1}^m \frac{a_i}{a_1 + \dots + a_m} \cdot \frac{x_i}{a_i} \log \frac{x_i}{a_i} \geqslant \frac{x_1 + \dots + x_m}{a_1 + \dots + a_m} \log \frac{x_1 + \dots + x_m}{a_1 + \dots + a_m},$$

whence

$$\sum_{i=1}^m x_i \log \frac{a_i}{x_i} \leqslant (x_1 + \dots + x_m) \log \frac{a_1 + \dots + a_m}{x_1 + \dots + x_m}.$$

Hence

$$\sum_{j=1}^{n-1} \tau_j \log \frac{B_n L + B' L_n + \tau_j}{\tau_j} \leqslant (\tau_1 + \dots + \tau_{n-1}) \log \left(1 + \frac{(n-1)(B_n L + B' L_n)}{\tau_1 + \dots + \tau_{n-1}}\right) \tag{2.19}$$
$$\leqslant T \log \left(1 + \frac{(n-1)(B_n L + B' L_n)}{T}\right),$$

where the last inequality follows from the fact that

$$g(x) = x \log\left(1 + \frac{a}{x}\right) \quad (a > 0)$$

increases for $x > 0$. On multiplying (2.18) for $j = 1, \dots, n-1$ and using (2.19), the lemma follows at once.

By an integral valued polynomial we mean a polynomial $f(x) \in \boldsymbol{C}[x]$ such that

$$f(m) \in \boldsymbol{Z} \quad \text{for every } m \in \boldsymbol{Z}.$$

Write $\delta f(x)$ for $f(x) - f(x-1)$. Then

$$\begin{aligned} &\delta\Delta(x;0) = 0, \\ &\delta\Delta(x;k) = \Delta(x;k-1) \quad (k \geqslant 1), \end{aligned} \tag{2.20}$$

for if $k \geqslant 2$ then

$$\delta\Delta(x;k) = \frac{(x+1)\dots(x+k)}{k!} - \frac{x\dots(x+k-1)}{k!}$$
$$= \frac{(x+1)\dots(x+k-1)(x+k-x)}{k!} = \Delta(x;k-1),$$

and $\delta\Delta(x;0) = 0$, $\delta\Delta(x;1) = \Delta(x;0)$ are obvious. Let $\boldsymbol{N} = \{m \in \boldsymbol{Z} \mid m \geqslant 0\}$.

Lemma 2.5. *Suppose $m \in \boldsymbol{N}$, $a \in \boldsymbol{C}$, $a \neq 0$. Then*

$$\det(\Delta(aj;k))_{0 \leqslant j,k \leqslant m} \neq 0.$$

Proof. The case $m = 0$ is trivial. So we may assume $m \geqslant 1$. Suppose that the determinant equals to zero, we proceed to deduce a contradiction. Thus there exist complex numbers $\lambda_0, \lambda_1, \dots, \lambda_m$, not all zero, such that

$$\sum_{k=0}^m \lambda_k \Delta(aj;k) = 0, \qquad j = 0, 1, \dots, m.$$

Hence the polynomial

$$\sum_{k=0}^{m} \lambda_k \Delta(x; k),$$

being of degree at most m, has $m+1$ zeros at aj with $j = 0, 1, \ldots, m$. So $\sum_{k=0}^{m} \lambda_k \Delta(x; k)$ is identically zero, a contradiction to the fact that $\Delta(x; 0)$, $\Delta(x; 1), \ldots, \Delta(x; m)$ are linearly independent over $\boldsymbol{C}$. This proves the lemma.

LEMMA 2.6. *Every integral valued polynomial $f(x)$ of degree $k > 0$ can be expressed as*

$$f(x) = a_k \Delta(x; k) + a_{k-1} \Delta(x; k-1) + \ldots + a_1 \Delta(x; 1) + a_0 \Delta(x; 0), \tag{2.21}$$

where $a_0, \ldots, a_k$ are rational integers.

Proof. By Lemma 2.5 with $a = 1$, there exists a unique $(k+1)$-tuple $(a_0, \ldots, a_k) \in \boldsymbol{C}^{k+1}$ such that (2.21) holds. It remains only to show that $a_0, \ldots, a_k$ are rational integers. By (2.20), (2.21) we get

$$\delta f(x) = a_k \Delta(x; k-1) + a_{k-1} \Delta(x; k-2) + \ldots + a_1.$$

Write

$$\delta^2 f(x) = \delta(\delta f(x)), \quad \ldots, \quad \delta^k f(x) = \delta(\delta^{k-1} f(x)).$$

Then

$$f(-1) = a_0, \quad (\delta f(x))_{x=-1} = a_1, \quad \ldots, \quad (\delta^k f(x))_{x=-1} = a_k.$$

Since $f(x)$ is integral valued, so are $\delta f(x), \delta^2 f(x), \ldots, \delta^k f(x)$. Hence a_0, a_1, $\ldots, a_k$ are rational integers. This completes the proof of the lemma.

LEMMA 2.7. *For every positive integer n, we have*

$$n! > \sqrt{2\pi}\, n^{n+1/2} e^{-n}.$$

Proof. By a formula for $\Gamma(x)$ at p. 253 of Whittaker and Watson [34], we have

$$n! = \Gamma(n+1) = (n+1)^{n+1/2} e^{-(n+1)} (2\pi)^{1/2} e^{\theta/(12(n+1))}, \quad \theta > 0.$$

Since $(1+(1/n))^{n+1/2} > e$ for $n = 1, 2, \ldots$, we obtain

$$n! > \sqrt{2\pi}\, n^{n+1/2} e^{-n}.$$

Chapter 3. A proposition towards the proof of Theorem 1

In this chapter we prove a proposition towards the proof of Theorem 1. The proof follows the main lines of Baker [6] and Waldschmidt [33].

We use the notation introduced for Theorem 1 and let $\varkappa$ and θ be defined as in Lemma 1.2. Put

$$G = N\mathfrak{p} - 1 = p^{f_\mathfrak{p}} - 1$$

and let $\zeta \in K_\mathfrak{p}$ be the Gth primitive root of unity fixed in Section 1.3. By the fact that $\mathrm{ord}_\mathfrak{p} \alpha_j = 0$ $(1 \leqslant j \leqslant n)$ (see (0.5)) and Lemma 1.3, there exists $(r'_1, \ldots, r'_n) \in \boldsymbol{Z}^n$ with $0 \leqslant r'_j < G$ $(1 \leqslant j \leqslant n)$ such that

$$e_\mathfrak{p} \,\mathrm{ord}_p(\alpha_j \zeta^{r'_j} - 1) \geqslant 1 \quad (1 \leqslant j \leqslant n).$$

Let $r_1, \ldots, r_n$ be the rational integers such that

$$r_j \equiv p^\varkappa r'_j (\mathrm{mod}\ G), \quad 0 \leqslant r_j < G \quad (1 \leqslant j \leqslant n).$$

Then we see, by Lemma 1.2, that

$$\mathrm{ord}_p(\alpha_j^{p^\varkappa} \zeta^{r_j} - 1) > \theta + \frac{1}{p-1} \quad (1 \leqslant j \leqslant n). \tag{3.1}$$

For later references, we give an expression (the following formula (3.3)) for

$$(\alpha_j^{p^\varkappa} \zeta^{r_j})^{1/q} = \exp\left(\frac{1}{q} \log(\alpha_j^{p^\varkappa} \zeta^{r_j})\right),$$

where the logarithmic and exponential functions are p-adic functions, which are well defined by (3.1) and the fact that $\mathrm{ord}_p q = 0$ (see (0.1)). By Section 1.1, (d), we have

$$((\alpha_j^{p^\varkappa} \zeta^{r_j})^{1/q})^q = \alpha_j^{p^\varkappa} \zeta^{r_j}. \tag{3.2}$$

On comparing (3.2) with $(\alpha_j'^{p^\varkappa} \zeta^{b r_j})^q = \alpha_j^{p^\varkappa} \zeta^{r_j}$, where $\alpha'_j \in \boldsymbol{C}_p$ is a qth root of α_j and b is defined by $bq \equiv 1$ (mod G) and $0 \leqslant b < G$, and on noting that $(p^\varkappa, q) = 1$ (see (0.1)), it is possible to choose a qth root $\alpha_j^{1/q} \in \boldsymbol{C}_p$ of α_j such that

$$(\alpha_j^{p^\varkappa} \zeta^{r_j})^{1/q} = (\alpha_j^{1/q})^{p^\varkappa} \zeta^{b r_j} \quad (1 \leqslant j \leqslant n). \tag{3.3}$$

3.1. Statement of the proposition. We define $h_j = h_j(n, q; c_0, c_2)$ $(0 \leqslant j \leqslant 7)$, $h_8 = h_8(n, q; c_0, c_2, c_3)$, $\varepsilon_j = \varepsilon_j(n, q; c_0, c_2)$ $(j = 1, 2)$ by the following 11 formulas, which will be referred as (3.4):

$$h_0 = n \log(2^{11} n q),$$

$$h_1 = 2^5 c_0 (2c_2 q)^n (q-1) \frac{n^{2n+1}}{n!} h_0,$$

$$h_2 = 2^5 c_0 (2c_2 q)^{n-1} (q-1) \frac{n^{2n-1}}{n!},$$

$$1 + \varepsilon_1 = \left(1 - \frac{1}{h_2}\right)^{-n},$$

$$
(3.4)\qquad
\begin{aligned}
h_3 &= \frac{h_1-1}{n^2},\\
1+\varepsilon_2 &= e^{1/h_3},\\
h_4 &= \frac{h_1}{h_0+1},\\
h_5 &= \frac{2^8 c_0(1+\varepsilon_1)(1+\varepsilon_2)}{\sqrt{2\pi n}\left(1-\frac{1}{32n}\right)},\\
h_6 &= \frac{2^6 h_1}{n},\\
\frac{1}{h_7} &= \frac{9\cdot 10^{-15}}{h_0 h_1}+\frac{(n+1)\log(2^6 h_0 h_1)}{2^6 h_0 h_1},\\
h_8 &= c_2 n(q-1)\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right),
\end{aligned}
$$

where $\log(2^{11}nq)$ and $\log(2^6 h_0 h_1)$ denote the usual logarithms. (In the sequel, it is easy to distinguish from the context what the symbol log (or exp) means: the usual or p-adic logarithmic (or exponential) function.)

In this chapter we suppose c_0, c_1, c_2, c_3, c_4 to be real numbers satisfying the following conditions (3.5), (3.6) and (3.7):

$$
(3.5)\qquad 2\leqslant c_0\leqslant 2^4,\quad 2\leqslant c_1\leqslant 7/2,\quad 8/3\leqslant c_2\leqslant 14,\quad 2^5\leqslant c_3\leqslant 2^8,\quad 2^5\leqslant c_4\leqslant 2^8;
$$

$$
\begin{aligned}
(3.6)\qquad &\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2\\
&\geqslant \left(\frac{1}{h_6}+\frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\\
&+\left\{1+\left(1+\frac{1}{h_0}\right)\log 3\right\}\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}\\
&+\left(1+\frac{1}{h_4}\right)\left\{4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}+\frac{1}{n}\left(1+\frac{1}{c_0-1}\right)\right.\\
&\left.+\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\right\}\frac{1}{c_4};
\end{aligned}
$$

$$
\begin{aligned}
(3.7)\qquad c_1 &\geqslant \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\\
&+\left\{2+\frac{1}{h_8}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}+\frac{n\log(h_0+1)}{h_0}\right\}\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_3}.
\end{aligned}
$$

The existence of such real numbers $c_0, \ldots, c_4$ will be proved in Chapter 5.

Put

$$
(3.8)\qquad V^*_{n-1} = \max\left(p^{f_p}, \left(2^{11} n q^{\frac{n+1}{n-1}} D^{\frac{n}{n-1}} V^+_{n-1}\right)^n\right),
$$

$$
(3.9)\qquad W^* = \max\left(W, n\log(2^{11} nqD)\right).
$$

Let U be a real number satisfying

$$
\begin{aligned}
(3.10)\qquad U &= (1+\varepsilon_1)(1+\varepsilon_2)c_0 c_1 c_2^n c_3 c_4 \frac{n^{2n+1}}{n!} q^{2n}(q-1)\\
&\times \frac{G(2+1/(p-1))^n}{e_p(f_p\log p)^{n+2}} D^{n+2} V_1\ldots V_n W^*\log V^*_{n-1}.
\end{aligned}
$$

PROPOSITION 1. *Suppose that* (0.5)–(0.8) *hold. Then*

$$
\mathrm{ord}_p(\alpha_1^{b_1}\ldots\alpha_n^{b_n}-1) < U.
$$

3.2. Notations. The following 8 formulas will be referred as (3.11):

$$
\begin{aligned}
Y &= \frac{e_p f_p\log p}{q^n D}\cdot U,\\
S &= q\left[\frac{c_3 nDW^*}{f_p\log p}\right],\\
T &= \left[\frac{Uf_p\log p}{q^n D}\cdot\frac{1}{c_1 c_3 W^*\theta}\right] = \left[\frac{Y}{c_1 c_3 W^* e_p\theta}\right],\\
L_{-1} &= [W^*],\\
L_0 &= \left[\frac{Ue_p f_p\log p}{q^n D}\cdot\frac{1}{c_1 c_4(L_{-1}+1)\log V^*_{n-1}}\right] = \left[\frac{Y}{c_1 c_4(L_{-1}+1)\log V^*_{n-1}}\right],\\
L_j &= \left[\frac{Ue_p f_p\log p}{q^n D}\cdot\frac{1}{c_1 c_2 np^\times S V_j}\right] = \left[\frac{Y}{c_1 c_2 np^\times S V_j}\right]\quad (1\leqslant j\leqslant n),\\
L &= \max_{1\leqslant j<n} L_j = L_1\quad (\text{see } (0.2)),\\
X_0 &= \left\{D\prod_{j=-1}^{n}(L_j+1)\right\}3^{T(L_{-1}+1)}\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}\\
&\times\left(1+\frac{(n-1)(B_n L_1+B' L_n)}{T}\right)^T\exp\left\{p^\times S\sum_{j=1}^{n} L_j V_j + nD\max_{1\leqslant j\leqslant n} V_j\right\}.
\end{aligned}
$$

For later convenience we proceed to prove the following inequalities (3.12)–(3.27):

$$
(3.12)\qquad (L_{-1}+1)(L_0+1)\prod_{j=1}^{n}(L_j+1-G)\geqslant c_0 G\left(1-\frac{1}{q}\right)S\binom{T+n}{n},
$$

(3.13) $$\frac{1}{n}q^{n-1}ST\theta > \left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\frac{1}{c_1}U,$$

(3.14) $$p^{\varkappa}S\sum_{j=1}^{n} L_j V_j \leqslant \frac{1}{c_1 c_2}Y,$$

(3.15) $$T(L_{-1}+1) \leqslant \left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1 c_3}Y,$$

(3.16) $$T\log\left(1+\frac{(n-1)q(B_n L_1+B' L_n)}{T}\right) \leqslant \left(2+\frac{1}{p-1}\right)\frac{1}{c_1 c_3}Y,$$

(3.17) $$(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right) \leqslant \left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_1 c_4}U,$$

(3.18) $$(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right) \leqslant \left(1+\frac{1}{h_4}\right)\frac{1}{n}\cdot\frac{1}{c_1 c_4}Y,$$

(3.19) $$(L_{-1}+1)(L_0+1)\log(qL_n) \leqslant \left(1+\frac{1}{h_4}\right)\left(2+\frac{1}{2^{11}nq}+\frac{\log h_5}{h_0}\right)\frac{1}{c_1 c_4}Y,$$

(3.20) $$nD\max_{1\leqslant j\leqslant n} V_j \leqslant \frac{1}{h_6}Y,$$

(3.21) $$\log(D(L_{-1}+1)\ldots(L_n+1)) \leqslant \frac{1}{h_7}Y,$$

(3.22) $$\frac{T\log(L_{-1}+1)}{\log p} \leqslant \frac{\log(h_0+1)}{h_0}\cdot\frac{2+1/(p-1)}{q^n}\cdot\frac{1}{c_1 c_3}U.$$

In (3.23)–(3.25), J, k are integers with $0 \leqslant J \leqslant \left[\frac{\log L_n}{\log q}\right]$, $0 \leqslant k \leqslant n-1$.

(3.23) $$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\mathrm{ord}_p b_n \leqslant \left(1+\frac{1}{h_8}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1 c_3}U,$$

(3.24) $$\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)q^{J+k}S\left(\frac{1}{p-1}+\left(1-\frac{1}{q}\right)\theta\right) \leqslant \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}U,$$

(3.25) $$\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\frac{\log(q^{J+k}S)}{\log p} \leqslant \left(1+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1 c_3}U,$$

(3.26) $$L_1+\ldots+L_{n-1} < \tfrac{1}{2}T,$$

(3.27) $$(L_{-1}+1)(L_0+1) < \tfrac{1}{4}ST.$$

Proof of (3.12). Note that

(3.28) $$\log V_{n-1}^* \geqslant \max(f_p\log p,\ h_0)$$

(see (3.8)). By (1.7), (3.28), $DV_j \geqslant f_p\log p$ $(1\leqslant j\leqslant n)$ (see (0.2)), $D \geqslant e_p$ and (3.10), we see that

(3.29) $$\frac{U e_p f_p \log p}{q^n D c_1 c_2 n p^{\varkappa} S V_j G} \geqslant h_2 \quad (1\leqslant j\leqslant n),$$

whence

(3.30) $$L_j+1-G > \frac{U e_p f_p \log p}{q^n D c_1 c_2 n p^{\varkappa} S V_j} - G \geqslant \frac{U e_p f_p \log p}{q^n D c_1 c_2 n p^{\varkappa} S V_j}\left(1-\frac{1}{h_2}\right) \quad (1\leqslant j\leqslant n).$$

By (3.30), (3.11) and $1+\varepsilon_1 = \left(1-\frac{1}{h_2}\right)^{-n}$ (see (3.4)) we get

(3.31) $$(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}(L_j+1-G) > \left(\frac{U e_p f_p \log p}{q^n D}\right)^{n+1}\frac{1}{c_1 c_4 \log V_{n-1}^*}\cdot\frac{1}{(c_1 c_2 n p^{\varkappa} S)^n V_1\ldots V_n}(1+\varepsilon_1)^{-1}.$$

Further, by (0.2), (3.28) and the fact that

$$G = p^{f_p}-1 \geqslant f_p\log p, \quad D \geqslant e_p, \quad \theta \leqslant 1$$

(see (1.7)), we obtain

(3.32) $$\frac{U f_p \log p}{q^n D c_1 c_3 W^* \theta} \geqslant h_1.$$

This and (3.11) yield

$$\frac{n^2}{T} \leqslant \frac{n^2}{h_1-1} = \frac{1}{h_3},$$

whence by $1+\varepsilon_2 = e^{1/h_3}$ (see (3.4))

$$\binom{T+n}{n} \leqslant \left(1+\frac{n}{T}\right)^n\frac{T^n}{n!} \leqslant \exp\left(\frac{n^2}{T}\right)\frac{T^n}{n!} \leqslant e^{1/h_3}\frac{T^n}{n!} = (1+\varepsilon_2)\frac{T^n}{n!}.$$

Thus

(3.33) $$c_0 G\left(1-\frac{1}{q}\right)S\binom{T+n}{n} \leqslant (1+\varepsilon_2)c_0 G\left(1-\frac{1}{q}\right)S\frac{T^n}{n!}.$$

By (3.11) we have

(3.34) $$ST \leqslant \frac{n}{q^{n-1}\theta}\cdot\frac{1}{c_1}U, \quad S \leqslant \frac{c_3 n q D W^*}{f_p \log p}.$$

In virtue of (3.31), (3.33), (3.34), to prove (3.12) it suffices to show

$$U \geqslant (1+\varepsilon_1)(1+\varepsilon_2)c_0c_1c_2^nc_3c_4\frac{n^{2n+1}}{n!}q^{2n}(q-1)\frac{G}{e_p(f_p\log p)^{n+2}}\left(\frac{p^\varkappa}{e_p\theta}\right)^n \times D^{n+2}V_1\ldots V_nW^*\log V_{n-1}^*. \tag{3.35}$$

By (1.7), (3.35) follows from (3.10) at once. This proves (3.12).

Proof of (3.13). By (3.11), (0.4) and (3.9), we have

$$S > q\left(\frac{c_3nDW^*}{f_p\log p}-1\right) \geqslant \frac{c_3nqDW^*}{f_p\log p}\left(1-\frac{1}{c_3n}\right). \tag{3.36}$$

By (3.32) we get

$$T > \frac{Uf_p\log p}{q^nDc_1c_3W^*\theta}-1 \geqslant \frac{Uf_p\log p}{q^nDc_1c_3W^*\theta}\left(1-\frac{1}{h_1}\right). \tag{3.37}$$

Now (3.36) and (3.37) imply (3.13) immediately.

(3.14) is a direct consequence of the definition of L_j $(1 \leqslant j \leqslant n)$ (see (3.11)).

Proof of (3.15). By (3.9), $W^* \geqslant h_0$. Hence we see, by (3.11) and (1.7), that

$$T(L_{-1}+1) \leqslant \frac{Y(W^*+1)}{c_1c_3W^*e_p\theta} \leqslant \left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1c_3}Y.$$

Proof of (3.16). By (3.4), (3.5), we have $h_1 > 32n$, $c_3 \geqslant 32$. Hence

$$\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right) > 1-\frac{1}{c_3n}-\frac{1}{h_1} > 1-\frac{1}{n}.$$

By (1.4),

$$\frac{e_p\theta}{p^\varkappa} < \frac{p-1}{p} < 1.$$

So by (3.11) and (3.13) we see that

$$\begin{aligned}\frac{(n-1)qL_j}{T} &\leqslant (n-1)q\frac{Ue_pf_p\log p}{q^nD}\cdot\frac{1}{c_1c_2np^\varkappa STV_j}\\ &\leqslant \frac{e_p\theta}{p^\varkappa}\cdot\frac{f_p\log p}{DV_j}\cdot\frac{n-1}{c_2n^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)}\\ &\leqslant \frac{1}{c_2n}\cdot\frac{f_p\log p}{DV_j}.\end{aligned} \tag{3.38}$$

Hence, on noting that $c_2 \geqslant 8/3$ (see (3.5)), we get

$$\begin{aligned}\log\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right) &\leqslant \log\left(1+\frac{1}{c_2n}\frac{f_p\log p}{D}\left(\frac{B_n}{V_1}+\frac{B'}{V_n}\right)\right)\\ &\leqslant \log\left(1+\frac{3}{8n}\frac{f_p\log p}{D}\left(\frac{B_n}{V_1}+\frac{B'}{V_n}\right)\right) \leqslant W \leqslant W^*.\end{aligned}$$

This together with (1.7) implies (3.16):

$$T\log\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right) \leqslant TW^* \leqslant \frac{Y}{c_1c_3e_p\theta} \leqslant \left(2+\frac{1}{p-1}\right)\frac{Y}{c_1c_3}.$$

In order to prove (3.17)–(3.19), we first establish

$$(L_{-1}+1)(L_0+1) \leqslant \left(1+\frac{1}{h_4}\right)\frac{1}{\log V_{n-1}^*}\cdot\frac{Y}{c_1c_4}. \tag{3.39}$$

By $DV_j \geqslant f_p\log p$ (see (0.2)), $W^* \geqslant h_0$ (see (3.9)) and $G = p^{f_p}-1 \geqslant f_p\log p$, we have

$$\frac{Y}{c_1c_4(L_{-1}+1)\log V_{n-1}^*} \geqslant h_4,$$

whence

$$L_0+1 \leqslant \frac{Y}{c_1c_4(L_{-1}+1)\log V_{n-1}^*}\left(1+\frac{1}{h_4}\right)$$

and (3.39) follows at once.

Proof of (3.17). By (3.28) and $e_p \leqslant D$, we have

$$\frac{Y}{\log V_{n-1}^*} = \frac{Ue_pf_p\log p}{q^nD\log V_{n-1}^*} \leqslant \frac{U}{q^n}.$$

On noting the above inequality and the fact that $\theta \leqslant 1$ (see (1.7)), (3.39) implies (3.17) immediately.

Proof of (3.18). Note that by (3.11) and (3.5)

$$\begin{aligned}2e\left(2+\frac{S}{L_{-1}+1}\right) &\leqslant 2e\left(2+\frac{S}{W^*}\right) \leqslant 2e\left(2+\frac{c_3nqD}{f_p\log p}\right)\\ &\leqslant 2e\left(2+\frac{2^8nqD}{\log 2}\right) \leqslant 2e\left(1+\frac{2^8}{\log 2}\right)nqD\\ &\leqslant 2^{11}nqD \leqslant (V_{n-1}^*)^{1/n},\end{aligned}$$

where the last inequality follows from (3.8). This and (3.39) imply (3.18).

Proof of (3.19). By (3.10), (3.11) and (3.36), we see that

$$qL_n \leqslant \frac{(1+\varepsilon_1)(1+\varepsilon_2)c_0c_4}{(1-1/(c_3n))}c_2^{n-1}\frac{n^{2n-1}}{n!}q^n(q-1)\frac{G(2+1/(p-1))^n}{p^\varkappa(f_p\log p)^n}D^nV_1\ldots$$

$$\ldots V_{n-1}\log V_{n-1}^*.$$

On noting the facts that $c_4 \leqslant 2^8$, $c_2 \leqslant 14$ (see (3.5)), $n! \geqslant \sqrt{2\pi n}\,n^n e^{-n}$ (Lemma 2.7) and $V_1 \leqslant \ldots \leqslant V_{n-1} \leqslant V_{n-1}^+$ (see (0.2)), the above inequality gives

$$qL_n \leqslant \frac{(1+\varepsilon_1)(1+\varepsilon_2)c_0 2^8}{\sqrt{2\pi n}(1-1/(c_3n))}\left(14\left(\frac{3e}{\log 2}\right)^{\frac{n}{n-1}} nq^{\frac{n+1}{n-1}}D^{\frac{n}{n-1}}V_{n-1}^+\right)^{n-1} G\log V_{n-1}^*.$$

It is easy to check that

$$14\left(\frac{3e}{\log 2}\right)^{n/(n-1)} \leqslant 14\left(\frac{3e}{\log 2}\right)^2 < 2^{11}.$$

So by the definitions of V_{n-1}^* (see (3.8)) and h_5 (see (3.4)), we get

$$qL_n \leqslant h_5(V_{n-1}^*)^{2-1/n}\log V_{n-1}^*. \tag{3.40}$$

Now we show that

$$\log V_{n-1}^* \leqslant (V_{n-1}^*)^{(1+c)/n} \quad \text{with } c = 1/(2^{11}q). \tag{3.41}$$

Put

$$g(x) = 2^{11}q^{\frac{x+1}{x-1}}D^{\frac{x}{x-1}}V_{n-1}^+ \geqslant 2^{11}q \quad \text{for } x \geqslant 2.$$

By (3.8)

$$V_{n-1}^* \geqslant (ng(n))^n > n^n. \tag{3.42}$$

Note that

$$(x^{(1+c)/n} - \log x)' > 0 \quad \text{for } x > n^n. \tag{3.43}$$

By (3.42) and (3.43) we see that, in order to prove (3.41), it suffices to show

$$\left((ng(n))^n\right)^{(1+c)/n} \geqslant \log(ng(n))^n \quad (n \geqslant 2)$$

or equivalently

$$n^c(g(n))^{1+c} \geqslant \log n + \log g(n) \quad (n \geqslant 2). \tag{3.44}$$

Now by $g(x) \geqslant 2^{11}q$ $(x \geqslant 2)$ and recalling $c = 1/(2^{11}q)$ (see (3.41)), we obtain

$$n^c(g(n))^{1+c} - \log g(n) \geqslant (n^c-1)(g(n))^{1+c}$$

$$\geqslant (n^c-1)2^{11}q \geqslant 2^{11}qc\log n = \log n.$$

This proves (3.44), whence (3.41) follows. On combining (3.39), (3.40), (3.41) and (3.28), we obtain (3.19).

Proof of (3.20). By (3.8)–(3.11), (3.5), (3.4) and $G = p^{f_p}-1 \geqslant f_p\log p$, it is readily verified that

$$Y \geqslant 2^6 h_1 D\max(h_0, \max_{1\leqslant j\leqslant n} V_j). \tag{3.45}$$

This implies (3.20).

Proof of (3.21). Since $n \geqslant 2$, $q \geqslant 3$ (see (0.1)), we have, by (3.4), (3.5),

$$h_0 \geqslant 18.83, \quad h_2 \geqslant 2^{13}, \quad h_4 \geqslant 2^{19}\times\frac{18.83}{19.83}. \tag{3.46}$$

By (3.39), (3.46), (3.5) and (3.28), we get

$$(L_{-1}+1)(L_0+1) \leqslant \frac{1}{2^6\cdot 18.83}\cdot\left(1+\frac{19.83}{2^{19}\times 18.83}\right)Y. \tag{3.47}$$

By (3.36), (0.2), (3.5), (3.9) we see that

$$(c_1c_2np^\varkappa S)^n V_1\ldots V_n \geqslant (c_1c_2n)^n\left(1-\frac{1}{c_3n}\right)^n(c_3nqW^*)^n\frac{D^nV_1\ldots V_n}{(f_p\log p)^n} \tag{3.48}$$

$$\geqslant (c_1c_2(c_3n^2-n)qW^*)^n \geqslant \left(2\cdot\frac{8}{3}\cdot(2^7-2)\cdot 3\cdot 18.83\right)^n$$

$$= (37961.28)^n.$$

Now (3.29) yields

$$L_j+1 \leqslant \frac{Y}{c_1c_2np^\varkappa SV_j}\left(1+\frac{1}{h_2}\right) \quad (1 \leqslant j \leqslant n),$$

whence on applying (3.46) and (3.48), we get

$$(L_1+1)\ldots(L_n+1) \leqslant \left(\frac{1+2^{-13}}{37961.28}\right)^n Y^n \leqslant \left(\frac{1+2^{-13}}{37961.28}\right)^2 Y^n. \tag{3.49}$$

(3.47) and (3.49) imply

$$D(L_{-1}+1)\ldots(L_n+1) \leqslant 5.76\cdot 10^{-13}Y^{n+1}D.$$

This together with (3.45) implies

$$\frac{\log(D(L_{-1}+1)\ldots(L_n+1))}{Y} \leqslant \frac{\log(5.76\cdot 10^{-13}D)}{Y} + (n+1)\frac{\log Y}{Y}$$

$$\leqslant \frac{5.76\cdot 10^{-13}D}{2^6h_0h_1D} + (n+1)\frac{\log(2^6h_0h_1)}{2^6h_0h_1} = \frac{1}{h_7}.$$

Proof of (3.22). By the facts that

$$\left(\frac{\log(x+1)}{x}\right)' < 0 \quad \text{for } x \geqslant 2$$

and $W^* \geqslant h_0$ (see (3.9)), and by (3.11), (1.7), we see that

$$\begin{aligned}\frac{T\log(L_{-1}+1)}{\log p} &\leqslant \frac{U}{q^n}\cdot\frac{f_p}{D\theta}\cdot\frac{\log(W^*+1)}{W^*}\cdot\frac{1}{c_1c_3}\\ &\leqslant \frac{U}{q^n}\cdot\frac{1}{e_p\theta}\cdot\frac{\log(h_0+1)}{h_0}\cdot\frac{1}{c_1c_3}\\ &\leqslant \frac{\log(h_0+1)}{h_0}\cdot\frac{2+1/(p-1)}{q^n}\cdot\frac{1}{c_1c_3}U.\end{aligned}$$

Proof of (3.23). We may assume $\operatorname{ord}_p b_n \neq 0$, since if $\operatorname{ord}_p b_n = 0$ (3.23) is trivial. By (0.7), we have

$$\operatorname{ord}_p b_n \leqslant \frac{\log B_0}{\log p} \leqslant \frac{W}{\log p} \leqslant \frac{W^*}{\log p}.$$

By (0.2), (3.38) (using its second line) and the fact that $p^{\varkappa}/(e_p\theta) > 1$, we see that

$$(3.50)\qquad \left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T \geqslant \left(1-\frac{1}{q}\right)\frac{1}{n}\cdot\frac{T}{L_n} \geqslant h_8.$$

So by $e_pf_p \leqslant D$, (1.7) and (3.50), we obtain

$$\begin{aligned}\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\operatorname{ord}_p b_n &\leqslant \left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\left(1+\frac{1}{h_8}\right)\frac{W^*}{\log p}\\ &\leqslant \left(1+\frac{1}{h_8}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1c_3}U.\end{aligned}$$

Proof of (3.24). By (1.3), (1.4) we have $\theta > 1/p$, whence

$$\frac{1}{p-1} < \frac{p}{p-1}\theta$$

and

$$\frac{1}{p-1}+\left(1-\frac{1}{q}\right)\theta < \left(2+\frac{1}{p-1}-\frac{1}{q}\right)\theta < \left(2+\frac{1}{p-1}\right)\theta.$$

By (3.50), (3.11) and the fact that $k \leqslant n-1$, we see that

$$\begin{aligned}&\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)q^{J+k}S\left(\frac{1}{p-1}+\left(1-\frac{1}{q}\right)\theta\right)\\ &\quad< \left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\left(1+\frac{1}{h_8}\right)q^{J+n-1}S\left(2+\frac{1}{p-1}\right)\theta\\ &\quad< \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{n}q^{n-1}ST\theta \leqslant \left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}U.\end{aligned}$$

Proof of (3.25). By (3.9), $W^* \geqslant n\log(2^{11}nqD) \geqslant h_0$. So by (3.5) we get

$$\log\left(\frac{c_3nqD}{f_p\log p}\right) \leqslant \log\left(\frac{2^8}{\log 2}nqD\right) < \log(2^{11}nqD) \leqslant \frac{1}{n}W^*$$

and

$$\frac{\log S}{W^*} \leqslant \frac{1}{W^*}\left(\log\left(\frac{c_3nqD}{f_p\log p}\right)+\log W^*\right) \leqslant \frac{1}{n}+\frac{\log h_0}{h_0}.$$

Hence, by $e_pf_p \leqslant D$ and (1.7), we obtain

$$(3.51)\qquad T\frac{\log S}{\log p} \leqslant \frac{f_p}{D\theta}\cdot\frac{1}{q^n}\cdot\frac{U}{c_1c_3}\cdot\frac{\log S}{W^*} \leqslant \left(\frac{1}{n}+\frac{\log h_0}{h_0}\right)\frac{2+1/(p-1)}{q^n}\cdot\frac{1}{c_1c_3}U.$$

Similarly, by the fact that $k \leqslant n-1$,

$$\begin{aligned}(3.52)\qquad T\frac{\log q^k}{\log p} &\leqslant \frac{(n-1)T\log q}{\log p} \leqslant \frac{(n-1)T}{\log p}\cdot\frac{1}{n}W^* \leqslant \left(1-\frac{1}{n}\right)\frac{U}{q^nc_1c_3}\cdot\frac{f_p}{D\theta}\\ &\leqslant \left(1-\frac{1}{n}\right)\frac{2+1/(p-1)}{q^n}\cdot\frac{1}{c_1c_3}U.\end{aligned}$$

On noting that

$$\frac{\log q^J}{q^J} \leqslant \frac{\log q}{q} \quad \text{for } J \geqslant 0,$$

we get (again by $e_pf_p \leqslant D$, (1.7) and (3.9))

$$(3.53)\qquad \frac{T}{\log p}\cdot\frac{\log q^J}{q^J} \leqslant \frac{Uf_p}{q^nDc_1c_3W^*\theta}\cdot\frac{\log q}{q} \leqslant \frac{1}{h_0}\cdot\frac{\log q}{q}\cdot\frac{2+1/(p-1)}{q^n}\cdot\frac{1}{c_1c_3}U.$$

It follows from (3.51)–(3.53) that

$$\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\frac{\log(q^{J+k}S)}{\log p}\leqslant\frac{1}{n}\left(T\frac{\log S}{\log p}+T\frac{\log q^k}{\log p}+\frac{T}{\log p}\cdot\frac{\log q^J}{q^J}\right)$$
$$\leqslant\left(1+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+1/(p-1)}{n q^n}\cdot\frac{1}{c_1 c_3}U.$$

Proof of (3.26). By (3.38), (0.2) and $c_2 \geqslant 8/3$ (see (3.5)), $q \geqslant 3$ (see (0.1)), we have

$$\frac{L_j}{T}\leqslant\frac{1}{c_2 n(n-1)q}\leqslant\frac{1}{8n(n-1)}\qquad(1\leqslant j\leqslant n).$$

Hence

$$\frac{L_1+\ldots+L_{n-1}}{T}\leqslant\frac{1}{8n}<\frac{1}{2}.$$

This proves (3.26).

Proof of (3.27). By (3.13) and the facts that $\theta \leqslant 1$ (see (1.7)), $c_3 \geqslant 2^5$, $h_1 > 2$ (see (3.4), (3.5)), we have

$$\frac{1}{4}ST>\frac{1}{q^{n-1}\theta}\cdot\frac{1}{4}\cdot\left(n-\frac{1}{c_3}\right)\left(1-\frac{1}{h_1}\right)\frac{1}{c_1}U$$
$$>\frac{1}{4}\cdot(2-1)\cdot\left(1-\frac{1}{2}\right)\cdot\frac{U}{q^{n-1}c_1}=\frac{U}{8q^{n-1}c_1}.$$

On the other hand, by (3.39), (3.28) and the facts that $h_4 \geqslant 1$ (see (3.46)), $c_4 \geqslant 2^5$ (see (3.5)), $q \geqslant 3$ (see (0.1)), we obtain

$$(L_{-1}+1)(L_0+1)\leqslant\frac{2}{c_1c_4}\cdot\frac{1}{\log V_{n-1}^*}\cdot\frac{U e_p f_p \log p}{q^n D}$$
$$\leqslant\frac{1}{16q}\cdot\frac{U}{q^{n-1}c_1}\leqslant\frac{U}{48q^{n-1}c_1}.$$

Now (3.27) follows from the above two inequalities.

So far we have established the inequalities (3.12)–(3.27). Now we introduce more notation. For $(J, \lambda_{-1}, \ldots, \lambda_n, \tau_0, \ldots, \tau_{n-1}) \in N^{2n+3}$ set

$$(3.54)\quad \Lambda_J(z,\tau)=\Delta(q^{-J}z+\lambda_{-1};\,L_{-1}+1,\,\lambda_0+1,\,\tau_0)\prod_{j=1}^{n-1}\Delta(b_n\lambda_j-b_j\lambda_n;\,\tau_j),$$

where $\Delta(z;k)$ and $\Delta(z;k,l,m)$ are defined by (2.11) and (2.12). In the sequel of this chapter, we abbreviate $(\lambda_{-1},\ldots,\lambda_n)$ as λ, $(\tau_0,\ldots,\tau_{n-1})$ as τ and write $|\tau|=\tau_0+\ldots+\tau_{n-1}$. Using a remark from Mignotte and Waldschmidt [24], § 4.2, we can fix a basis $\xi_1, \ldots, \xi_D$ of $K = \boldsymbol{Q}(\alpha_1, \ldots, \alpha_n)$ over $\boldsymbol{Q}$ of the shape

$$(3.55)\qquad \xi_d=\alpha_1^{k_{1d}}\ldots\alpha_n^{k_{nd}}$$

with

$$(k_{1d},\ldots,k_{nd})\in N^n\quad\text{and}\quad\sum_{j=1}^n k_{jd}\leqslant D-1\quad(1\leqslant d\leqslant D).$$

3.3. Construction of the rational integers $p_d(\lambda)$. We recall that $r_1, \ldots, r_n$ are the rational integers introduced in the beginning of this chapter, $G = p^{f_p}-1$, X_0 is defined in (3.11).

LEMMA 3.1. *For $d = 1, \ldots, D$ and $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$ in the range*

$$(3.56)\quad 0\leqslant\lambda_j\leqslant L_j\quad(-1\leqslant j\leqslant n),\qquad r_1\lambda_1+\ldots+r_n\lambda_n\equiv 0\ (\mathrm{mod}\ G)$$

there exist rational integers $p_d(\lambda)$ with

$$0<\max_{d,\lambda}|p_d(\lambda)|\leqslant X_0^{1/(c_0-1)}$$

such that

$$(3.57)\qquad \sum_{\lambda}\sum_{d=1}^{D}p_d(\lambda)\,\xi_d\Lambda_0(s,\tau)\prod_{j=1}^n(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j s}=0$$

for all $(s, \tau_0, \ldots, \tau_{n-1}) \in N^{n+1}$ satisfying

$$1\leqslant s\leqslant S,\quad (s,q)=1,\quad |\tau|\leqslant T,$$

where $\sum_\lambda$ ranges over (3.56).

Remark. In the rest of this chapter s always denotes a rational integer and τ a point $(\tau_0, \ldots, \tau_{n-1}) \in N^n$. The expression "for $(s, \tau_0, \ldots, \tau_{n-1}) \in N^{n+1}$" will be omitted.

Proof. Write

$$P_{d,\lambda;s,\tau}(x_1,\ldots,x_n)=(v(L_{-1}+1))^{\tau_0}\Lambda_0(s,\tau)\,x_1^{p^\varkappa\lambda_1 s+k_{1d}}\ldots x_n^{p^\varkappa\lambda_n s+k_{nd}}$$
$$=(v(L_{-1}+1))^{\tau_0}\Delta(s+\lambda_{-1};\,L_{-1}+1,\,\lambda_0+1,\,\tau_0)$$
$$\times\prod_{j=1}^{n-1}\Delta(b_n\lambda_j-b_j\lambda_n;\,\tau_j)\prod_{j=1}^n x_j^{p^\varkappa\lambda_j s+k_{jd}}$$

for d, λ, s, τ with $1 \leqslant d \leqslant D$, λ in the range (3.56), $1 \leqslant s \leqslant S$, $(s, q) = 1$ and $|\tau| \leqslant T$. By Lemmas 2.3 and 2.4 we see that each $P_{d,\lambda;s,\tau}$ is a monomial in $x_1, \ldots, x_n$ with rational integer coefficient, whose absolute value is at most

$$3^{(L_{-1}+1)\tau_0}e^{T-\tau_0}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T-\tau_0}\right)^{T-\tau_0}\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}$$
$$\leqslant 3^{(L_{-1}+1)T}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)^T\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}$$

Further

$$\deg_{x_j} P_{d,\lambda;s,\tau} \leqslant p^{\varkappa} S L_j + D \quad (1 \leqslant j \leqslant n).$$

On noting that

$$\zeta^{r_1\lambda_1 s + \ldots + r_n\lambda_n s} = \zeta^{(r_1\lambda_1 + \ldots + r_n\lambda_n)s} = 1$$

for $\lambda_1, \ldots, \lambda_n$ satisfying the congruence in (3.56), we see that (3.57) is equivalent to

$$(3.57)' \quad \sum_{\lambda}\sum_{d} P_{d,\lambda;s,\tau}(\alpha_1, \ldots, \alpha_n) p_d(\lambda) = 0, \quad 1 \leqslant s \leqslant S, \quad (s, q) = 1, \quad |\tau| \leqslant T.$$

In (3.57)′ there are $\left(1-\frac{1}{q}\right)S\binom{T+n}{n}$ equations and at least

$$D(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}\left[\frac{L_j+1}{G}\right]\cdot G^{n-1}\,\text{g.c.d.}(r_1, \ldots, r_n, G)$$
$$\geqslant \frac{1}{G}D(L_{-1}+1)(L_0+1)\prod_{j=1}^{n}(L_j+1-G)$$

unknowns $p_d(\lambda)$. By (3.12), we can apply Lemma 2.2 to $\alpha_1, \ldots, \alpha_n$, the field $K = Q(\alpha_1, \ldots, \alpha_n)$ and the polynomials $P_{d,\lambda;s,\tau}$. Then the lemma follows at once.

3.4. The main inductive argument. For rational integers $r^{(J)}$, $L_j^{(J)}$ $(-1 \leqslant j \leqslant n)$ and $p_d^{(J)}(\lambda) = p_d^{(J)}(\lambda_{-1}, \ldots, \lambda_n)$, which will be constructed in the following "main inductive argument", set

$$(3.58) \qquad \varphi_J(z, \tau) = \sum_{\lambda}{}^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d \Lambda_J(z, \tau) \prod_{j=1}^{n} (\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\lambda_j z},$$

where $\sum_{\lambda}^{(J)}$ is taken over the range of $\lambda = (\lambda_{-1}, \ldots, \lambda_n)$:

$$(3.59) \quad 0 \leqslant \lambda_j \leqslant L_j^{(J)} \quad (-1 \leqslant j \leqslant n), \quad r_1\lambda_1 + \ldots + r_n\lambda_n \equiv r^{(J)} (\mathrm{mod}\, G).$$

Note that, by (3.1), the p-adic functions

$$(\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\lambda_j z} = \exp\left(\lambda_j z \log(\alpha_j^{p^{\varkappa}}\zeta^{r_j})\right) \quad (1 \leqslant j \leqslant n)$$

are normal.

THE MAIN INDUCTIVE ARGUMENT. *Suppose that there are algebraic numbers $\alpha_1, \ldots, \alpha_n$ and rational integers $b_1, \ldots, b_n$ satisfying* (0.5)–(0.8), *such that*

$$(3.60) \qquad \mathrm{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) \geqslant U.$$

Then for every rational integer J with

$$0 \leqslant J \leqslant \left[\frac{\log L_n}{\log q}\right] + 1,$$

there exist rational integers $r^{(J)}, L_j^{(J)}$ $(-1 \leqslant j \leqslant n)$ with

$$0 \leqslant r^{(J)} < G, \quad \text{g.c.d.}(r_1, \ldots, r_n, G) | r^{(J)},$$

$$L_{-1}^{(J)} = L_{-1}, \quad L_0^{(J)} = L_0, \quad 0 \leqslant L_j^{(J)} \leqslant q^{-J} L_j \quad (1 \leqslant j \leqslant n),$$

and rational integers $p_d^{(J)}(\lambda)$ for $d = 1, \ldots, D$ and λ in the range (3.59), *not all zero, with absolute values not exceeding $X_0^{1/(c_0-1)}$, such that*

$$\varphi_J(s, \tau) = 0 \quad \text{for } 1 \leqslant s \leqslant q^J S,\ (s, q) = 1,\ |\tau| \leqslant q^{-J} T.$$

The main inductive argument will be proved by an induction on J. On taking $r^{(0)} = 0$, $L_j^{(0)} = L_j$ $(-1 \leqslant j \leqslant n)$, $p_d^{(0)}(\lambda) = p_d(\lambda)$, which are constructed in Lemma 3.1, we see, by Lemma 3.1, that the case $J = 0$ is true. In the rest of this section, we suppose the main inductive argument is valid for some J with $0 \leqslant J \leqslant \left[\frac{\log L_n}{\log q}\right]$, we are going to prove it for $J+1$. So we always keep the hypothesis (3.60). We first prove the following Lemmas 3.2, 3.3, 3.4, then deduce from Lemma 3.4 the main inductive argument for $J+1$.

Let

$$\gamma_j = \lambda_j - \frac{b_j}{b_n}\lambda_n \quad (1 \leqslant j \leqslant n-1)$$

and

$$p^{(J)}(\lambda) = \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d.$$

Set

$$f_J(z, \tau) = \sum_{\lambda}{}^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d \Lambda_J(z, \tau) \prod_{j=1}^{n-1} (\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\gamma_j z}$$
$$= \sum_{\lambda}{}^{(J)} p^{(J)}(\lambda) \Lambda_J(z, \tau) \prod_{j=1}^{n-1} (\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\gamma_j z}.$$

Note that, by (3.1) and (0.7), the p-adic functions

$$(\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\gamma_j p^{-\theta} z} = \exp\left(\gamma_j p^{-\theta} z \log(\alpha_j^{p^{\varkappa}}\zeta^{r_j})\right) \quad (1 \leqslant j \leqslant n-1)$$

are normal.

LEMMA 3.2. *For any τ with $|\tau| \leqslant T$ and any rational number $y > 0$ with $\mathrm{ord}_p y \geqslant 0$, we have*

$$\mathrm{ord}_p(\varphi_J(y, \tau) - f_J(y, \tau)) \geqslant U - \frac{T\log(L_{-1}+1)}{\log p} - \mathrm{ord}_p b_n.$$

Proof. We first show that

$$(3.61) \qquad b_1 r_1 + \ldots + b_n r_n \equiv 0 \,(\mathrm{mod}\, G).$$

We use the concept of congruence mod $\mathfrak{p}$ (introduced in § 1.3) on $O_{\mathfrak{p}} = \{\alpha \in K_{\mathfrak{p}} | \mathrm{ord}_{\mathfrak{p}}\alpha \geqslant 0\}$. Note that if $\alpha, \beta, \gamma, \delta$ in $O_{\mathfrak{p}}$ satisfy $\alpha \equiv \beta (\mathrm{mod}\, \mathfrak{p})$, $\gamma \equiv \delta (\mathrm{mod}\, \mathfrak{p})$, then $\alpha\gamma \equiv \beta\delta \pmod{\mathfrak{p}}$; and if $\mathrm{ord}_{\mathfrak{p}}\alpha = \mathrm{ord}_{\mathfrak{p}}\beta = 0$, $\alpha \equiv \beta \pmod{\mathfrak{p}}$, then $\alpha^{-1} \equiv \beta^{-1} \pmod{\mathfrak{p}}$. Hence from the congruences

$$\alpha_j \zeta^{r_j'} \equiv 1 \pmod{\mathfrak{p}} \qquad (1 \leqslant j \leqslant n)$$

(see the beginning of this chapter) and $\mathrm{ord}_{\mathfrak{p}}\alpha_j = 0$ $(1 \leqslant j \leqslant n)$ (see (0.5)) we get

$$\alpha_j^{b_j} \zeta^{b_j r_j'} \equiv 1 \pmod{\mathfrak{p}} \qquad (1 \leqslant j \leqslant n),$$

whence

$$\zeta^{-b_j r_j'} \equiv \alpha_j^{b_j} \pmod{\mathfrak{p}} \qquad (1 \leqslant j \leqslant n).$$

This together with (3.60) and the fact that $U \geqslant 2$ implies

$$\zeta^{-(b_1 r_1' + \ldots + b_n r_n')} \equiv \alpha_1^{b_1} \ldots \alpha_n^{b_n} \equiv 1 \pmod{\mathfrak{p}}.$$

Since $\zeta \in K_{\mathfrak{p}}$ is a primitive Gth root of unity, we obtain, by Hasse [17], p. 153, 155, 220,

$$b_1 r_1' + \ldots + b_n r_n' \equiv 0 \pmod{G}.$$

On recalling $r_j \equiv p^{\varkappa} r_j' \pmod{G}$, (3.61) follows at once.

Next we show that

$$\mathrm{ord}_p\Big(\prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{-\frac{b_j}{b_n}\lambda_n y} - 1\Big) \geqslant U - \mathrm{ord}_p b_n. \tag{3.62}$$

By (0.7), (3.1), § 1.1 (b), we see that

$$\mathrm{ord}_p\left(-\frac{b_j}{b_n}\lambda_n y \log(\alpha_j^{p^\varkappa}\zeta^{r_j})\right) \geqslant \mathrm{ord}_p \log(\alpha_j^{p^\varkappa}\zeta^{r_j})$$
$$= \mathrm{ord}_p(\alpha_j^{p^\varkappa}\zeta^{r_j} - 1) > \theta + \frac{1}{p-1}.$$

From this inequality and by § 1.1 (a), (b), (d), (3.61), (3.60) and the fact that $U \geqslant 16\, W^* \geqslant 16$, we obtain

$$\prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{-\frac{b_j}{b_n}\lambda_n y} = \prod_{j=1}^{n} \exp\left(-\frac{b_j}{b_n}\lambda_n y \log(\alpha_j^{p^\varkappa}\zeta^{r_j})\right)$$
$$= \exp\left(-\frac{\lambda_n}{b_n} y \sum_{j=1}^{n} b_j \log(\alpha_j^{p^\varkappa}\zeta^{r_j})\right)$$
$$= \exp\left(-\frac{\lambda_n}{b_n} y \sum_{j=1}^{n} \log(\alpha_j^{p^\varkappa}\zeta^{r_j})^{b_j}\right)$$
$$= \exp\left(-\frac{\lambda_n}{b_n} y \log \prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{b_j}\right)$$
$$= \exp\left(-\frac{\lambda_n}{b_n} y \log(\alpha_1^{b_1} \ldots \alpha_n^{b_n})^{p^\varkappa}\right)$$
$$= \exp\left(-\frac{\lambda_n}{b_n} y p^\varkappa \log(\alpha_1^{b_1} \ldots \alpha_n^{b_n})\right).$$

On noting that if $\mathrm{ord}_p b_n > 0$ then

$$U - \mathrm{ord}_p b_n \geqslant U - \frac{\log B_0}{\log p} \geqslant U - 2W^* \geqslant \frac{7}{8}U > \frac{1}{p-1}$$

and using (3.60), §1.1 (b), we get

$$\mathrm{ord}_p\left(-\frac{\lambda_n}{b_n} y p^\varkappa \log(\alpha_1^{b_1} \ldots \alpha_n^{b_n})\right) \geqslant \mathrm{ord}_p \log(\alpha_1^{b_1} \ldots \alpha_n^{b_n}) - \mathrm{ord}_p b_n$$
$$= \mathrm{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) - \mathrm{ord}_p b_n \geqslant U - \mathrm{ord}_p b_n$$
$$> \frac{1}{p-1}.$$

Therefore by §1.1 (a)

$$\mathrm{ord}_p\Big(\prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{-\frac{b_j}{b_n}\lambda_n y} - 1\Big) = \mathrm{ord}_p\left(\exp\left(-\frac{\lambda_n}{b_n} y p^\varkappa \log(\alpha_1^{b_1} \ldots \alpha_n^{b_n})\right) - 1\right)$$
$$= \mathrm{ord}_p\left(-\frac{\lambda_n}{b_n} y p^\varkappa \log(\alpha_1^{b_1} \ldots \alpha_n^{b_n})\right)$$
$$\geqslant U - \mathrm{ord}_p b_n.$$

This proves (3.62).

We assert that

$$\mathrm{ord}_p(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y} = 0 \qquad (1 \leqslant j \leqslant n),$$

for the inequality

$$\mathrm{ord}_p(\lambda_j y \log(\alpha_j^{p^\varkappa}\zeta^{r_j})) \geqslant \mathrm{ord}_p \log(\alpha_j^{p^\varkappa}\zeta^{r_j}) = \mathrm{ord}_p(\alpha_j^{p^\varkappa}\zeta^{r_j} - 1)$$
$$> \theta + \frac{1}{p-1}$$

implies

$$\mathrm{ord}_p((\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y} - 1) = \mathrm{ord}_p(\exp(\lambda_j y \log(\alpha_j^{p^\varkappa}\zeta^{r_j})) - 1)$$
$$= \mathrm{ord}_p(\lambda_j y \log(\alpha_j^{p^\varkappa}\zeta^{r_j})) > \theta + \frac{1}{p-1},$$

whence

$$\mathrm{ord}_p(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y} = \min\{\mathrm{ord}_p 1, \mathrm{ord}_p((\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y} - 1)\} = 0.$$

On combining the above assertion and (3.62), and noting, by §1.1 (d), that

$$\prod_{j=1}^{n-1} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j y} - \prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y} = \Big\{\prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y}\Big\}\Big(\prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{-\frac{b_j}{b_n}\lambda_n y} - 1\Big),$$

we obtain

$$\operatorname{ord}_p\Big\{\prod_{j=1}^{n-1}(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j y}-\prod_{j=1}^{n}(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y}\Big\}\geqslant U-\operatorname{ord}_p b_n. \tag{3.63}$$

Write $y=k/h$, where $h>0$, $k>0$ are coprime rational integers. Then $\operatorname{ord}_p h=0$, since $\operatorname{ord}_p y\geqslant 0$. Note also that $\operatorname{ord}_p q=0$ (see (0.1)). Now by Lemma 2.3 we have

$$(q^J h)^{2(L_{-1}+1)(L_0+1)}\big(v(L_{-1}+1)\big)^T \Lambda_J(y,\tau)\in \mathbf{Z},$$

whence

$$\operatorname{ord}_p \Lambda_J(y,\tau)\geqslant -T\operatorname{ord}_p v(L_{-1}+1)\geqslant -T\frac{\log(L_{-1}+1)}{\log p}. \tag{3.64}$$

Obviously for any d with $1\leqslant d\leqslant D$ and λ in the range (3.59), we have, by (0.5),

$$\operatorname{ord}_p(p_d^{(J)}(\lambda)\xi_d)\geqslant 0. \tag{3.65}$$

Now on noting

$$f_J(y,\tau)-\varphi_J(y,\tau)=\sum_\lambda{}^{(J)}\sum_{d=1}^{D}p_d^{(J)}(\lambda)\xi_d\Lambda_J(y,\tau)\Big(\prod_{j=1}^{n-1}(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j y}-\prod_{j=1}^{n}(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y}\Big),$$

the lemma follows from (3.63)–(3.65) immediately.

LEMMA 3.3. *For $k=0,1,\ldots,n-1$, we have*

$$\varphi_J(s,\tau)=0 \tag{3.66}$$

for $1\leqslant s\leqslant q^{J+k}S$, $(s,q)=1$, $|\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right)q^{-J}T.$

Proof. We argue by a further induction on k. By the main inductive hypothesis for J, (3.66) with $k=0$ is true. We assume (3.66) is valid for some k with $0\leqslant k\leqslant n-1$. We shall prove it for $k+1$ if $k<n-1$ and include the case $k=n-1$ for later use. Thus, we see, by Lemma 3.2, that

$$\operatorname{ord}_p f_J(s,\tau)\geqslant U-T\frac{\log(L_{-1}+1)}{\log p}-\operatorname{ord}_p b_n \tag{3.67}$$

for $1\leqslant s\leqslant q^{J+k}S$, $(s,q)=1$, $|\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right)q^{-J}T.$

Note that, by (3.1) and (0.7), the p-adic function

$$\prod_{j=1}^{n-1}(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j p^{-\theta}z}$$

is normal. Further by (2.15) and $\operatorname{ord}_p q=0$ we see that

$$p^{(L_{-1}+1)(L_0+1)\theta}\big((L_{-1}+1)!\big)^{L_0+1}\Lambda_J(p^{-\theta}z,\tau)$$

is a normal function, whence so is

$$p^{(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)}\Lambda_J(p^{-\theta}z,\tau).$$

Thus by the definition of $f_J(z,\tau)$,

$$F_J(z,\tau):=p^{(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)}f_J(p^{-\theta}z,\tau) \tag{3.68}$$

for

$$|\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$$

are normal functions. We now apply Lemma 1.4 to each function $F_J(z,\tau)$ in (3.68), taking

$$R=q^{J+k}S,\qquad M=\left[\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\right]+1. \tag{3.69}$$

Note that by (3.68)

$$\frac{1}{m!}\frac{d^m}{dz^m}F_J(sp^\theta,\tau)=p^{(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)-m\theta}\frac{1}{m!}\frac{d^m}{dz^m}f_J(s,\tau). \tag{3.70}$$

It is also easy to verify that

$$\begin{aligned}&\frac{1}{\mu_0!}\frac{d^{\mu_0}}{dz^{\mu_0}}\Delta(q^{-J}z+\lambda_{-1};\,L_{-1}+1,\,\lambda_0+1,\,\tau_0)\\&\quad=q^{-J\mu_0}\binom{\tau_0+\mu_0}{\mu_0}\Delta(q^{-J}z+\lambda_{-1};\,L_{-1}+1,\,\lambda_0+1,\,\tau_0+\mu_0).\end{aligned} \tag{3.71}$$

Further we note that for any $t,m\in\mathbf{N}$, $\Delta(x;t)x^m$ is an integral valued polynomial of degree $t+m$, whence, by Lemma 2.6, there are $a_l^{(t,m)}\in\mathbf{Z}$ $(l=0,1,\ldots,t+m)$, such that

$$\Delta(x;t)x^m=\sum_{l=0}^{t+m}a_l^{(t,m)}\Delta(x;l). \tag{3.72}$$

We abbreviate $(\mu_0,\ldots,\mu_{n-1})\in\mathbf{N}^n$ to μ and write $|\mu|$ for $\mu_0+\ldots+\mu_{n-1}$, and recall

$$p^{(J)}(\lambda)=\sum_{d=1}^{D}p_d^{(J)}(\lambda)\xi_d,\qquad \gamma_j=\lambda_j-\frac{b_j}{b_n}\lambda_n.$$

Now by (3.71), (3.72) we obtain

$$(3.73)\quad \frac{1}{m!}\frac{d^m}{dz^m}f_J(z,\tau)$$

$$= \sum_{|\mu|=m}\sum_{\lambda}{}^{(J)} p^{(J)}(\lambda)\left\{\frac{1}{\mu_0!}\frac{d^{\mu_0}}{dz^{\mu_0}}\Delta(q^{-J}z+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0)\right\}$$

$$\times\prod_{j=1}^{n-1}\Delta(b_n\gamma_j;\tau_j)\prod_{j=1}^{n-1}\frac{1}{\mu_j!}\frac{d^{\mu_j}}{dz^{\mu_j}}(\alpha_j^{p^\kappa}\zeta^{r_j})^{\gamma_j z}$$

$$= \sum_{|\mu|=m} q^{-J\mu_0}\binom{\tau_0+\mu_0}{\mu_0} b_n^{-(\mu_1+\ldots+\mu_{n-1})}\prod_{j=1}^{n-1}\frac{(\log(\alpha_j^{p^\kappa}\zeta^{r_j}))^{\mu_j}}{\mu_j!}$$

$$\times\sum_{\lambda}{}^{(J)} p^{(J)}(\lambda)\,\Delta(q^{-J}z+\lambda_{-1};L_{-1}+1,\lambda_0+1,\tau_0+\mu_0)$$

$$\times\left\{\prod_{j=1}^{n-1}(\Delta(b_n\gamma_j;\tau_j)(b_n\gamma_j)^{\mu_j})\right\}\prod_{j=1}^{n-1}(\alpha_j^{p^\kappa}\zeta^{r_j})^{\gamma_j z}$$

$$= \sum_{|\mu|=m} q^{-J\mu_0}\binom{\tau_0+\mu_0}{\mu_0} b_n^{-(m-\mu_0)}\left\{\prod_{j=1}^{n-1}\frac{(\log(\alpha_j^{p^\kappa}\zeta^{r_j}))^{\mu_j}}{\mu_j!}\right\}$$

$$\times\sum_{\sigma_1=0}^{\tau_1+\mu_1}\cdots\sum_{\sigma_{n-1}=0}^{\tau_{n-1}+\mu_{n-1}}\left\{\prod_{j=1}^{n-1}a_{\sigma_j}^{(\tau_j,\mu_j)}\right\}f_J(z,\tau_0+\mu_0,\sigma_1,\ldots,\sigma_{n-1}).$$

By (3.1) and §1.1 (b),

$$\operatorname{ord}_p\prod_{j=1}^{n-1}\frac{(\log(\alpha_j^{p^\kappa}\zeta^{r_j}))^{\mu_j}}{\mu_j!}\geqslant\sum_{j=1}^{n-1}\left\{\mu_j\left(\theta+\frac{1}{p-1}\right)-\frac{\mu_j}{p-1}\right\}\geqslant\theta(\mu_1+\ldots+\mu_{n-1})\geqslant 0.$$

For

$$|\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T,\quad |\mu|\leqslant m\leqslant M-1=\left[\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\right]$$

and

$$(\sigma_1,\ldots,\sigma_{n-1})\in N^{n-1}\quad\text{with } \sigma_j\leqslant\tau_j+\mu_j\quad(1\leqslant j\leqslant n-1),$$

we have

$$\tau_0+\mu_0+\sigma_1+\ldots+\sigma_{n-1}\leqslant\sum_{j=0}^{n-1}(\tau_j+\mu_j)=|\tau|+|\mu|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right)q^{-J}T,$$

whence by (3.67),

$$\operatorname{ord}_p f_J(s,\tau_0+\mu_0,\sigma_1,\ldots,\sigma_{n-1})\geqslant U-\frac{T\log(L_{-1}+1)}{\log p}-\operatorname{ord}_p b_n$$

for all $f_J(z,\tau_0+\mu_0,\sigma_1,\ldots,\sigma_{n-1})$ appearing in (3.73) and $1\leqslant s\leqslant q^{J+k}S$, $(s,q)=1$. On combining the above observations, (3.73) yields

$$\operatorname{ord}_p\left(\frac{1}{m!}\frac{d^m}{dz^m}f_J(s,\tau)\right)\geqslant U-\frac{T\log(L_{-1}+1)}{\log p}-\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\operatorname{ord}_p b_n$$

for

$$0\leqslant m\leqslant M-1,\quad 1\leqslant s\leqslant q^{J+k}S=R,\quad (s,q)=1,$$

$$|\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T.$$

This together with (3.70), (3.22), (3.23) implies

$$(3.74)\quad \min_{\substack{1\leqslant s\leqslant R,(s,q)=1\\0\leqslant t\leqslant M-1}}\left\{\operatorname{ord}_p\left(\frac{1}{t!}\frac{d^t}{dz^t}F_J(sp^\theta,\tau)\right)+t\theta\right\}$$

$$\geqslant U+(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)-\frac{T\log(L_{-1}+1)}{\log p}$$

$$-\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\operatorname{ord}_p b_n$$

$$\geqslant U-\left\{1+\frac{1}{h_8}+\frac{n\log(h_0+1)}{h_0}\right\}\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}U$$

for

$$|\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T,$$

where

$$R=q^{J+k}S,\quad M=\left[\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\right]+1$$

(see (3.69)).

On the other hand, by (3.24), (3.25), we see that

$$(3.75)\quad \left(1-\frac{1}{q}\right)RM\theta+M\operatorname{ord}_pR!+(M-1)\frac{\log R}{\log p}$$

$$\leqslant\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)q^{J+k}S\left(\left(1-\frac{1}{q}\right)\theta+\frac{1}{p-1}\right)+\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\frac{\log(q^{J+k}S)}{\log p}$$

$$\leqslant\left(1+\frac{1}{h_8}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}U+\left(1+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}U.$$

Now we see from (3.74), (3.75), (3.7) that each $F_J(z, \tau)$ in (3.68) satisfies the condition (1.8) with R, M given by (3.69). Thus by Lemma 1.4 and (3.68) we obtain

$$\begin{aligned}\operatorname{ord}_p f_J\left(\frac{s}{q}, \tau\right) &\geqslant \operatorname{ord}_p F_J\left(\frac{s}{q}p^{\theta}, \tau\right)-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)\\ &\geqslant \left(1-\frac{1}{q}\right)RM\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)\\ &> \left(1-\frac{1}{q}\right)^2\frac{1}{n}q^kST\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)\end{aligned}$$

for $s\in \boldsymbol{Z}$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$.

By the second inequality in (3.74), we have

$$\begin{aligned}&U-\frac{T\log(L_{-1}+1)}{\log p}-\operatorname{ord}_p b_n+(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)\\ &\qquad\geqslant U-\left\{1+\frac{1}{h_8}+\frac{n\log(h_0+1)}{h_0}\right\}\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_1c_3}U.\end{aligned}$$

The right-hand side of the above inequality is, by (3.7), at least the right-hand side of (3.75). Thus by Lemma 3.2 and the fact that $\operatorname{ord}_p q=0$, by the above observation and by (3.75), we get for $s \geqslant 1$

$$\begin{aligned}\operatorname{ord}_p\left(\varphi_J\left(\frac{s}{q}, \tau\right)-f_J\left(\frac{s}{q}, \tau\right)\right) &\geqslant U-\frac{T\log(L_{-1}+1)}{\log p}-\operatorname{ord}_p b_n\\ &> \left(1-\frac{1}{q}\right)RM\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)\\ &> \left(1-\frac{1}{q}\right)^2\frac{1}{n}q^kST\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right).\end{aligned}$$

Hence

$$\begin{aligned}(3.76)\quad \operatorname{ord}_p\varphi_J\left(\frac{s}{q}, \tau\right) &\geqslant \min\left(\operatorname{ord}_p f_J\left(\frac{s}{q}, \tau\right), \operatorname{ord}_p\left(\varphi_J\left(\frac{s}{q}, \tau\right)-f_J\left(\frac{s}{q}, \tau\right)\right)\right)\\ &> \left(1-\frac{1}{q}\right)^2\frac{1}{n}q^kST\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)\\ &> \frac{U}{c_1}q^{k+1-n}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)\right.\\ &\qquad\left.-\frac{1}{q^{k+1}}\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{c_4}\right\}\end{aligned}$$

for $s \geqslant 1$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$, where the third inequality follows from (3.13) and (3.17). From now on we assume that $0 \leqslant k \leqslant n-2$.

On the other hand, by (3.59), we see that for $1 \leqslant s \leqslant q^{J+k+1}S$, $(s, q)=1$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$,

$$\begin{aligned}&\zeta^{-r^{(J)}s}q^{J\cdot 2(L_{-1}+1)(L_0+1)}(v(L_{-1}+1))^{\tau_0}\varphi_J(s, \tau)\\ &=\sum_{\lambda}{}^{(J)}\sum_{d=1}^{D}p_d^{(J)}(\lambda)q^{2J(L_{-1}+1)(L_0+1)}(v(L_{-1}+1))^{\tau_0}\Delta(q^{-J}s+\lambda_{-1};L_{-1}+1, \lambda_0+1, \tau_0)\\ &\qquad\qquad\times\left\{\prod_{j=1}^{n-1}\Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)\right\}\prod_{j=1}^{n}\alpha_j^{p^{\varkappa}\lambda_js+k_{jd}},\end{aligned}$$

which is the value at the point $(\alpha_1, \ldots, \alpha_n)$ of a polynomial, say, $Q_{J;s,\tau}(x_1, \ldots, x_n)$ in $\boldsymbol{Z}[x_1, \ldots, x_n]$ of degree at most

$$p^{\varkappa}L_j^{(J)}q^{J+k+1}S+D \leqslant p^{\varkappa}q^{k+1}SL_j+D$$

in x_j $(1 \leqslant j \leqslant n)$. Note that by the main inductive hypothesis for J and Lemmas 2.3, 2.4, for $1 \leqslant d \leqslant D$, λ satisfying (3.59), $1 \leqslant s \leqslant q^{J+k+1}S$, $(s, q)=1$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$, we have

$$|p_d^{(J)}(\lambda)| \leqslant X_0^{1/(c_0-1)},$$

$$q^{2J(L_{-1}+1)(L_0+1)} \leqslant L_n^{2(L_{-1}+1)(L_0+1)},$$

$$\begin{aligned}|\Delta(q^{-J}s+\lambda_{-1};L_{-1}+1, \lambda_0+1, \tau_0)| &\leqslant \left(2e\left(2+\frac{q^{k+1}S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}\\ &\leqslant \left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{q^{k+1}(L_{-1}+1)(L_0+1)},\end{aligned}$$

$$\begin{aligned}&(v(L_{-1}+1))^{\tau_0}\prod_{j=1}^{n-1}|\Delta(b_n\lambda_j-b_j\lambda_n;\tau_j)|\\ &\qquad\leqslant 3^{(L_{-1}+1)\tau_0}e^{T-\tau_0}\left(1+\frac{(n-1)(B_nL^{(J)}+B'L_n^{(J)})}{q^{-J}T}\right)^T\\ &\qquad\leqslant 3^{(L_{-1}+1)T}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)^T,\end{aligned}$$

where $L^{(J)}=\max_{1\leqslant j<n}L_j^{(J)}$. So the polynomial $Q_{J;s,\tau}(x_1, \ldots, x_n)$ has its length at most

$$\begin{aligned}&\left(D\prod_{j=-1}^{n}(L_j+1)\right)\cdot X_0^{1/(c_0-1)}L_n^{2(L_{-1}+1)(L_0+1)}\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{q^{k+1}(L_{-1}+1)(L_0+1)}\\ &\qquad\times 3^{(L_{-1}+1)T}\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{\mathrm{T}}\right)^T.\end{aligned}$$

Now assume there exist s,τ with

$$1\leqslant s\leqslant q^{J+k+1}S,\quad (s,q)=1,\quad |\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$$

such that

$$\varphi_J(s,\tau)\neq 0,$$

and we proceed to deduce a contradiction. By Lemma 2.1 and the definition of X_0 (see (3.11)), and by (3.14)–(3.16), (3.18)–(3.21), the assumption $\varphi_J(s,\tau)\neq 0$ implies that

$$\begin{aligned}
&\operatorname{ord}_p\varphi_J(s,\tau)\\
&\leqslant \operatorname{ord}_p\left(\zeta^{-r(J)s}q^{2J(L_{-1}+1)(L_0+1)}\left(v(L_{-1}+1)\right)^{\tau_0}\varphi_J(s,\tau)\right)\\
&\leqslant \frac{D}{e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}\Big\{\log\Big(D\prod_{j=-1}^{n}(L_j+1)\Big)\\
&\quad+\frac{1}{c_0-1}\log X_0+\log 3\cdot T(L_{-1}+1)+T\log\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)\\
&\quad+2(L_{-1}+1)(L_0+1)\log L_n+q^{k+1}(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)\\
&\quad+p^{\varkappa}q^{k+1}S\sum_{j=1}^{n}L_jV_j+nD\max_{1\leqslant j\leqslant n}V_j\Big\}\\
&\leqslant q^{k+1-n}\frac{q^nD}{e_{\mathfrak{p}}f_{\mathfrak{p}}\log p}\Big\{\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)\Big(\log\Big(D\prod_{j=-1}^{n}(L_j+1)\Big)+nD\max_{1\leqslant j\leqslant n}V_j\Big)\\
&\quad+\left(1+\frac{1}{q(c_0-1)}\right)p^{\varkappa}S\sum_{j=1}^{n}L_jV_j+\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)\log 3\cdot T(L_{-1}+1)\\
&\quad+\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)T\log\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)\\
&\quad+\left(1+\frac{1}{q(c_0-1)}\right)(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)\\
&\quad+\frac{1}{q}\cdot 2(L_{-1}+1)(L_0+1)\log L_n\Big\}\\
&\leqslant\frac{U}{c_1}q^{k+1-n}\Big\{\left(\frac{1}{h_6}+\frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\\
&\quad+\left(1+\left(1+\frac{1}{h_0}\right)\log 3\right)\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}\\
&\quad+\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{c_0-1}\right)\frac{1}{n}\cdot\frac{1}{c_4}+\frac{1}{q}\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}\right)\frac{1}{c_4}\Big\}.
\end{aligned}$$

This together with (3.6) implies

$$\begin{aligned}
(3.77)\qquad \operatorname{ord}_p\varphi_J(s,\tau)\leqslant\frac{U}{c_1}q^{k+1-n}&\Big\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)\\
&-\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_4}\\
&-\left(1-\frac{1}{q}\right)\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}\right)\frac{1}{c_4}\Big\}.
\end{aligned}$$

On noting that

$$\begin{aligned}
&\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}+\left(1-\frac{1}{q}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}\right)\\
&\quad>\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}+4\left(1-\frac{1}{q}\right)>\left(1+\frac{1}{p-1}\right)\frac{1}{q}\geqslant\left(1+\frac{1}{p-1}\right)\frac{1}{q^{k+1}},
\end{aligned}$$

(3.77) yields

$$\begin{aligned}
&\operatorname{ord}_p\varphi_J(s,\tau)\\
&<\frac{U}{c_1}q^{k+1-n}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)-\frac{1}{q^{k+1}}\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{c_4}\right\},
\end{aligned}$$

contradicting (3.76). This contradiction proves

$$\varphi_J(s,\tau)=0$$

for

$$1\leqslant s\leqslant q^{J+k+1}S,\quad (s,q)=1,\quad |\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T.$$

Thus the proof of the lemma is complete.

LEMMA 3.4.

$$\varphi_J\left(\frac{s}{q},\tau\right)=0$$

for

$$1\leqslant s\leqslant q^{J+1}S,\quad (s,q)=1,\quad |\tau|\leqslant q^{-(J+1)}T.$$

Proof. We recall that (3.76) holds for $k=n-1$. This is

$$\text{(3.78)}\quad \operatorname{ord}_p\varphi_J\left(\frac{s}{q},\tau\right)>\left(1-\frac{1}{q}\right)^2\frac{1}{n}q^{n-1}ST\theta-(L_{-1}+1)(L_0+1)\left(\theta+\frac{1}{p-1}\right)$$
$$>\frac{U}{c_1}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)-\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_4}\right\}$$

for $s\geqslant 1$, $|\tau|\leqslant q^{-(J+1)}T$.

On the other hand, on noting that, by §1.1 (d) and (3.3), we have for $(\lambda_1,\ldots,\lambda_n)\in N^n$ satisfying

$$r_1\lambda_1+\ldots+r_n\lambda_n\equiv r^{(J)}(\operatorname{mod}G),$$

$$\text{(3.79)}\quad \prod_{j=1}^{n}(\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_js/q}=\prod_{j=1}^{n}((\alpha_j^{p^\varkappa}\zeta^{r_j})^{1/q})^{\lambda_js}$$
$$=\prod_{j=1}^{n}\{(\alpha_j^{1/q})^{p^\varkappa}\zeta^{br_j}\}^{\lambda_js}$$
$$=\{\prod_{j=1}^{n}(\alpha_j^{1/q})^{p^\varkappa\lambda_js}\}\cdot\zeta^{bs(r_1\lambda_1+\ldots+r_n\lambda_n)}$$
$$=\zeta^{bsr^{(J)}}\prod_{j=1}^{n}(\alpha_j^{1/q})^{p^\varkappa\lambda_js},$$

we see that for $1\leqslant s\leqslant q^{J+1}S$, $(s,q)=1$, $|\tau|\leqslant q^{-(J+1)}T$

$$\zeta^{-bsr^{(J)}}(q^{J+1})^{2(L_{-1}+1)(L_0+1)}(\nu(L_{-1}+1))^{\tau_0}\varphi_J(s/q,\tau)$$
$$=\sum_{\lambda}{}^{(J)}\sum_{d=1}^{D}p_d^{(J)}(\lambda)q^{2(J+1)(L_{-1}+1)(L_0+1)}\Delta(q^{-(J+1)}s+\lambda_{-1};\,L_{-1}+1,\,\lambda_0+1,\,\tau_0)$$
$$\times(\nu(L_{-1}+1))^{\tau_0}\prod_{j=1}^{n-1}\Delta(b_n\lambda_j-b_j\lambda_n;\,\tau_j)\prod_{j=1}^{n}(\alpha_j^{1/q})^{p^\varkappa\lambda_js+qk_{jd}}$$

is the value at the point $(\alpha_1^{1/q},\ldots,\alpha_n^{1/q})$ of a polynomial, say, $Q^*_{J;s,\tau}(x_1,\ldots,x_n)$ in $\mathbf{Z}[x_1,\ldots,x_n]$ of degree at most

$$p^\varkappa q^{J+1}SL_j^{(J)}+qD\leqslant p^\varkappa qSL_j+qD$$

in x_j $(1\leqslant j\leqslant n)$. By the main inductive hypothesis for J, Lemmas 2.3, 2.4 we have for $1\leqslant s\leqslant q^{J+1}S$, $(s,q)=1$, $|\tau|\leqslant q^{-(J+1)}T$, $1\leqslant d\leqslant D$, λ in the range (3.59),

$$|p_d^{(J)}(\lambda)|\leqslant X_0^{1/(c_0-1)},$$
$$q^{2(J+1)(L_{-1}+1)(L_0+1)}\leqslant(qL_n)^{2(L_{-1}+1)(L_0+1)},$$
$$|\Delta(q^{-(J+1)}s+\lambda_{-1};\,L_{-1}+1,\,\lambda_0+1,\,\tau_0)|\leqslant\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)},$$
$$(\nu(L_{-1}+1))^{\tau_0}\prod_{j=1}^{n-1}|\Delta(b_n\lambda_j-b_j\lambda_n;\,\tau_j)|$$
$$\leqslant 3^{(L_{-1}+1)\tau_0}e^{\frac{1}{q}T-\tau_0}\left(1+\frac{(n-1)(B_nL_1^{(J)}+B'L_n^{(J)})}{q^{-(J+1)}T}\right)^{q^{-(J+1)}T}$$
$$\leqslant 3^{\frac{1}{q}T(L_{-1}+1)}\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right)^{\frac{1}{q}T}.$$

So the polynomial $Q^*_{J;s,\tau}(x_1,\ldots,x_n)$ has length not exceeding

$$\{D\prod_{j=-1}^{n}(L_j+1)\}X_0^{1/(c_0-1)}3^{\frac{1}{q}T(L_{-1}+1)}\left(1+\frac{(n-1)q(B_nL_1+B'L_n)}{T}\right)^{\frac{1}{q}T}$$
$$\times\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)^{(L_{-1}+1)(L_0+1)}(qL_n)^{2(L_{-1}+1)(L_0+1)}.$$

Now we assume that there exist s, τ satisfying

$$1\leqslant s\leqslant q^{J+1}S,\quad (s,q)=1,\quad |\tau|\leqslant q^{-(J+1)}T$$

such that

$$\varphi_J(s/q,\tau)\neq 0,$$

and we proceed to deduce a contradiction. In Lemma 2.1, let $E=K(\alpha_1^{1/q},\ldots,\alpha_n^{1/q})$, $\mathfrak{P}$ be a prime ideal of O_E lying above $\mathfrak{p}$. Thus

$$[E:Q]=[E:K][K:Q]=q^nD$$

(see (0.6)) and

$$e_{\mathfrak{P}}\geqslant e_{\mathfrak{p}},\quad f_{\mathfrak{P}}\geqslant f_{\mathfrak{p}}.$$

Note that $h(\alpha_j^{1/q})=\dfrac{1}{q}h(\alpha_j)$. Then by Lemma 2.1 and the definition of X_0 (see (3.11)), and by (3.14)–(3.16), (3.18)–(3.21), (3.6), we see that

$$\varphi_J\left(\frac{s}{q},\tau\right)\neq 0\quad\text{with } 1\leqslant s\leqslant q^{J+1}S,\quad (s,q)=1,\quad |\tau|\leqslant q^{-(J+1)}T$$

implies

$$
\begin{aligned}
&\operatorname{ord}_p\varphi_J\left(\frac{s}{q}, \tau\right)\\
&\leqslant \operatorname{ord}_p\left\{\zeta^{-bsr^{(J)}} q^{2(J+1)(L_{-1}+1)(L_0+1)}\left(\nu(L_{-1}+1)\right)^{\tau_0}\varphi_J\left(\frac{s}{q}, \tau\right)\right\}\\
&\leqslant \frac{q^n D}{e_{\mathfrak{p}} f_{\mathfrak{p}} \log p}\Bigg\{\log\Big(D\prod_{j=-1}^{n}(L_j+1)\Big)+\frac{1}{c_0-1}\log X_0+p^{\varkappa} S\sum_{j=1}^{n} L_j V_j+nD\max_{1\leqslant j\leqslant n} V_j\\
&\quad+(\log 3)\frac{1}{q}T(L_{-1}+1)+\frac{1}{q}T\log\left(1+\frac{(n-1)q(B_n L_1+B' L_n)}{T}\right)\\
&\quad+(L_{-1}+1)(L_0+1)\log\left(2e\left(2+\frac{S}{L_{-1}+1}\right)\right)+2(L_{-1}+1)(L_0+1)\log(qL_n)\Bigg\}\\
&\leqslant \frac{U}{c_1}\Bigg\{\left(\frac{1}{h_6}+\frac{1}{h_7}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\\
&\quad+\left(1+\left(1+\frac{1}{h_0}\right)\log 3\right)\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}\\
&\quad+\left(1+\frac{1}{h_4}\right)\left(4+\frac{1}{2^{10}nq}+\frac{2\log h_5}{h_0}+\frac{1}{n}\left(1+\frac{1}{c_0-1}\right)\right)\frac{1}{c_4}\Bigg\}\\
&\leqslant \frac{U}{c_1}\left\{\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)-\left(1+\frac{1}{h_4}\right)\left(1+\frac{1}{p-1}\right)\frac{1}{q^n}\cdot\frac{1}{c_4}\right\},
\end{aligned}
$$

a contradiction to (3.78). This contradiction proves

$$\varphi_J\left(\frac{s}{q}, \tau\right)=0 \quad \text{for } 1\leqslant s\leqslant q^{J+1}S, \quad (s, q)=1, \quad |\tau|\leqslant q^{-(J+1)}T.$$

The proof of the lemma is thus complete.

LEMMA 3.5. *The main inductive argument is true for $J+1$.*

Proof. Similarly to (3.79) we have for $(\mu_1, \ldots, \mu_n, r)\in N^{n+1}$ satisfying $r_1\mu_1+\ldots+r_n\mu_n\equiv r \pmod G$ the equality

$$\prod_{j=1}^{n}(\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\mu_j s/q}=\zeta^{bsr}\prod_{j=1}^{n}(\alpha_j^{1/q})^{p^{\varkappa}\mu_j s}. \tag{3.80}$$

Writing

$$\mu_j=\lambda_j^*+q\lambda_j, \quad 0\leqslant\lambda_j^*<q \quad (1\leqslant j\leqslant n),$$

we see that

$$(\alpha_j^{1/q})^{p^{\varkappa}\mu_j s}=\alpha_j^{p^{\varkappa}\lambda_j s}(\alpha_j^{1/q})^{p^{\varkappa}\lambda_j^* s} \quad (1\leqslant j\leqslant n). \tag{3.81}$$

By Lemma 3.4, (3.80), (3.81), we obtain

$$
\begin{aligned}
&\sum_{\lambda_1^*=0}^{q-1}\cdots\sum_{\lambda_n^*=0}^{q-1}\prod_{j=1}^{n}(\alpha_j^{1/q})^{p^{\varkappa}\lambda_j^* s}\\
&\times\sum_{\lambda_{-1}=0}^{L_{-1}^{(J)}}\sum_{\lambda_0=0}^{L_0^{(J)}}\sum_{\lambda_1,\ldots,\lambda_n}\sum_{d=1}^{D} p_d^{(J)}(\lambda_{-1}, \lambda_0, \lambda_1^*+q\lambda_1, \ldots, \lambda_n^*+q\lambda_n)\xi_d\\
&\times\Delta(q^{-(J+1)}s+\lambda_{-1}; L_{-1}+1, \lambda_0+1, \tau_0)\\
&\times\prod_{j=1}^{n-1}\Delta\left(q(b_n\lambda_j-b_j\lambda_n)+(b_n\lambda_j^*-b_j\lambda_n^*); \tau_j\right)\prod_{j=1}^{n}\alpha_j^{p^{\varkappa}\lambda_j s}=0
\end{aligned}
\tag{3.82}
$$

for $1\leqslant s\leqslant q^{J+1}S$, $(s, q)=1$, $|\tau|\leqslant q^{-(J+1)}T$, where $\sum_{\lambda_1,\ldots,\lambda_n}$ ranges over the rational integers $\lambda_1, \ldots, \lambda_n$ satisfying

$$0\leqslant\lambda_j\leqslant L_j^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*):=\left[\frac{L_j^{(J)}-\lambda_j^*}{q}\right] \quad (1\leqslant j\leqslant n) \tag{3.83}$$

and

$$\sum_{j=1}^{n} r_j(\lambda_j^*+q\lambda_j)\equiv r^{(J)} \pmod G. \tag{3.84}$$

We emphasize that, by (0.1) $(q, G)=1$, hence (3.84) is equivalent to

$$r_1\lambda_1+\ldots+r_n\lambda_n\equiv r^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*) \pmod G, \tag{3.84$'$}$$

where $r^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*)$ is the unique solution of the congruence

$$qx\equiv r^{(J)}-(r_1\lambda_1^*+\ldots+r_n\lambda_n^*) \pmod G$$

in the range $0\leqslant x<G$. Now by the main inductive hypothesis for J, there exists a n-tuple $\lambda_1^*, \ldots, \lambda_n^*$ with $0\leqslant\lambda_j^*<q$ $(1\leqslant j\leqslant n)$, such that the rational integers

$$p_d^{(J)}(\lambda_{-1}, \lambda_0, \lambda_1^*+q\lambda_1, \ldots, \lambda_n^*+q\lambda_n)$$

for $1\leqslant d\leqslant D$, $0\leqslant\lambda_j\leqslant L_j^{(J)}$ $(j=-1, 0)$, $\lambda_1, \ldots, \lambda_n$ satisfying (3.83), (3.84)$'$, are not all zero. Fix this n-tuple $\lambda_1^*, \ldots, \lambda_n^*$, take

$$r^{(J+1)}=r^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*),$$

which is obviously divisible by g.c.d.$(r_1, \ldots, r_n, G)$, and set

$$L_j^{(J+1)}=L_j^{(J)}=L_j \quad (j=-1, 0), \qquad L_j^{(J+1)}=L_j^{(J+1)}(\lambda_1^*, \ldots, \lambda_n^*) \quad (1\leqslant j\leqslant n),$$

$$p_d^{(J+1)}(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_n)=p_d^{(J)}(\lambda_{-1}, \lambda_0, \lambda_1^*+q\lambda_1, \ldots, \lambda_n^*+q\lambda_n)$$

for

$$
\begin{aligned}
&1\leqslant d\leqslant D, \quad 0\leqslant\lambda_j\leqslant L_j^{(J+1)} \quad (-1\leqslant j\leqslant n),\\
&r_1\lambda_1+\ldots+r_n\lambda_n\equiv r^{(J+1)} \pmod G.
\end{aligned}
\tag{3.85}
$$

By the condition (0.6) and the fact that $(p^{\varkappa}s, q) = 1$, we obtain from (3.82) that

$$\sum_{\lambda}^{(J+1)} \sum_{d=1}^{D} p_d^{(J+1)}(\lambda)\xi_d \Delta(q^{-(J+1)}s+\lambda_{-1}; L_{-1}+1, \lambda_0+1, \tau_0) \times \prod_{j=1}^{n-1} \Delta(q(b_n\lambda_j-b_j\lambda_n)+(b_n\lambda_j^*-b_j\lambda_n^*); \tau_j)\cdot \prod_{j=1}^{n} \alpha_j^{p^{\varkappa}\lambda_j s} = 0 \tag{3.86}$$

for $1 \leqslant s \leqslant q^{J+1}S$, $(s, q) = 1$, $|\tau| \leqslant q^{-(J+1)}T$, where $\sum_{\lambda}^{(J+1)}$ denotes the summation over the λ's in (3.85). By Lemma 2.6 for each j with $1 \leqslant j \leqslant n-1$ and $0 \leqslant k \leqslant \tau_j$

$$\Delta(q(b_n\lambda_j-b_j\lambda_n)+(b_n\lambda_j^*-b_j\lambda_n^*); k)$$

is a linear combination of the $k+1$ numbers

$$\Delta(b_n\lambda_j-b_j\lambda_n; t), \qquad t = 0, 1, \ldots, k,$$

with coefficients independent of $\lambda_1, \ldots, \lambda_n$, where the coefficient of $\Delta(b_n\lambda_j-b_j\lambda_n; k)$ is non-zero. Hence for each j with $1 \leqslant j \leqslant n-1$, $\Delta(b_n\lambda_j-b_j\lambda_n; \tau_j)$ is a linear combination of the τ_j+1 numbers

$$\Delta(q(b_n\lambda_j-b_j\lambda_n)+(b_n\lambda_j^*-b_j\lambda_n^*); k), \qquad k = 0, 1, \ldots, \tau_j,$$

with coefficients independent of $\lambda_1, \ldots, \lambda_n$. By this observation we see that (3.86) implies

$$\zeta^{-sr^{(J+1)}}\varphi_{J+1}(s, \tau) = 0$$

for

$$1 \leqslant s \leqslant q^{J+1}S, \quad (s, q) = 1, \quad |\tau| \leqslant q^{-(J+1)}T.$$

This completes the proof of the lemma.

Thus we have established the main inductive argument for $J = 0, 1, \ldots, \ldots, \left[\dfrac{\log L_n}{\log q}\right]+1$.

We should like to make some remarks on van der Poorten [26]. Recall that

$$g_{\mathfrak{p}} = \left[\frac{1}{2}+\frac{e_{\mathfrak{p}}}{p-1}\right], \qquad G_{\mathfrak{p}} = N\mathfrak{p}^{g_{\mathfrak{p}}}\cdot(N\mathfrak{p}-1)$$

and let ζ' be a $G_{\mathfrak{p}}$-th primitive root of unity in $\boldsymbol{C}_p$. It is asserted in [26], p. 35 that for $\alpha \in K$ with $\operatorname{ord}_{\mathfrak{p}}\alpha = 0$ there is an integer r, $0 \leqslant r < G_{\mathfrak{p}}$ such that

$$\operatorname{ord}_{\mathfrak{p}}(\alpha\zeta'^r-1) \geqslant g_{\mathfrak{p}}+1. \tag{3.87}$$

Note that this is false. A simple counter-example is the following. Take $K = \boldsymbol{Q}$, $\mathfrak{p} = 3\boldsymbol{Z}$, then $e_{\mathfrak{p}} = g_{\mathfrak{p}} = 1$, $G_{\mathfrak{p}} = 6$. Let ζ' be a 6-th primitive root of unity. Take $\alpha = 2/5$, then $\operatorname{ord}_{\mathfrak{p}}\alpha = 0$ and it is readily verified that

$$\operatorname{ord}_{\mathfrak{p}}(\alpha\zeta'^r-1) \leqslant 1 < g_{\mathfrak{p}}+1 \qquad \text{for } r = 0, 1, \ldots, G_{\mathfrak{p}}-1.$$

We should also point out that the assertion (3.87) does hold for the special case where $g_{\mathfrak{p}} = 0$, by virtue of our Lemma 1.3; but even in this special case, there are still some inaccuracies in [26]. For instance, in the proof of Lemma 7 in [26], pp. 46, 47, which corresponds to our Lemma 3.5, the author of [26] does not put an additional restriction on q that

$$(q, G_{\mathfrak{p}}) = 1, \tag{3.88}$$

which seems to be essential to make his proof work. On the other hand, if one does assume (3.88), then by Hasse [17], p. 220, $K_{\mathfrak{p}}$, whence K, does not contain the qth primitive roots of unity, and we cannot understand the arguments related to Kummer theory in Section 5 of [26], pp. 49–51. The same remark extends to the proofs of Theorems 2, 3, 4 of [26].

3.5. The completion of the proof of Proposition 1. We suppose that Proposition 1 is false, that is, there exist algebraic numbers $\alpha_1, \ldots, \alpha_n$ and rational integers $b_1, \ldots, b_n$ satisfying (0.5)–(0.8) such that

$$\operatorname{ord}_{\mathfrak{p}}(\alpha_1^{b_1} \ldots \alpha_n^{b_n}-1) \geqslant U,$$

then we proceed to deduce a contradiction. By the main inductive argument for

$$J = J_0 = \left[\frac{\log L_n}{\log q}\right]+1,$$

we have

$$\varphi_{J_0}(s, \tau) = 0 \quad \text{for } 1 \leqslant s \leqslant q^{J_0}S, \quad (s, q) = 1, \quad |\tau| \leqslant q^{-J_0}T. \tag{3.89}$$

Since $0 \leqslant L_n^{(J_0)} \leqslant q^{-J_0}L_n$, we see that $L_n^{(J_0)} = 0$. Further if $\tau = (\tau_0, \ldots, \tau_{n-1})$ satisfies

$$0 \leqslant \tau_0 \leqslant \tfrac{1}{2}q^{-J_0}T, \quad 0 \leqslant \tau_j \leqslant L_j^{(J_0)} \quad (1 \leqslant j \leqslant n-1),$$

then we see, by (3.26), that

$$\begin{aligned} |\tau| &= \tau_0 + \ldots + \tau_{n-1} \leqslant \tfrac{1}{2}q^{-J_0}T+L_1^{(J_0)} + \ldots + L_{n-1}^{(J_0)} \\ &\leqslant \tfrac{1}{2}q^{-J_0}T+q^{-J_0}(L_1 + \ldots + L_{n-1}) \leqslant q^{-J_0}T. \end{aligned}$$

By these observations, (3.89) implies (writing again $p^{(J_0)}(\lambda) = \sum_{d=1}^{D} p_d^{(J_0)}(\lambda)\xi_d$)

$$\sum_{\lambda_{n-1}=0}^{L_{n-1}^{(J_0)}} \Big\{ \sum_{\lambda_{-1}=0}^{L_{-1}^{(J_0)}} \cdots \sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} p^{(J_0)}(\lambda_{-1}, \ldots, \lambda_{n-1}, 0)\Delta(q^{-J_0}s+\lambda_{-1}; L_{-1}+1, \lambda_0+1, \tau_0) \times \Big(\prod_{j=1}^{n-2} \Delta(b_n\lambda_j; \tau_j)\Big)\cdot \prod_{j=1}^{n-1} (\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\lambda_j s}\Big\} \Delta(b_n\lambda_{n-1}; \tau_{n-1}) = 0 \tag{3.90}$$

for $1 \leqslant s \leqslant q^{J_0}S$, $(s, q) = 1$, $0 \leqslant \tau_0 \leqslant \frac{1}{2}q^{-J_0}T$, $0 \leqslant \tau_j \leqslant L_j^{(J_0)}$ $(1 \leqslant j \leqslant n-1)$, where we have set

$$p^{(J_0)}(\lambda_{-1}, \ldots, \lambda_{n-1}, 0) = 0$$

for $\lambda_{-1}, \ldots, \lambda_{n-1}$ satisfying

$$0 \leqslant \lambda_j \leqslant L_j^{(J_0)} \ (-1 \leqslant j \leqslant n-1) \quad \text{and} \quad r_1\lambda_1 + \ldots + r_{n-1}\lambda_{n-1} \not\equiv r^{(J_0)} (\operatorname{mod} G).$$

By Lemma 2.5 we have

$$\det\big(\Delta(b_n\lambda_{n-1}; \tau_{n-1})\big)_{0 \leqslant \lambda_{n-1}, \tau_{n-1} \leqslant L_{n-1}^{(J_0)}} \neq 0.$$

So (3.90) implies that for each λ_{n-1} with $0 \leqslant \lambda_{n-1} \leqslant L_{n-1}^{(J_0)}$

$$\sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} \Big\{ \sum_{\lambda_{-1}=0}^{L_{-1}^{(J_0)}} \cdots \sum_{\lambda_{n-3}=0}^{L_{n-3}^{(J_0)}} p^{(J_0)}(\lambda_{-1}, \ldots, \lambda_{n-1}, 0)\Delta(q^{-J_0}s+\lambda_{-1}; L_{-1}+1, \lambda_0+1, \tau_0)$$
$$\times \Big(\prod_{j=1}^{n-3} \Delta(b_n\lambda_j; \tau_j)\Big) \prod_{j=1}^{n-2} (\alpha_j^{p^\kappa}\zeta^{r_j})^{\lambda_j s}\Big\} \Delta(b_n\lambda_{n-2}; \tau_{n-2}) = 0$$

for $1 \leqslant s \leqslant q^{J_0}S$, $(s, q) = 1$, $0 \leqslant \tau_0 \leqslant \frac{1}{2}q^{-J_0}T$, $0 \leqslant \tau_j \leqslant L_j^{(J_0)}$ $(1 \leqslant j \leqslant n-2)$. On repeating this argument $n-1$ times and noting

$$L_j^{(J_0)} = L_j \quad (j = -1, 0),$$

we obtain

$$\sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(J_0)}(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_{n-1}, 0)\Delta(q^{-J_0}s+\lambda_{-1}; L_{-1}+1, \lambda_0+1, \tau_0) = 0$$

for

$$0 \leqslant \lambda_j \leqslant L_j^{(J_0)} \quad (1 \leqslant j \leqslant n-1)$$

and

$$1 \leqslant s \leqslant q^{J_0}S, \quad (s, q) = 1, \quad 0 \leqslant \tau_0 \leqslant \tfrac{1}{2}q^{-J_0}T.$$

This implies that each polynomial

(3.91) $$Q_{\lambda_1, \ldots, \lambda_{n-1}}(x) = \sum_{\lambda_{-1}=0}^{L_{-1}} \sum_{\lambda_0=0}^{L_0} p^{(J_0)}(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_{n-1}, 0)\Delta(x+\lambda_{-1}; L_{-1}+1, \lambda_0+1, 0)$$

with $0 \leqslant \lambda_j \leqslant L_j^{(J_0)}$ $(1 \leqslant j \leqslant n-1)$ has at least

$$\left(1-\frac{1}{q}\right)q^{J_0}S([\tfrac{1}{2}q^{-J_0}T]+1) > \tfrac{1}{2}\left(1-\frac{1}{q}\right)ST > \tfrac{1}{4}ST$$

zeros. But (3.27) yields

$$\tfrac{1}{4}ST > (L_{-1}+1)(L_0+1) \geqslant \deg Q_{\lambda_1, \ldots, \lambda_{n-1}}(x).$$

So

(3.92) $$Q_{\lambda_1, \ldots, \lambda_{n-1}}(x) = 0 \quad \text{for } 0 \leqslant \lambda_j \leqslant L_j^{(J_0)} \quad (1 \leqslant j \leqslant n-1).$$

According to Lemma 2.3, the polynomials

$$\Delta(x+\lambda_{-1}; L_{-1}+1, \lambda_0+1, 0) = \big(\Delta(x+\lambda_{-1}; L_{-1}+1)\big)^{\lambda_0+1},$$
$$0 \leqslant \lambda_{-1} \leqslant L_{-1}, \quad 0 \leqslant \lambda_0 \leqslant L_0$$

are linearly independent. Thus (3.91) and (3.92) imply

$$p^{(J_0)}(\lambda_{-1}, \lambda_0, \lambda_1, \ldots, \lambda_{n-1}, 0) = 0 \quad \text{for } 0 \leqslant \lambda_j \leqslant L_j^{(J_0)} \ (-1 \leqslant j \leqslant n-1),$$

that is,

$$p_d^{(J_0)}(\lambda_{-1}, \ldots, \lambda_{n-1}, 0) = 0 \quad \text{for } 1 \leqslant d \leqslant D, 0 \leqslant \lambda_j \leqslant L_j^{(J_0)} \quad (-1 \leqslant j \leqslant n-1),$$

contradicting the construction in the main inductive argument. This contradiction proves Proposition 1.

Chapter 4. A proposition towards the proof of Theorem 2

In this chapter we prove a proposition towards the proof of Theorem 2. The proof goes along the same line as in Chapter 3. Since we do not introduce the polynomials $\Delta(x; k, l, m)$ in our auxiliary functions, we have some simplification. We use the notations introduced for Theorem 2 and those introduced at the beginning of Chapter 3.

4.1. Statement of the proposition. We define

$$h_j = h_j(n, q; c_0, c_2) \quad (0 \leqslant j \leqslant 5), \quad h_6 = h_6(n, q; c_0, c_2, c_3),$$
$$\varepsilon_j = \varepsilon_j(n, q; c_0, c_2) \quad (j = 1, 2)$$

by the following 9 formulas, which will be referred as (4.1):

$$h_0 = n\log(2^{11}nq),$$

$$h_1 = 16c_0(2c_2q)^n(q-1)\frac{n^{2n+2}}{n!}h_0,$$

$$h_2 = 16c_0(2c_2q)^{n-1}(q-1)\frac{n^{2n}}{n!}, \quad 1+\varepsilon_1 = \left(1-\frac{1}{h_2}\right)^{-n},$$

(4.1) $$h_3 = \frac{h_1-1}{(n-1)^2}, \quad 1+\varepsilon_2 = e^{h_3^{-1}},$$

$$h_4 = \frac{2^5h_1}{n},$$

$$h_5^{-1} = \frac{1.02\times 10^{-10}}{h_0h_1} + \frac{n\log(2^5h_0h_1)}{2^5h_0h_1},$$

$$h_6 = c_2n(q-1)\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right).$$

In this chapter we suppose c_0, c_1, c_2, c_3 to be real numbers satisfying the following conditions (4.2), (4.3), (4.4):

$$2 \leqslant c_0 \leqslant 2^4, \quad 2 \leqslant c_1 \leqslant 7/2, \quad c_2 \geqslant 5/2, \quad 2^4 \leqslant c_3 \leqslant 2^8; \tag{4.2}$$

$$\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2 \geqslant \left(\frac{1}{h_4}+\frac{1}{h_5}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2} + \left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}; \tag{4.3}$$

$$c_1 \geqslant \left(1+\frac{1}{h_6}\right)\left(2+\frac{1}{p-1}\right)+\left\{2+\frac{1}{h_6}+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right\}\cdot\frac{2+\frac{1}{p-1}}{nq^n}\cdot\frac{1}{c_3}. \tag{4.4}$$

The existence of such real numbers c_0, c_1, c_2, c_3 will be proved in Chapter 5. Let

$$W^* = \max(W, n\log(2^{11} nqD)), \tag{4.5}$$

where W is a real number satisfying (0.9), and let U be a real number satisfying

$$U = (1+\varepsilon_1)(1+\varepsilon_2)c_0 c_1 c_2^n c_3^2 \frac{n^{2n+2}}{n!} q^{2n}(q-1) \times \frac{G(2+1/(p-1))^n}{(f_p \log p)^{n+2}} D^{n+2} V_1 \ldots V_n (W^*)^2. \tag{4.6}$$

Proposition 2. *Suppose that* (0.5)–(0.8) *hold. Then*

$$\operatorname{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) < U.$$

4.2. Notations. The following 6 formulas will be referred as (4.7).

$$\begin{aligned} Y &= \frac{e_p f_p \log p}{q^n D} U, \\ S &= q\left[\frac{c_3 nDW^*}{f_p \log p}\right], \\ T &= \left[\frac{U f_p \log p}{q^n D}\cdot\frac{1}{c_1 c_3 W^* \theta}\right] = \left[\frac{Y}{c_1 c_3 W^* e_p \theta}\right], \\ L_j &= \left[\frac{U e_p f_p \log p}{q^n D}\cdot\frac{1}{c_1 c_2 np^\varkappa S V_j}\right] = \left[\frac{Y}{c_1 c_2 np^\varkappa S V_j}\right] \quad (1 \leqslant j \leqslant n), \\ L &= \max_{1 \leqslant j < n} L_j = L_1 \ (\text{see } (0.2)), \end{aligned} \tag{4.7}$$

$$X_0 = \Big(D\prod_{j=1}^n (L_j+1)\Big) e^T \left(1+\frac{(n-1)(B_n L_1 + B' L_n)}{T}\right)^T \times \exp\Big(p^\varkappa S \sum_{j=1}^n L_j V_j + nD \max_{1 \leqslant j \leqslant n} V_j\Big).$$

The following 11 inequalities (4.8)–(4.18), which can be established in almost the same way as in § 3.2, will be required later. We give only the proofs of (4.11) and (4.14), and omit the proof of the rest.

$$(L_1+1-G)\ldots(L_n+1-G) \geqslant c_0 G\left(1-\frac{1}{q}\right) S\binom{T+n-1}{n-1}, \tag{4.8}$$

$$\frac{1}{n} q^{n-1} ST\theta > \left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\frac{1}{c_1} U, \tag{4.9}$$

$$p^\varkappa S \sum_{j=1}^n L_j V_j \leqslant \frac{1}{c_1 c_2} Y, \tag{4.10}$$

$$T \leqslant \frac{1}{h_0}\left(2+\frac{1}{p-1}\right)\frac{1}{c_1 c_3} Y, \tag{4.11}$$

$$T \log\left(1+\frac{(n-1)q(B_n L_1 + B' L_n)}{T}\right) \leqslant \left(2+\frac{1}{p-1}\right)\frac{1}{c_1 c_3} Y, \tag{4.12}$$

$$nD \max_{1 \leqslant j \leqslant n} V_j \leqslant \frac{1}{h_4} Y, \tag{4.13}$$

$$\log(D(L_1+1)\ldots(L_n+1)) \leqslant \frac{1}{h_5} Y, \tag{4.14}$$

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n} q^{-J} T + 1\right)\operatorname{ord}_p b_n \leqslant \left(1+\frac{1}{h_6}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1 c_3} U, \tag{4.15}$$

$$\left(\left(1-\frac{1}{q}\right)\frac{1}{n} q^{-J} T + 1\right) q^{J+k} S\left(\frac{1}{p-1}+\left(1-\frac{1}{q}\right)\theta\right) \leqslant \left(1+\frac{1}{h_6}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1} U, \tag{4.16}$$

$$\left(1-\frac{1}{q}\right)\frac{1}{n} q^{-J} T \frac{\log(q^{J+k} S)}{\log p} \leqslant \left(1+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1 c_3} U. \tag{4.17}$$

(In (4.15)–(4.17), J, k are integers with $0 \leqslant J \leqslant \frac{\log L_n}{\log q}$, $0 \leqslant k \leqslant n-1$.)

$$L_1 + \ldots + L_{n-1} \leqslant T. \tag{4.18}$$

Proof of (4.11). By (4.5), $W^* \geqslant n\log(2^{11} nqD) \geqslant h_0$. Hence the definition of T in (4.7) and (1.7) imply

$$T \leqslant \frac{Y}{c_1 c_3 W^* e_p \theta} \leqslant \frac{1}{h_0}\left(2+\frac{1}{p-1}\right)\frac{1}{c_1 c_3} Y.$$

Proof of (4.14). By (0.1), (0.2), (0.9), (4.1), (4.2), (4.5)–(4.7), we have

$$q \geqslant 3, \quad W^* \geqslant h_0 \geqslant 2\log(2^{11}\cdot 2\cdot 3) \geqslant 18.832, \quad h_2 \geqslant 2^9\cdot 15,$$

$$Y \geqslant 2^5 h_0 h_1 D,$$

$$\frac{Y}{c_1 c_2 np^{\varkappa} SV_j} \geqslant h_0 h_2,$$

$$c_1 c_2 np^{\varkappa} SV_j \geqslant c_1 c_2 n(c_3 n-1) qW^* \geqslant 17513.76.$$

Thus we see that

$$\begin{aligned}\prod_{j=1}^{n}(L_j+1) &\leqslant \prod_{j=1}^{n}\left(\frac{Y}{c_1 c_2 np^{\varkappa} SV_j}+1\right)\\ &\leqslant \prod_{j=1}^{n}\left\{\frac{Y}{c_1 c_2 np^{\varkappa} SV_j}\left(1+\frac{1}{h_0 h_2}\right)\right\}\\ &\leqslant Y^n\left(\frac{1+6.9143\cdot 10^{-6}}{17513.76}\right)^2\\ &\leqslant 3.2603\cdot 10^{-9}\, Y^n.\end{aligned}$$

So

$$\begin{aligned}\frac{\log\left(D\prod_{j=1}^{n}(L_j+1)\right)}{Y} &\leqslant \frac{1}{Y}(\log(3.2603\cdot 10^{-9} D)+n\log Y)\\ &\leqslant \frac{1.02\cdot 10^{-10}}{h_0 h_1}+\frac{n\log(2^5 h_0 h_1)}{2^5 h_0 h_1} = h_5^{-1}.\end{aligned}$$

This proves (4.14).

In the sequel we abbreviate $(\lambda_1, \ldots, \lambda_n) \in N^n$ as λ, $(\tau_1, \ldots, \tau_{n-1}) \in N^{n-1}$ as τ, and write

$$|\tau| = \tau_1 + \ldots + \tau_{n-1},$$

$$\Lambda(\tau) = \prod_{j=1}^{n-1} \Delta(b_n\lambda_j - b_j\lambda_n;\ \tau_j).$$

We also use the basis $\xi_1, \ldots, \xi_D$ of $K = Q(\alpha_1, \ldots, \alpha_n)$ over Q of the shape (3.55).

4.3. Construction of the rational integers $p_d(\lambda)$.

LEMMA 4.1. *For $d = 1, \ldots, D$ and $\lambda = (\lambda_1, \ldots, \lambda_n)$ satisfying*

(4.19) $$0 \leqslant \lambda_j \leqslant L_j \quad (1 \leqslant j \leqslant n), \quad r_1\lambda_1 + \ldots + r_n\lambda_n \equiv 0 \pmod G$$

there exist rational integers $p_d(\lambda)$ with

$$0 < \max_{d,\lambda}|p_d(\lambda)| \leqslant X_0^{1/(c_0-1)}$$

such that

$$\sum_{\lambda}\sum_{d=1}^{D} p_d(\lambda)\xi_d \Lambda(\tau)\prod_{j=1}^{n}(\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\lambda_j s} = 0$$

for $1 \leqslant s \leqslant S$, $(s, q) = 1$, $|\tau| \leqslant T$, where $\sum_{\lambda}$ denotes the summation over the range (4.19).

Proof. Similar to the proof of Lemma 3.1.

4.4. The main inductive argument. For rational integers $r^{(J)}$, $L_j^{(J)}$ $(1 \leqslant j \leqslant n)$ and $p_d^{(J)}(\lambda) = p_d^{(J)}(\lambda_1, \ldots, \lambda_n)$, which will be constructed in the following "main inductive argument", we set

(4.20) $$\varphi_J(z, \tau) = \sum_{\lambda}^{(J)}\sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d \Lambda(\tau)\prod_{j=1}^{n}(\alpha_j^{p^{\varkappa}}\zeta^{r_j})^{\lambda_j z},$$

where $\sum_{\lambda}^{(J)}$ denotes the summation over the range of $\lambda = (\lambda_1, \ldots, \lambda_n)$:

(4.21) $$0 \leqslant \lambda_j \leqslant L_j^{(J)} \quad (1 \leqslant j \leqslant n), \quad r_1\lambda_1 + \ldots + r_n\lambda_n \equiv r^{(J)} \pmod G.$$

THE MAIN INDUCTIVE ARGUMENT. *Suppose that there are algebraic numbers $\alpha_1, \ldots, \alpha_n$ and rational integers $b_1, \ldots, b_n$, satisfying (0.5)–(0.8), such that*

(4.22) $$\mathrm{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) \geqslant U.$$

Then for every rational integer J with

$$0 \leqslant J \leqslant \left[\frac{\log L_n}{\log q}\right]+1$$

there exist rational integers $r^{(J)}$, $L_j^{(J)}$ $(1 \leqslant j \leqslant n)$ with

$$0 \leqslant r^{(J)} < G, \quad \text{g.c.d.}(r_1, \ldots, r_n, G) | r^{(J)},$$

$$0 \leqslant L_j^{(J)} \leqslant q^{-J} L_j \quad (1 \leqslant j \leqslant n),$$

and rational integers $p_d^{(J)}(\lambda)$ for $d = 1, \ldots, D$ and λ satisfying (4.21), not all zero, with absolute values not exceeding $X_0^{1/(c_0-1)}$, such that

$$\varphi_J(s, \tau) = 0 \quad \text{for } 1 \leqslant s \leqslant q^J S, \quad (s, q) = 1, \quad |\tau| \leqslant q^{-J} T.$$

The proof of the main inductive argument is similar to that in § 3.4. So we only give a detailed sketch. We prove it by an induction on J. On taking $r^{(0)}=0$, $L_j^{(0)}=L_j$ $(1 \leqslant j \leqslant n)$, $p_d^{(0)}(\lambda)=p_d(\lambda)(1 \leqslant d \leqslant D$, λ satisfying (4.21)), we see, by Lemma 4.1, that the case $J=0$ is true. In the remaining part of this section, we assume the main inductive argument is valid for some J with

$$0 \leqslant J \leqslant \left[\frac{\log L_n}{\log q}\right],$$

and we shall prove it for $J+1$. So we always keep the hypothesis (4.22). We first show the following Lemmas 4.2, 4.3, 4.4, then deduce the main inductive argument for $J+1$.

Let

$$\gamma_j = \lambda_j - \frac{b_j}{b_n}\lambda_n \quad (1 \leqslant j \leqslant n-1)$$

and put

$$f_J(z, \tau) = \sum_\lambda^{(J)} \sum_{d=1}^{D} p_d^{(J)}(\lambda)\,\xi_d\,\Lambda(\tau) \prod_{j=1}^{n-1} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j z}.$$

We write $p^{(J)}(\lambda)$ for $\sum_{d=1}^{D} p_d^{(J)}(\lambda)\,\xi_d$.

LEMMA 4.2. *For any $\tau=(\tau_1, \ldots, \tau_{n-1})$ with $|\tau| \leqslant T$ and any $y \in \mathbf{Q}$, $y>0$, with $\operatorname{ord}_p y \geqslant 0$, we have*

$$\operatorname{ord}_p(\varphi_J(y, \tau)-f_J(y, \tau)) \geqslant U-\operatorname{ord}_p b_n.$$

Proof. By the definitions of $\varphi_J(z, \tau)$ and $f_J(z, \tau)$, we have

$$\varphi_J(y, \tau)-f_J(y, \tau) = \sum_\lambda^{(J)} p^{(J)}(\lambda)\,\Lambda(\tau)\Big\{\prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y} - \prod_{j=1}^{n-1} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j y}\Big\}.$$

It is easy to see $\operatorname{ord}_p \Lambda(\tau) \geqslant 0$ (since $\Lambda(\tau) \in \mathbf{Z}$) and $\operatorname{ord}_p p^{(J)}(\lambda) \geqslant 0$ by (0.5). Similarly to the proof of (3.63), we can readily show that

$$\operatorname{ord}_p\Big\{\prod_{j=1}^{n} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\lambda_j y} - \prod_{j=1}^{n-1} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j y}\Big\} \geqslant U-\operatorname{ord}_p b_n.$$

Now the lemma follows from the above observations at once.

LEMMA 4.3. *For $k=0, 1, \ldots, n-1$, we have*

$$\varphi_J(s, \tau)=0 \tag{4.23}$$

for $1 \leqslant s \leqslant q^{J+k}S$, $(s, q)=1$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right)q^{-J}T$.

Proof. We argue by an induction on k. By the main inductive hypothesis for J, (4.23) with $k=0$ is true. Assuming (4.23) is valid for some k with $0 \leqslant k \leqslant n-1$, we shall prove it for $k+1$ if $k<n-1$ and include the case $k=n-1$ for later use. Thus, we see, by Lemma 4.2, that

$$\operatorname{ord}_p f_J(s, \tau) \geqslant U-\operatorname{ord}_p b_n \tag{4.24}$$

for $1 \leqslant s \leqslant q^{J+k}S$, $(s, q)=1$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k}{n}\right)q^{-J}T$. By (0.7), (3.1) and the remark below the proof of Lemma 1.1,

$$\prod_{j=1}^{n-1} (\alpha_j^{p^\varkappa}\zeta^{r_j})^{\gamma_j p^{-\theta} z}$$

is a p-adic normal function, whence so are

$$F_J(z, \tau)=f_J(p^{-\theta}z, \tau) \quad \text{for } |\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T. \tag{4.25}$$

We now apply Lemma 1.4 to each $F_J(z, \tau)$ in (4.25), taking

$$R=q^{J+k}S, \quad M=\left[\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T\right]+1. \tag{4.26}$$

Similarly to the proof of Lemma 3.3, we see, by (4.24), (4.25) and (4.15), that

$$\min_{\substack{1 \leqslant s \leqslant R, (s,q)=1\\ 0 \leqslant t \leqslant M-1}} \left\{\operatorname{ord}_p\left(\frac{1}{t!}\frac{d^t}{dz^t}F_J(sp^\theta, \tau)\right)+t\theta\right\} \tag{4.27}$$

$$\geqslant U-\left(\left(1-\frac{1}{q}\right)\frac{1}{n}q^{-J}T+1\right)\operatorname{ord}_p b_n \geqslant U-\left(1+\frac{1}{h_6}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1c_3}U$$

for $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$, where R, M are given by (4.26). On the other hand, by (4.16) and (4.17)

$$\left(1-\frac{1}{q}\right)RM\theta+M\operatorname{ord}_p R!+(M-1)\frac{\log R}{\log p} \tag{4.28}$$

$$\leqslant \left(1+\frac{1}{h_6}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_1}U+\left(1+\frac{\log h_0}{h_0}+\frac{1}{h_0}\cdot\frac{\log q}{q}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1c_3}U.$$

By (4.27), (4.28), (4.4), we see that each $F_J(z, \tau)$ in (4.25) satisfies the condition (1.8) of Lemma 1.4 with R, M given by (4.26). So Lemma 1.4 and (4.25) imply

$$\operatorname{ord}_p f_J\left(\frac{s}{q}, \tau\right) \geqslant \left(1-\frac{1}{q}\right)RM\theta > \left(1-\frac{1}{q}\right)^2\frac{1}{n}q^k ST\theta$$

for $s \in \mathbf{Z}$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$.

By the second inequality of (4.27), we have

$$U-\operatorname{ord}_p b_n \geqslant U-\left(1+\frac{1}{h_6}\right)\frac{2+1/(p-1)}{nq^n}\cdot\frac{1}{c_1c_3}U.$$

The right-hand side of the above inequality is, by (4.4), at least the right-hand side of (4.28). Thus by Lemma 4.2 and the fact that $\operatorname{ord}_p q = 0$, by the above observation and by (4.28), we get for $s \geqslant 1$

$$\operatorname{ord}_p\left(\varphi_J\left(\frac{s}{q},\tau\right)-f_J\left(\frac{s}{q},\tau\right)\right)\geqslant U-\operatorname{ord}_p b_n>\left(1-\frac{1}{q}\right)RM\theta>\left(1-\frac{1}{q}\right)^2\frac{1}{n}q^k ST\theta.$$

Hence

$$(4.29)\quad \operatorname{ord}_p\varphi_J\left(\frac{s}{q},\tau\right)\geqslant\min\left(\operatorname{ord}_p f_J\left(\frac{s}{q},\tau\right),\ \operatorname{ord}_p\left(\varphi_J\left(\frac{s}{q},\tau\right)-f_J\left(\frac{s}{q},\tau\right)\right)\right)$$
$$>\left(1-\frac{1}{q}\right)^2\frac{1}{n}q^k ST\theta$$
$$>\frac{U}{c_1}q^{k+1-n}\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)$$

for $s \geqslant 1$, $|\tau| \leqslant \left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$, where the last inequality follows from (4.9). From now on we assume that $0 \leqslant k \leqslant n-2$.

Now assuming there exist s, τ with

$$1\leqslant s\leqslant q^{J+k+1}S,\quad (s,q)=1,\quad |\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T$$

such that

$$\varphi_J(s,\tau)\neq 0,$$

we proceed to deduce a contradiction. On applying Lemma 2.1, in a way similar to that in the proof of Lemma 3.3, to the polynomial in $\mathbf{Z}[x_1,\ldots,x_n]$, whose value at the point $(\alpha_1,\ldots,\alpha_n)$ is $\zeta^{-r(J)s}\varphi_J(s,\tau)$, we see, by (4.10)–(4.14), $|p_d^{(J)}(\lambda)| \leqslant X_0^{1/(c_0-1)}$ (from the main inductive argument for J) and the definition of X_0 in (4.7), that

$$\operatorname{ord}_p\varphi_J(s,\tau)=\operatorname{ord}_p(\zeta^{-r(J)s}\varphi_J(s,\tau))$$
$$\leqslant\frac{D}{e_pf_p\log p}\Big\{\log(D(L_1+1)\ldots(L_n+1))+\frac{1}{c_0-1}\log X_0+T$$
$$+T\log\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)$$
$$+p^{\varkappa}q^{k+1}S\sum_{j=1}^{n}L_jV_j+nD\max_{1\leqslant j\leqslant n}V_j\Big\}$$
$$\leqslant q^{k+1-n}\frac{q^nD}{e_pf_p\log p}\Big\{\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)(\log(D(L_1+1)\ldots(L_n+1))+nD\max_{1\leqslant j\leqslant n}V_j)$$
$$+\left(1+\frac{1}{q(c_0-1)}\right)p^{\varkappa}S\sum_{j=1}^{n}L_jV_j$$
$$+\frac{1}{q}\left(1+\frac{1}{c_0-1}\right)\left(T+T\log\left(1+\frac{(n-1)(B_nL_1+B'L_n)}{T}\right)\right)\Big\}$$
$$\leqslant\frac{U}{c_1}q^{k+1-n}\Big\{\left(\frac{1}{h_4}+\frac{1}{h_5}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}$$
$$+\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}\Big\}$$
$$\leqslant\frac{U}{c_1}q^{k+1-n}\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2$$

(where the last inequality follows from (4.3)), contrary to (4.29). This contradiction proves

$$\varphi_J(s,\tau)=0\quad\text{for } 1\leqslant s\leqslant q^{J+k+1}S,\quad (s,q)=1,$$
$$|\tau|\leqslant\left(1-\left(1-\frac{1}{q}\right)\frac{k+1}{n}\right)q^{-J}T,$$

thereby establishes the lemma.

LEMMA 4.4.

$$\varphi_J\left(\frac{s}{q},\tau\right)=0$$

for $1\leqslant s\leqslant q^{J+1}S$, $(s,q)=1$, $|\tau|\leqslant q^{-(J+1)}T$.

Proof. We recall that (4.29) holds for $k=n-1$. This is

$$(4.30)\quad \operatorname{ord}_p\varphi_J\left(\frac{s}{q},\tau\right)>\frac{U}{c_1}\left(1-\frac{1}{q}\right)^2\left(1-\frac{1}{c_3n}\right)\left(1-\frac{1}{h_1}\right)$$

for $s\geqslant 1$, $|\tau|\leqslant q^{-(J+1)}T$. Assuming that there exist s, τ with

$$1\leqslant s\leqslant q^{J+1}S,\quad (s,q)=1,\quad |\tau|\leqslant q^{-(J+1)}T$$

such that

$$\varphi_J\left(\frac{s}{q}, \tau\right) \neq 0,$$

we proceed to deduce a contradiction. On applying Lemma 2.1, in a way similar to that in the proof of Lemma 3.4, to the polynomial in $\boldsymbol{Z}[x_1, \ldots, x_n]$, whose value at the point $(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q})$ is $\zeta^{-bsr^{(J)}} \varphi_J(s/q, \tau)$ (recalling that b is introduced in (3.3)) and whose degree in x_j $(1 \leqslant j \leqslant n)$ is at most

$$p^{\varkappa} L_j^{(J)} q^{J+1} S + qD \leqslant q(p^{\varkappa} SL_j + D) \quad (1 \leqslant j \leqslant n),$$

and on utilizing (4.10)–(4.14), $|p_d^{(J)}(\lambda)| \leqslant X_0^{1/(c_0-1)}$ (from the main inductive argument for J) and the definition of X_0 in (4.7), we obtain

$$\begin{aligned}
&\operatorname{ord}_p \varphi_J\left(\frac{s}{q}, \tau\right) \\
&\leqslant \frac{q^n D}{e_p f_p \log p}\left\{\left(1+\frac{1}{c_0-1}\right)\left(\log\big(D(L_1+1)\ldots(L_n+1)\big)+nD \max_{1\leqslant j\leqslant n} V_j\right)\right. \\
&\qquad +\left(1+\frac{1}{c_0-1}\right)p^{\varkappa} S \sum_{j=1}^{n} L_j V_j \\
&\qquad \left. +\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(T+T\log\left(1+\frac{(n-1)q(B_n L_1+B' L_n)}{T}\right)\right)\right\} \\
&\leqslant \frac{U}{c_1}\left\{\left(\frac{1}{h_4}+\frac{1}{h_5}\right)\left(1+\frac{1}{c_0-1}\right)c_1+\left(1+\frac{1}{c_0-1}\right)\frac{1}{c_2}\right. \\
&\qquad \left. +\left(\frac{1}{q}+\frac{1}{c_0-1}\right)\left(1+\frac{1}{h_0}\right)\left(2+\frac{1}{p-1}\right)\frac{1}{c_3}\right\} \\
&\leqslant \frac{U}{c_1}\left(1-\frac{1}{c_3 n}\right)\left(1-\frac{1}{h_1}\right)\left(1-\frac{1}{q}\right)^2
\end{aligned}$$

(where the last inequality follows from (4.3)), contrary to (4.30). This contradiction proves

$$\varphi_J\left(\frac{s}{q}, \tau\right) = 0 \quad \text{for } 1 \leqslant s \leqslant q^{J+1} S,\ (s, q) = 1,\ |\tau| \leqslant q^{-(J+1)} T,$$

thereby establishes the lemma.

Lemma 4.5. *The main inductive argument is true for* $J+1$.

Proof. Similar to the proof of Lemma 3.5.

Thus we have established the main inductive argument for

$$J = 0, 1, \ldots, \left[\frac{\log L_n}{\log q}\right]+1.$$

4.5. The completion of the proof of Proposition 2. We assume that Proposition 2 is false, that is, there exist algebraic numbers $\alpha_1, \ldots, \alpha_n$ and rational integers $b_1, \ldots, b_n$ satisfying (0.5)–(0.8), such that

$$\operatorname{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) \geqslant U,$$

and we proceed to deduce a contradiction. By the main inductive argument for

$$J = J_0 = \left[\frac{\log L_n}{\log q}\right]+1,$$

we have

$$\varphi_{J_0}(s, \tau) = 0 \quad \text{for } 1 \leqslant s \leqslant q^{J_0} S,\ (s, q) = 1,\ |\tau| \leqslant q^{-J_0} T. \tag{4.31}$$

Since $0 \leqslant L_n^{(J_0)} \leqslant q^{-J_0} L_n$, we see that $L_n^{(J_0)} = 0$. Further if $\tau = (\tau_1, \ldots, \tau_{n-1})$ satisfies

$$0 \leqslant \tau_j \leqslant L_j^{(J_0)} \quad (1 \leqslant j \leqslant n-1),$$

then by (4.18)

$$|\tau| = \tau_1 + \ldots + \tau_{n-1} \leqslant q^{-J_0}(L_1 + \ldots + L_{n-1}) \leqslant q^{-J_0} T.$$

Thus (4.31) implies (writing $p^{(J)}(\lambda) = \sum_{d=1}^{D} p_d^{(J)}(\lambda)\xi_d$)

$$\begin{aligned}
\sum_{\lambda_{n-1}=0}^{L_{n-1}^{(J_0)}} \Big\{ \sum_{\lambda_1=0}^{L_1^{(J_0)}} \cdots \sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} p^{(J)}(\lambda_1, \ldots, \lambda_{n-1}, 0) \prod_{j=1}^{n-2} \Delta(b_n \lambda_j; \tau_j) \\
\times \prod_{j=1}^{n-1} (\alpha_j^{p^{\varkappa}} \zeta^{r_j})^{\lambda_j s} \Big\} \Delta(b_n \lambda_{n-1}; \tau_{n-1}) = 0
\end{aligned} \tag{4.32}$$

for $1 \leqslant s \leqslant q^{J_0} S$, $(s, q) = 1$, $0 \leqslant \tau_j \leqslant L_j^{(J_0)}$ $(1 \leqslant j \leqslant n-1)$, where we have set

$$p^{(J_0)}(\lambda_1, \ldots, \lambda_{n-1}, 0) = 0$$

for $\lambda_1, \ldots, \lambda_{n-1}$ satisfying $0 \leqslant \lambda_j \leqslant L_j^{(J_0)}$ $(1 \leqslant j \leqslant n-1)$ and $r_1 \lambda_1 + \ldots + r_{n-1} \lambda_{n-1} \not\equiv r^{(J_0)} \pmod{G}$. By Lemma 2.5 we have

$$\det\big(\Delta(b_n \lambda_{n-1}; \tau_{n-1})\big)_{0 \leqslant \lambda_{n-1}, \tau_{n-1} \leqslant L_{n-1}^{(J_0)}} \neq 0.$$

So (4.32) implies that for each λ_{n-1} with $0 \leqslant \lambda_{n-1} \leqslant L_{n-1}^{(J_0)}$

$$\begin{aligned}
\sum_{\lambda_{n-2}=0}^{L_{n-2}^{(J_0)}} \Big\{ \sum_{\lambda_1=0}^{L_1^{(J_0)}} \cdots \sum_{\lambda_{n-3}=0}^{L_{n-3}^{(J_0)}} p^{(J_0)}(\lambda_1, \ldots, \lambda_{n-1}, 0) \prod_{j=1}^{n-3} \Delta(b_n \lambda_j; \tau_j) \\
\times \prod_{j=1}^{n-2} (\alpha_j^{p^{\varkappa}} \zeta^{r_j})^{\lambda_j s} \Big\} \Delta(b_n \lambda_{n-2}; \tau_{n-2}) = 0
\end{aligned}$$

for $1 \leqslant s \leqslant q^{J_0} S$, $(s, q) = 1$, $0 \leqslant \tau_j \leqslant L_j^{(J_0)}$ $(1 \leqslant j \leqslant n-2)$.

Repeating this argument $n-1$ times, we obtain

$$p^{(J_0)}(\lambda_1, \ldots, \lambda_{n-1}, 0) = 0 \quad \text{for } 0 \leqslant \lambda_j \leqslant L_j^{(J_0)} \quad (1 \leqslant j \leqslant n-1),$$

contrary to the construction in the main inductive argument. This contradiction proves Proposition 2.

Chapter 5. Completion of the proofs of Theorems 1 and 2

5.1. Solving the system of inequalities (3.5)—(3.7). We solve the system of inequalities (3.5)–(3.7) in the following cases:

(1.a) $p = 2,\ 2 \leqslant n \leqslant 7$;

(1.b) $p = 2,\ n \geqslant 8$;

(2.a) $p = 3,\ 2 \leqslant n \leqslant 7$;

(2.b) $p = 3,\ n \geqslant 8$;

(3.a) $p \geqslant 5,\ 2 \leqslant n \leqslant 6$;

(3.b) $p \geqslant 5,\ n = 7$;

(3.c) $p \geqslant 5,\ n \geqslant 8$.

We abbreviate $h_i(n, q; c_0, c_2)$ as h_i $(0 \leqslant i \leqslant 7)$, $h_8(n, q; c_0, c_2, c_3)$ as h_8, $\varepsilon_i(n, q; c_0, c_2)$ as ε_i $(i = 1, 2)$.

We first deal with the cases (1.a), (2.a), (3.a), (3.b). In these cases

$$n \geqslant 2, \quad q \geqslant 3$$

and we fix

$$c_0 = 8, \quad c_2 = 56/15.$$

Then we have the following inequalities:

$$h_0 \geqslant h_0(2, 3) \geqslant 18.832756, \quad 1/h_0 \leqslant 5.3099 \cdot 10^{-2}, \quad h_0(2, 3) \leqslant 18.832758,$$

$$\frac{\log h_0}{h_0} \leqslant 1.5587732 \cdot 10^{-1}, \quad \frac{\log(h_0+1)}{h_0} \leqslant 0.1586245,$$

$$h_1 \geqslant h_1(2, 3; 8, 56/15) \geqslant 7.74103 \cdot 10^7, \quad 1/h_1 \leqslant 1.291818 \cdot 10^{-8},$$

$$h_2(2, 3; 8, 56/15) \geqslant \tfrac{7}{5} \cdot 2^{15}, \quad (h_2(2, 3; 8, 56/15))^{-1} \leqslant 2.17983 \cdot 10^{-5},$$

$$1 + \varepsilon_1 \leqslant 1 + \varepsilon_1(2, 3; 8, 56/15) \leqslant (1 - 2.17983 \cdot 10^{-5})^{-2} \leqslant 1 + 4.35986 \cdot 10^{-5},$$

$$(h_3(2, 3; 8, 56/15))^{-1} \leqslant \frac{4}{7.74103 \cdot 10^7 - 1} \leqslant 5.167273 \cdot 10^{-8},$$

$$1 + \varepsilon_2 \leqslant 1 + \varepsilon_2(2, 3; 8, 56/15) \leqslant 1 + 5.167274 \cdot 10^{-8},$$

$$(1+\varepsilon_1)(1+\varepsilon_2) \leqslant 1 + 4.366 \cdot 10^{-5},$$

$$1/h_4 \leqslant (h_4(2, 3; 8, 56/15))^{-1} \leqslant 1.291818 \cdot 10^{-8} \cdot 19.832758 \leqslant 2.5620315 \cdot 10^{-7},$$

$$\log h_5 \leqslant \log h_5(2, 3; 8, 56/15) \leqslant 6.3749002,$$

$$1/h_6 \leqslant (h_6(2, 3; 8, 56/15))^{-1} \leqslant 4.03694 \cdot 10^{-10},$$

$$1/h_7 \leqslant (h_7(2, 3; 8, 56/15))^{-1} \leqslant 8.1217 \cdot 10^{-10}.$$

The above inequalities will be repeatedly used in the cases (1.a), (2.a), (3.a), (3.b).

Case (1.a): $p = 2,\ 2 \leqslant n \leqslant 7$. It is easy to verify that

$$c_0 = 8, \quad c_1 = 3.2119513, \quad c_2 = 56/15, \quad c_3 = 47.766502, \quad c_4 = 79.102681$$

satisfy the system of inequalities (3.5)–(3.7).

Case (2.a): $p = 3,\ 2 \leqslant n \leqslant 7$. By (0.1), we have $q \geqslant 5$. It is easy to verify that

$$c_0 = 8, \quad c_1 = 2.5889785, \quad c_2 = 56/15, \quad c_3 = c_4 = 32$$

satisfy the system of inequalities (3.5)–(3.7).

Case (3.a): $p \geqslant 5,\ 2 \leqslant n \leqslant 6$. It is easy to verify that

$$c_0 = 8, \quad c_1 = 1.0723192\left(2 + \frac{1}{p-1}\right), \quad c_2 = \frac{56}{15},$$

$$c_3 = 16.457689\left(2 + \frac{1}{p-1}\right), \quad c_4 = 77.89776$$

satisfy the system of inequalities (3.5)–(3.7).

Case (3.b): $p \geqslant 5,\ n = 7$. It is easy to verify that

$$c_0 = 8, \quad c_1 = 1.0192253\left(2 + \frac{1}{p-1}\right), \quad c_2 = \frac{56}{15},$$

$$c_3 = 16\left(2 + \frac{1}{p-1}\right), \quad c_4 = 69.994513$$

satisfy the system of inequalities (3.5)–(3.7).

Now we treat the cases (1.b), (2.b), (3.c). In these cases $n \geqslant 8$, $q \geqslant 3$ and we fix $c_0 = 16$, $c_2 = 8/3$. Then it is easy to establish the following inequalities:

$$h_0 \geqslant h_0(8, 3) \geqslant 86.42138, \quad 1/h_0 \leqslant 1.157122 \cdot 10^{-2}, \quad h_0(8, 3) \leqslant 86.421384,$$

$$\frac{\log h_0}{h_0} \leqslant 5.1598793 \cdot 10^{-2}, \quad \frac{\log(h_0+1)}{h_0} \leqslant 5.1731917 \cdot 10^{-2},$$

$$h_1 \geqslant h_1(8, 3;\, 16, 8/3) \geqslant 2.1226 \cdot 10^{25}, \quad 1/h_1 \leqslant 4.711204 \cdot 10^{-26},$$

$$h_2(8, 3;\, 16, 8/3) = \frac{2^{76}}{5 \cdot 7 \cdot 9}, \quad (h_2(8, 3;\, 16, 8/3))^{-1} \leqslant 4.1689994 \cdot 10^{-21},$$

$$1+\varepsilon_1 \leqslant 1+\varepsilon_1(8, 3;\, 16, 8/3) \leqslant (1-4.1689994 \cdot 10^{-21})^{-8} \leqslant 1+3.3352 \cdot 10^{-20},$$

$$(h_3(8, 3;\, 16, 8/3))^{-1} \leqslant \frac{8^2}{2.1226 \cdot 10^{25}-1} \leqslant 3.0151703 \cdot 10^{-24},$$

$$1+\varepsilon_2 \leqslant 1+\varepsilon_2(8, 3;\, 16, 8/3) \leqslant 1+3.0151704 \cdot 10^{-24},$$

$$(1+\varepsilon_1)(1+\varepsilon_2) \leqslant 1+4 \cdot 10^{-20},$$

$$\frac{1}{h_4} \leqslant (h_4(8, 3;\, 16, 8/3))^{-1} \leqslant \frac{87.421384}{2.1226 \cdot 10^{25}} \leqslant 4.1185992 \cdot 10^{-24},$$

$$\log h_5 \leqslant \log h_5(8, 3;\, 16, 8/3) \leqslant 6.3630211,$$

$$1/h_6 \leqslant (h_6(8, 3;\, 16, 8/3))^{-1} \leqslant 5.889006 \cdot 10^{-27},$$

$$1/h_7 \leqslant (h_7(8, 3;\, 16, 8/3))^{-1} \leqslant 5.13132 \cdot 10^{-27}.$$

The above inequalities will be repeatedly used in the cases (1.b), (2.b), (3.c).

Case (1.b): $p = 2$, $n \geqslant 8$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 3.0703894, \quad c_2 = 8/3, \quad c_3 = 116.51153, \quad c_4 = 192.64207$$

satisfy the system of the inequalities (3.5)–(3.7).

Case (2.b): $p = 3$, $n \geqslant 8$. By (0.1) we have $q \geqslant 5$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 2.52941225, \quad c_2 = 8/3, \quad c_3 = 32, \quad c_4 = 35.671814$$

satisfy the system of inequalities (3.5)–(3.7).

Case (3.c): $p \geqslant 5$, $n \geqslant 8$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 1.0234756\left(2+\frac{1}{p-1}\right), \quad c_2 = 8/3,$$

$$c_3 = 39.253842\left(2+\frac{1}{p-1}\right), \quad c_4 = 192.63692$$

satisfy the system of inequalities (3.5)–(3.7).

On summing up all the cases (1.a)–(3.c) and applying Proposition 1, we obtain the following

PROPOSITION 3. *Let*

$$\varepsilon = \varepsilon(n) = \begin{cases} 4.366 \cdot 10^{-5}, & 2 \leqslant n \leqslant 7, \\ 4 \cdot 10^{-20}, & n \geqslant 8 \end{cases}$$

and c_0, c_1, c_2, c_3, c_4 be positive numbers given by the following two tables.

Case		c_0	c_1	c_2	c_3	c_4
$p=2$	$2 \leqslant n \leqslant 7$	8	3.2119513	$\frac{56}{15}$	47.766502	79.102681
	$n \geqslant 8$	16	3.0703894	$\frac{8}{3}$	116.51153	192.64207
$p=3$	$2 \leqslant n \leqslant 7$	8	2.5889785	$\frac{56}{15}$	32	32
	$n \geqslant 8$	16	2.52941225	$\frac{8}{3}$	32	35.671814

Case		c_0	$c_1\Big/\left(2+\frac{1}{p-1}\right)$	c_2	$c_3\Big/\left(2+\frac{1}{p-1}\right)$	c_4
$p \geqslant 5$	$2 \leqslant n \leqslant 6$	8	1.0723192	$\frac{56}{15}$	16.457689	77.89776
	$n = 7$	8	1.0192253	$\frac{56}{15}$	16	69.994513
	$n \geqslant 8$	16	1.0234756	$\frac{8}{3}$	39.253842	192.63692

Let

$$U = (1+\varepsilon)c_0 c_1 c_2^n c_3 c_4 \frac{n^{2n+1}}{n!} q^{2n}(q-1) \frac{G(2+1/(p-1))^n}{e_p(f_p \log p)^{n+2}} D^{n+2} V_1 \ldots V_n W^* \log V_{n-1}^*.$$

Suppose that (0.5)–(0.8) *hold. Then*

$$\operatorname{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) < U.$$

5.2. Solving the system of inequalities (4.2)–(4.4). We solve the system of inequalities (4.2)–(4.4) in the following cases:

(1.a) $p = 2$, $2 \leqslant n \leqslant 7$,
(1.b) $p = 2$, $n \geqslant 8$,
(2.a) $p = 3$, $2 \leqslant n \leqslant 7$,
(2.b) $p = 3$, $n \geqslant 8$,
(3.a) $p \geqslant 5$, $2 \leqslant n \leqslant 7$,
(3.b) $p \geqslant 5$, $n \geqslant 8$.

We abbreviate $h_i(n, q;\, c_0, c_2)$ $(0 \leqslant i \leqslant 5)$ as h_i, $h_6(n, q;\, c_0, c_2, c_3)$ as h_6, $\varepsilon_i(n, q;\, c_0, c_2)$ $(i = 1, 2)$ as ε_i.

We first deal with the cases (1.a), (2.a), (3.a). In these cases $n \geqslant 2$, $q \geqslant 3$ and we fix $c_0 = 16$, $c_2 = 8/3$. Then we have the following inequalities

$$h_0 \geqslant h_0(2, 3) \geqslant 18.832756, \quad \frac{1}{h_0} \leqslant 5.3099 \cdot 10^{-2}, \quad \frac{\log h_0}{h_0} \leqslant 1.5587732 \cdot 10^{-1},$$

$$h_1 \geqslant h_1(2, 3;\, 16, 8/3) \geqslant 78990303, \quad 1/h_1 \leqslant 1.26598 \cdot 10^{-8},$$

$$(h_2(2, 3;\, 16, 8/3))^{-1} = 2^{-16} \leqslant 1.52588 \cdot 10^{-5},$$

$$1+\varepsilon_1 \leqslant 1+\varepsilon_1(2, 3; 16, 8/3) \leqslant 1+3.05192\cdot 10^{-5},$$
$$(h_3(2, 3; 16, 8/3))^{-1} \leqslant 1.26598\cdot 10^{-8},$$
$$1+\varepsilon_2 \leqslant 1+\varepsilon_2(2, 3; 16, 8/3) \leqslant 1+1.266\cdot 10^{-8},$$
$$(1+\varepsilon_1)(1+\varepsilon_2) \leqslant 1+3.0532\cdot 10^{-5},$$
$$1/h_4 \leqslant (h_4(2, 3; 16, 8/3))^{-1} \leqslant 7.91238\cdot 10^{-10},$$
$$1/h_5 \leqslant (h_5(2, 3; 16, 8/3))^{-1} \leqslant 1.03297\cdot 10^{-9}.$$

The above inequalities will be repeatedly used in the cases (1.a), (2.a), (3.a).

Case (1.a): $p = 2$, $2 \leqslant n \leqslant 7$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 3.2968387, \quad c_2 = 8/3, \quad c_3 = 33.433683$$

satisfy the system of inequalities (4.2)–(4.4).

Case (2.a): $p = 3$, $2 \leqslant n \leqslant 7$. By (0.1) we have $q \geqslant 5$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 2.62791175, \quad c_2 = 8/3, \quad c_3 = 16$$

satisfy the system of inequalities (4.2)–(4.4).

Case (3.a): $p \geqslant 5$, $2 \leqslant n \leqslant 7$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 1.1010155\left(2+\frac{1}{p-1}\right), \quad c_2 = 8/3, \quad c_3 = 11.977897\left(2+\frac{1}{p-1}\right)$$

satisfy the system of inequalities (4.2)–(4.4).

Remark. Note that the inequalities for $h_0, \ldots, h_5, h_6, \varepsilon_1, \varepsilon_2$ we used in the cases (1.a), (2.a), (3.a) depend on the fact that $n \geqslant 2$, but not on $n \leqslant 7$. Hence the solutions c_0, c_1, c_2, c_3 of the system of inequalities (4.2)–(4.4), which we obtained in the cases (1.a), (2.a), (3.a), are also the solutions of the system (4.2)–(4.4) for the cases (1.b), (2.b), (3.b).

Now we treat the cases (1.b), (2.b), (3.b). In these cases $n \geqslant 8$, $q \geqslant 3$ and we fix $c_0 = 16$, $c_2 = 5/2$. Then we have the following inequalities

$$h_0 \geqslant h_0(8, 3) \geqslant 86.42138, \quad 1/h_0 \leqslant 1.157122\cdot 10^{-2},$$
$$\frac{\log h_0}{h_0} \leqslant 5.1598793\cdot 10^{-2},$$
$$h_1 \geqslant h_1(8, 3; 16, 5/2) \geqslant 5.06661\cdot 10^{25}, \quad 1/h_1 \leqslant 1.974\cdot 10^{-26},$$
$$h_2(8, 3; 16, 5/2) \geqslant 6.1068935\cdot 10^{20}, \quad (h_2(8, 3; 16, 5/2))^{-1} \leqslant 1.637494\cdot 10^{-21},$$
$$1+\varepsilon_1 \leqslant 1+\varepsilon_1(8, 3; 16, 5/2) \leqslant 1+1.31\cdot 10^{-20},$$

$$h_3(8, 3; 16, 5/2) \geqslant \frac{1}{49}\cdot 5.0666\cdot 10^{25}, \quad (h_3(8, 3; 16, 5/2))^{-1} \leqslant 9.67112\cdot 10^{-25},$$
$$1+\varepsilon_2 \leqslant 1+\varepsilon_2(8, 3; 16, 5/2) \leqslant 1+9.68\cdot 10^{-25},$$
$$(1+\varepsilon_1)(1+\varepsilon_2) \leqslant 1+1.4\cdot 10^{-20},$$
$$1/h_4 \leqslant (h_4(8, 3; 16, 5/2))^{-1} \leqslant 4.935\cdot 10^{-27},$$
$$1/h_5 \leqslant (h_5(8, 3; 16, 5/2))^{-1} \leqslant 3.835\cdot 10^{-27}.$$

The above inequalities will be repeatedly used in the cases (1.b), (2.b), (3.b).

Case (1.b): $p = 2$, $n \geqslant 8$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 3.0751334, \quad c_2 = 5/2, \quad c_3 = 71.406058$$

satisfy the system of inequalities (4.2)–(4.4).

Case (2.b): $p = 3$, $n \geqslant 8$. By (0.1) we have $q \geqslant 5$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 2.5314965, \quad c_2 = 5/2, \quad c_3 = 16$$

satisfy the system of inequalities (4.2)–(4.4).

Case (3.b): $p \geqslant 5$, $n \geqslant 8$. It is easy to verify that

$$c_0 = 16, \quad c_1 = 1.0250654\left(2+\frac{1}{p-1}\right), \quad c_2 = 5/2, \quad c_3 = 24.322856\left(2+\frac{1}{p-1}\right)$$

satisfy the system of inequalities (4.2)–(4.4).

On summing up all the cases (1.a)–(3.b) and the remark at the end of the discussion of the case (3.a), and applying Proposition 2, we obtain the following

PROPOSITION 4. (i) *Let*

$$\varepsilon = \varepsilon(n) = \begin{cases} 1+3.0532\cdot 10^{-5}, & 2 \leqslant n \leqslant 7, \\ 1+1.4\cdot 10^{-20}, & n \geqslant 8 \end{cases}$$

and c_0, c_1, c_2, c_3 *be positive numbers given by the following two tables:*

Case		c_0	c_1	c_2	c_3
$p = 2$	$2 \leqslant n \leqslant 7$	16	3.2968387	$\frac{8}{3}$	33.433683
	$n \geqslant 8$	16	3.0751334	$\frac{5}{2}$	71.406058
$p = 3$	$2 \leqslant n \leqslant 7$	16	2.62791175	$\frac{8}{3}$	16
	$n \geqslant 8$	16	2.5314965	$\frac{5}{2}$	16

Case		c_0	$c_1/\left(2+\frac{1}{p-1}\right)$	c_2	$c_3/\left(2+\frac{1}{p-1}\right)$
$p \geqslant 5$	$2 \leqslant n \leqslant 7$	16	1.1010155	$\frac{8}{3}$	11.977897
	$n \geqslant 8$	16	1.0250654	$\frac{5}{2}$	24.322856

Let

$$(5.1)\quad U = (1+\varepsilon)c_0 c_1 c_2^n c_3^2 \frac{n^{2n+2}}{n!} q^{2n}(q-1)\frac{G(2+1/(p-1))^n}{(f_p \log p)^{n+2}} D^{n+2} V_1 \ldots V_n (W^*)^2.$$

Suppose that (0.5)–(0.8) *hold. Then*

$$(5.2)\quad \operatorname{ord}_p(\alpha_1^{b_1} \ldots \alpha_n^{b_n} - 1) < U.$$

(ii) *Suppose that* (0.5)–(0.8) *hold. If in* (5.1), ε, c_0, c_1, c_2, c_3 *take the values (given in the above two tables) for the cases* $p = 2$, $2 \leqslant n \leqslant 7$; $p = 3$, $2 \leqslant n \leqslant 7$; $p \geqslant 5$, $2 \leqslant n \leqslant 7$, *respectively, then* (5.2) *holds also for the cases* $p = 2$, $n \geqslant 8$; $p = 3$, $n \geqslant 8$; $p \geqslant 5$, $n \geqslant 8$, *respectively.*

5.3. Estimates for $\log V_{n-1}^*$ and W^*.

LEMMA 5.1. *Let*

$$v_2 = 5.2336533,\quad v_3 = 3.81275,\quad v_4 = 3.2814667,\quad v_5 = 2.9909667,$$

$$v_6 = 2.8030858,\quad v_7 = 2.66939,\quad v_n = 2.5681639 \quad (n \geqslant 8);$$

$$w_2 = 3.7909562,\quad w_3 = 3.2245056,\quad w_4 = 2.9347108,\quad w_5 = 2.7523294,$$

$$w_6 = 2.6242173,\quad w_7 = 2.5278708,\quad w_n = 2.4519668 \quad (n \geqslant 8).$$

Then for $n \geqslant 2$ *we have*

$$(5.3)\quad \log V_{n-1}^* \leqslant v_n n \log(nq)\cdot\left(\log(4DV_{n-1}^+) + \frac{f_p \log p}{8n}\right),$$

$$W^* \leqslant w(n)\, n \log(nq)\cdot\left(\frac{W}{6n} + \log(4D)\right),$$

where

$$(5.4)\quad w(n) = \frac{\log(2^{11}\cdot 3n)}{\log 4\cdot \log(3n)}$$

and

$$(5.5)\quad W^* \leqslant w_n n \log(nq)\cdot\left(\frac{W}{6n} + \log(4D)\right).$$

Proof. Note that by $q \geqslant 3$ we have

$$(5.6)\quad \log(2^{11} nq^{\frac{n+1}{n-1}} D^{\frac{n}{n-1}} V_{n-1}^+) = \log(2^9 nq^{\frac{n+1}{n-1}}) + \log(4D^{\frac{n}{n-1}} V_{n-1}^+)$$

$$= \frac{n+1}{n-1}\log(2^{\frac{9(n-1)}{n+1}} n^{-\frac{2}{n+1}} nq) + \frac{n}{n-1}\log((4V_{n-1}^+)^{-\frac{1}{n}} 4DV_{n-1}^+)$$

$$\leqslant \log(nq)\cdot\log(4DV_{n-1}^+)$$

$$\times\left\{\frac{n+1}{n-1}\left(\frac{\log(2^{\frac{9(n-1)}{n+1}} n^{-\frac{2}{n+1}})}{\log 4\cdot\log(nq)} + \frac{1}{\log 4}\right) + \frac{n}{n-1}\cdot\frac{1}{\log(nq)}\right\}$$

$$\leqslant \log(nq)\cdot\log(4DV_{n-1}^+)\frac{\log(2^{9(n-1)} n^{-2}) + (n+1)\log(3n) + n\log 4}{\log 4\cdot(n-1)\log(3n)}$$

$$= \log(nq)\cdot\log(4DV_{n-1}^+)$$

$$\times\frac{(n-1)\log n + \log(2^{9(n-1)}) + (n+1)\log 3 + n\log 4}{\log 4\cdot(n-1)\log(3n)}$$

$$= \log(nq)\cdot\log(4DV_{n-1}^+)\, v(n) \text{ (say).}$$

It is easy to verify that $v(n)$ decreases monotonically and by a direct computation we see that

$$(5.7)\quad v(n) \leqslant v_n \quad (n \geqslant 2).$$

Now by the definition of V_{n-1}^* (see (3.8)) and by (5.6), (5.7), we have

$$\log V_{n-1}^* \leqslant n\log(2^{11} nq^{\frac{n+1}{n-1}} D^{\frac{n}{n-1}} V_{n-1}^+) + f_p \log p$$

$$\leqslant v_n n\log(nq)\left(\log(4DV_{n-1}^+) + \frac{f_p \log p}{n v_n \log(3n)}\right).$$

This together with the fact that $v_n \log(3n) \geqslant 8$ $(n \geqslant 2)$, which can be verified by a direct calculation, yields (5.3) at once.

Further, we have

$$(5.8)\quad \log(2^{11} nqD) = \log(nq)\cdot\log(4D)\cdot\frac{\log 2^9 + \log(nq) + \log(4D)}{\log(nq)\cdot\log(4D)}$$

$$\leqslant \log(nq)\cdot\log(4D)\left\{\frac{\log 2^9}{\log 4\cdot\log(3n)} + \frac{1}{\log 4} + \frac{1}{\log(3n)}\right\}$$

$$= \log(nq)\cdot\log(4D)\frac{\log(2^{11}\cdot 3n)}{\log 4\cdot\log(3n)}$$

$$= w(n)\log(nq)\cdot\log(4D).$$

Obviously $w(n)$ decreases monotonically and by a direct calculation we see that

$$(5.9)\quad w(n) \leqslant w_n \quad (n \geqslant 2).$$

Now by the definition of W^* (see (3.9) and (4.5)) and by (5.8), we get

$$W^* \leqslant W + n\log(2^{11} nqD) \leqslant w(n)\, n \log(nq)\left(\frac{W}{nw(n)\log(3n)} + \log(4D)\right).$$

This together with the fact that

$$w(n)\log(3n) = \frac{\log(2^{11}\cdot 3n)}{\log 4} > 6 \quad (n \geqslant 2)$$

implies (5.4) immediately. Now (5.5) follows from (5.4) and (5.9). The proof of the lemma is thus complete.

5.4. Completion of the proofs of Theorems 1 and 2.

Completion of the proof of Theorem 1. By Proposition 3, Lemma 5.1 and Lemma 2.7, we see that, in order to prove Theorem 1, it suffices to show

$$(5.10) \qquad (1+\varepsilon) c_0 c_1 c_3 c_4 v_n w_n / \sqrt{2\pi} \leqslant C_1(p, n),$$

where ε, c_0, c_1, c_3, c_4 are given in Proposition 3 and v_n, w_n are given in Lemma 5.1. We can easily prove (5.10) by a direct calculation, thereby complete the proof of Theorem 1.

Completion of the proof of Theorem 2. Theorem 2 is a direct consequence of Proposition 4, Lemma 5.1 and Lemma 2.7.

(1) $p = 2$. If $2 \leqslant n \leqslant 17$, it suffices to show that

$$(5.11) \qquad (1+\varepsilon) c_0 c_1 c_3^2 w_n^2 / \sqrt{2\pi} \leqslant C_2(2, n),$$

where c_0, c_1, c_3, ε are given by Proposition 4, (ii).

If $n \geqslant 18$, on noting that $w(n) \geqslant w(18)$, it suffices to show that

$$(5.12) \qquad (1+\varepsilon) c_0 c_1 c_3^2 (w(18))^2 / \sqrt{2\pi} \leqslant C_2(2, n),$$

where c_0, c_1, c_3, ε are given by Proposition 4, (i), $w(18) \leqslant 2.1001457$ (see Lemma 5.1).

(2) $p = 3$. It suffices to show that

$$(5.13) \qquad (1+\varepsilon) c_0 c_1 c_3^2 w_n^2 / \sqrt{2\pi} \leqslant C_2(3, n),$$

where c_0, c_1, c_3, ε are given by Proposition 4, (i).

(3) $p \geqslant 5$. If $2 \leqslant n \leqslant 16$, it suffices to show that

$$(5.14) \qquad (1+\varepsilon) c_0 c_1 c_3^2 w_n^2 / \sqrt{2\pi} \leqslant C_2(p, n),$$

where c_0, c_1, c_3, ε are given by Proposition 4, (ii).

If $n \geqslant 17$, on noting that $w(n) \geqslant w(17)$, it suffices to show that

$$(5.15) \qquad (1+\varepsilon) c_0 c_1 c_3^2 (w(17))^2 / \sqrt{2\pi} \leqslant C_2(p, n),$$

where c_0, c_1, c_3, ε are given by Proposition 4, (i), and $w(17) \leqslant 2.1201893$ (see Lemma 5.1).

Now the inequalities (5.11)–(5.15) can be easily verified by a direct calculation. This completes the proof of Theorem 2.

Appendix. Hermite interpolation

Let E be an algebraically closed field of characteristic 0. Suppose that $n \geqslant 2$, $\tau_1 > 0, \ldots, \tau_n > 0$ are integers,

$$T = \tau_1 + \ldots + \tau_n.$$

Let $\beta_1, \ldots, \beta_n$ ($\beta_i \neq \beta_j$ for $1 \leqslant i < j \leqslant n$) and $q_{i,t}$ ($1 \leqslant i \leqslant n$, $0 \leqslant t < \tau_i$) be given elements in E.

Theorem A. *The unique polynomial $Q(z) \in E[z]$ of degree at most $T-1$ satisfying*

$$(1) \qquad Q^{(t-1)}(\beta_i) = q_{i,t-1} \quad (1 \leqslant i \leqslant n,\ 1 \leqslant t \leqslant \tau_i)$$

is given by the formula

$$(2) \qquad Q(z) = \sum_{h=1}^{n} \sum_{t=1}^{\tau_h} q_{h,t-1} (-1)^{\tau_h - t} \frac{(z-\beta_h)^{t-1}}{(t-1)!} \left\{ \prod_{\substack{k=1 \\ k \neq h}}^{n} \left(\frac{z-\beta_k}{\beta_h - \beta_k} \right)^{\tau_k} \right\}$$

$$\times \sum_{s=1}^{\tau_h - t} (-1)^{s-1} \sum_{\substack{\lambda_1 + \ldots + \lambda_{\tau_h - t} = \tau_h - t \\ \lambda_j = 0\,(j<s),\, \lambda_j \geqslant 1\,(j \geqslant s)}} \prod_{j=1}^{\tau_h - t} \frac{\left(\frac{\partial}{\partial y}\right)^{\lambda_j} \left\{ (z-y) \prod_{\substack{k=1 \\ k \neq h}}^{n} (y-\beta_k)^{\tau_k} \right\}_{y=\beta_h}}{\lambda_j! \prod_{\substack{k=1 \\ k \neq h}}^{n} (\beta_h - \beta_k)^{\tau_k}},$$

where the second line of (2) reads as 1 when $t = \tau_h$.

I am indebted to R. Tijdeman and R. J. Kooman for giving an elegant proof of Theorem A (see below). It is simpler than the proof given in the Appendix to Yu [37]. The argument is based on the following lemma.

Lemma. *If g is $k \geqslant 1$ times differentiable, then*

$$\frac{d^k}{dx^k}\left(\frac{1}{g}\right) = \sum_{j=1}^{k} (-1)^j \sum_{\substack{\lambda_1 + \ldots + \lambda_j = k \\ \lambda_1 > 0, \ldots, \lambda_j > 0}} \binom{k}{\lambda_1 \ldots \lambda_j} g^{(\lambda_1)} \ldots g^{(\lambda_j)} g^{-j-1}.$$

Proof. By induction on k. For $k = 1$ the formula is correct. Assume the formula is correct for $k = 1, \ldots, l-1$. Then

$$0 = \left(\frac{1}{g}\cdot g\right)^{(l)} = \sum_{h=0}^{l}\binom{l}{h} g^{(l-h)}\left(\frac{1}{g}\right)^{(h)}$$

$$= g\left(\frac{1}{g}\right)^{(l)} + \frac{1}{g}g^{(l)} + \sum_{h=1}^{l-1}\binom{l}{h} g^{(l-h)} \sum_{j=1}^{h}(-1)^j \sum_{\substack{\lambda_1+\ldots+\lambda_j=h\\ \lambda_1>0,\ldots,\lambda_j>0}} \binom{h}{\lambda_1 \ldots \lambda_j} g^{(\lambda_1)}\ldots g^{(\lambda_j)} g^{-j-1}$$

$$= g\left(\frac{1}{g}\right)^{(l)} + \frac{1}{g}g^{(l)} + \sum_{j=1}^{l-1}(-1)^j \sum_{\substack{\lambda_1+\ldots+\lambda_{j+1}=l\\ \lambda_1>0,\ldots,\lambda_{j+1}>0}} \binom{l}{\lambda_1 \ldots \lambda_{j+1}} g^{(\lambda_1)}\ldots g^{(\lambda_j)} g^{(\lambda_{j+1})} g^{-j-1}.$$

Hence

$$\left(\frac{1}{g}\right)^{(l)} = -\frac{g^{(l)}}{g^2} - \sum_{j=2}^{l}(-1)^{j-1} \sum_{\substack{\lambda_1+\ldots+\lambda_j=l\\ \lambda_1>0,\ldots,\lambda_j>0}} \binom{l}{\lambda_1 \ldots \lambda_j} g^{(\lambda_1)}\ldots g^{(\lambda_j)} g^{-j-1}$$

$$= \sum_{j=1}^{l}(-1)^j \sum_{\substack{\lambda_1+\ldots+\lambda_j=l\\ \lambda_1>0,\ldots,\lambda_j>0}} \binom{l}{\lambda_1 \ldots \lambda_j} g^{(\lambda_1)}\ldots g^{(\lambda_j)} g^{-j-1}.$$

Remark. A. Schinzel supplied further reference on the formula for kth derivative of $1/g(x)$, namely Faa di Bruno [10] and E. Goursat [15], p. 80.

Proof of Theorem A. According to Mahler [23], pp. 84–85, we have

$$Q(z) = \sum_{h=1}^{n}\sum_{t=1}^{\tau_h} q_{h,t-1}\frac{P(z)}{(t-1)!(\tau_h-t)!}\left(\frac{\partial}{\partial y}\right)^{\tau_h-t}\left\{\frac{(y-\beta_h)^{\tau_h}}{(z-y)P(y)}\right\}_{y=\beta_h}$$

where

$$P(z) = \prod_{h=1}^{n}(z-\beta_h)^{\tau_h}.$$

Put

$$g(y) = g_h(y) = \frac{(z-y)P(y)}{(y-\beta_h)^{\tau_h}} = (z-y)\prod_{\substack{k=1\\ k\neq h}}^{n}(y-\beta_k)^{\tau_k}.$$

By applying the lemma with $s = k-j+1$ and $k = \tau_h - t$ we obtain

$$\left\{\frac{\partial^{\tau_h-t}}{\partial y^{\tau_h-t}}\left(\frac{(y-\beta_h)^{\tau_h}}{(z-y)P(y)}\right)\right\}_{y=\beta_h} = \left\{\frac{\partial^{\tau_h-t}}{\partial y^{\tau_h-t}}\left(\frac{1}{g(y)}\right)\right\}_{y=\beta_h}$$

$$= \sum_{s=1}^{\tau_h-t}(-1)^{\tau_h-t-s+1} \sum_{\substack{\lambda_s+\ldots+\lambda_{\tau_h-t}=\tau_h-t\\ \lambda_s>0,\ldots,\lambda_{\tau_h-t}>0}} \binom{\tau_h-t}{\lambda_s \ldots \lambda_{\tau_h-t}}\left(\prod_{j=s}^{\tau_h-t} g^{(\lambda_j)}(y)\right)_{y=\beta_h}(g(\beta_h))^{-\tau_h+t+s-2}$$

$$= (-1)^{\tau_h-t}\frac{(\tau_h-t)!}{g(\beta_h)}\sum_{s=1}^{\tau_h-t}(-1)^{s-1} \sum_{\substack{\lambda_1+\ldots+\lambda_{\tau_h-t}=\tau_h-t\\ \lambda_j=0\,(j<s);\,\lambda_j>0\,(j\geq s)}} \prod_{j=1}^{\tau_h-t}\left(\frac{(g^{(\lambda_j)}(y))_{y=\beta_h}}{\lambda_j!\,g(\beta_h)}\right).$$

Hence

$$Q(z) = \sum_{h=1}^{n}\sum_{t=1}^{\tau_h} q_{h,t-1}\frac{(-1)^{\tau_h-t}}{(t-1)!}\cdot\frac{P(z)}{g_h(\beta_h)}\sum_{s=1}^{\tau_h-t}(-1)^{s-1}\sum_{(\lambda)}\prod_{j=1}^{\tau_h-t}\left(\frac{(g_h^{(\lambda_j)}(y))_{y=\beta_h}}{\lambda_j!\,g_h(\beta_h)}\right)$$

$$= \sum_{h=1}^{n}\sum_{t=1}^{\tau_h} q_{h,t-1}(-1)^{\tau_h-t}\frac{(z-\beta_h)^{t-1}}{(t-1)!}\left\{\prod_{\substack{k=1\\ k\neq h}}^{n}\left(\frac{z-\beta_k}{\beta_h-\beta_k}\right)^{\tau_k}\right\}$$

$$\times\sum_{s=1}^{\tau_h-t}(-1)^{s-1}\sum_{(\lambda)}\prod_{j=1}^{\tau_h-t}\left(\frac{(g_h^{(\lambda_j)}(y))_{y=\beta_h}}{\lambda_j!\prod_{\substack{k=1\\ k\neq h}}^{n}(\beta_h-\beta_k)^{\tau_k}}\right).$$

This proves Theorem A.

Remark. van der Poorten [25] gives a similar formula, but his conditions of summation and some signs are inaccurate; a simple counterexample can be obtained in the case $n = 2$, $\tau(1) = \tau(2) = 2$. Consequently, the interpolation formula in Lemma 1 of van der Poorten [26] is incorrect also.

References

[1] W. W. Adams, *Transcendental numbers in the p-adic domain*, Amer. J. Math. 88 (1966), 279–308.

[2] M. Anderson and D. W. Masser, *Lower bounds for heights on elliptic curves*, Math. Zeitschr. 174 (1980), 23–34.

[3] A. Baker, *Linear forms in the logarithms of algebraic numbers I, II, III*, Mathematika 13 (1966), 204–216; 14 (1967), 102–107, 220–228.

[4] — *Linear forms in the logarithms of algebraic numbers IV*, ibid. 15 (1968), 204–216.

[5] — *A sharpening of the bounds for linear forms in logarithms II*, Acta Arith. 24 (1973), 33–36.

[6] — *The theory of linear forms in logarithms*, in: *Transcendence theory: advances and applications*, edited by A. Baker and D. W. Masser, Academic Press, London 1977, pp. 1–27.

[7] A. Baker and J. Coates, *Fractional parts of powers of rationals*, Math. Proc. Camb. Phil. Soc. 77 (1975), 269–279.

[8] D. Bertrand, *Approximations diophantiennes p-adiques sur les courbes elliptiques admettant une multiplication complexe*, Compositio Math. 37 (1978), 21–50.

[9] A. Brumer, *On the units of algebraic number fields*, Mathematika 14 (1967), 121–124.

[10] G. Faà di Bruno, Quarterly J. Pure Appl. Math. 1 (1864), p. 359.

[11] P. J. Cijsouw and M. Waldschmidt, *Linear forms and simultaneous approximations*, Compositio Math. 34 (1977), 173–197.

[12] J. Coates, *An effective p-adic analogue of a theorem of Thue I; II: The greatest prime factor of a binary form*, Acta Arith. 15 (1969), 279–305; 16 (1970), 399–412.

[13] N. I. Feldman, *An improvement of the estimate of a linear form in the logarithms of algebraic numbers*, Mat. Sbornik 77 (1968), 423–436; = Math. USSR Sbornik 6 (1968), 393–406.

[14] A. O. Gelfond, *Sur la divisibilité de la différence des puissances de deux nombres entières par une puissance d'un idéal premier*, Mat. Sbornik 7 (1940), 7–26.

[15] E. Goursat, *Cours d'analyse mathématique*, 3 édition, Tome I, Paris 1917.

[16] D. Hanson, *On the product of the primes*, Canad. Math. Bull. 15 (1972), 33–37.

[17] H. Hasse, *Number theory*, Springer-Verlag, Berlin, Heidelberg, New York 1980.

[18] R. M. Kaufman, *Bounds for linear forms in the logarithms of algebraic numbers with p-adic metric*, Vestnik Moskov. Univ. Ser. I, 26 (1971), 3–10.

[19] J. H. Loxton, M. Mignotte, A. J. van der Poorten and M. Waldschmidt, *A lower bound for linear forms in the logarithms of algebraic numbers*, C. R. Math. Acad. Sci. Canada = Math. Report Acad. Sci. Canada 11 (1987), 119–124.

[20] K. Mahler, *Ein Beweis der Transzendenz der P-adischen Exponentialfunktion*, J. Reine Angew. Math. 169 (1932), 61–66.

[21] — *Über transzendente P-adische Zahlen*, Compositio Math. 2 (1935), 259–275.

[22] — *On some inequalities for polynomials in several variables*, J. London Math. Soc. 37 (1962), 341–344.

[23] — *On a class of entire functions*, Acta Math. Acad. Sci. Hungar. 18 (1967), 83–96.

[24] M. Mignotte and M. Waldschmidt, *Linear forms in two logarithms and Schneider's method*, Math. Annalen 231 (1978), 241–267.

[25] A. J. van der Poorten, *Hermite interpolation and p-adic exponential polynomials*, J. Austral. Math. Soc. 22 (1976), 12–26.

[26] — *Linear forms in logarithms in the p-adic case*, in: *Transcendence theory: advances and applications*, edited by A. Baker and D. W. Masser, Academic Press, London 1977, 29–57 pp.

[27] J. B. Rosser and L. Schoenfeld, *Approximate formulas for some functions of prime numbers*, Illinois J. Math. 6 (1962), 64–94.

[28] A. Schinzel, *On two theorems of Gelfond and some of their applications*, Acta Arith. 13 (1967), 177–236.

[29] L. G. Schnirelman, *On functions in normed algebraically closed fields*, Izv. Akad. Nauk SSSR, Ser. Mat. 5/6, 23 (1938), 487–496.

[30] V. G. Sprindžuk, *Concerning Baker's theorem on linear forms in logarithms*, Dokl. Akad. Nauk BSSR, 11 (1967), 767–769.

[31] — *Estimates of linear forms with p-adic logarithms of algebraic numbers*, Vesci Akad. Nauk BSSR, Ser. Fiz-Mat. (1968), no. 4, 5–14.

[32] R. Tijdeman, *On the equation of Catalan*, Acta Arith. 29 (1976), 197–209.

[33] M. Waldschmidt, *A lower bound for linear forms in logarithms*, ibid. 37 (1980), 257–283.

[34] E. T. Whittaker and G. N. Watson, *A course of modern analysis*, Reprint of 4th ed., Cambridge Univ. Press, 1969.

[35] G. Wüstholz, *A new approach to Baker's theorem on linear forms in logarithms* I, II, Lecture Notes in Math. 1290 (1987), 189–202, 203–211, III, in: *New advances in transcendence theory, the Proceedings of the Durham Symposium on Transcendental Number Theory, July 1986*, edited by A. Baker, Cambridge Univ. Press, Cambridge 1988, pp. 399–410.

[36] K. R. Yu, *Linear forms in logarithms in the p-adic case*, in: *New advances in transcendence theory, the Proceedings of the Durham Symposium on Transcendental Number Theory, July 1986*, edited by A. Baker, Cambridge Univ. Press, Cambridge 1988, pp. 411–434.

[37] — *Linear forms in the p-adic logarithms*, Max-Planck-Institut für Mathematik, Bonn, MPI/87-20.

MAX-PLANCK-INSTITUT FÜR MATHEMATIK
Gottfried-Claren-Str. 26
D-5300 Bonn 3, Federal Republic of Germany
INSTITUTE OF MATHEMATICS
ACADEMIA SINICA
Beijing 100080, China

Received on 4.6.1987
and in revised form on 12.2.1988 (1728)

ACTA ARITHMETICA
LIII (1989)

Integers with identical digits

by

T. N. SHOREY (Bombay)

In memory of Professor V. G. Sprindžuk

1. For an integer $\nu > 1$, we denote by $\omega(\nu)$ the number of distinct prime factors of ν and we write $\omega(1) = 0$. Let $N > 2$ be an integer. Let $S(N)$ be the set of all integers x with $1 < x < N-1$ such that N has all the digits equal to one in its x-adic expansion. We write $s(N)$ for the number of distinct elements of $S(N)$. Goormaghtigh in 1917 observed that $s(31) = s(8191) = 2$;

$$31 = \frac{2^5-1}{2-1} = \frac{5^3-1}{5-1}, \quad 8191 = \frac{2^{13}-1}{2-1} = \frac{90^3-1}{90-1}.$$

It has been conjectured that

$$s(N) \leqslant 1, \quad N \neq 31 \quad \text{and} \quad N \neq 8191. \tag{1}$$

A weaker conjecture states that $s(N) \leqslant 1$ whenever N is a prime number different from 31 and 8191. See Dickson [3], p. 703 and Guy [4], p. 45. For $x \in S(N)$, we have

$$N = \frac{x^\mu - 1}{x-1} \tag{2}$$

and

$$N-1 = x\frac{x^{\mu-1}-1}{x-1} \tag{3}$$

for some integer $\mu \geqslant 3$. We write

$$\mu = l(N; x) \geqslant 3. \tag{4}$$

We prove

THEOREM 1. *Let $N > 2$, $N \neq 31$ and $N \neq 8191$ be an integer satisfying $\omega(N-1) \leqslant 5$. There is at most one $y \in S(N)$ such that $l(N; y)$ is an odd integer.*