whence

$$\left\{r\left(1+NY^{-3}\left|\alpha-\frac{a}{rd}\right|\right)\right\}^{1/6} \ll H^{1/8} \ll Y$$

and (8.18) yields

$$(8.20)\qquad \sum_{\substack{r\leqslant H^{3/4}\\ (7.22)}} \sum_{a\leqslant rd} \int_{\mathfrak{R}(d,r,a)} \sum_{\substack{p\\ (8.10)}} \Xi_p(\lambda_1\alpha p^2)\, S(\alpha)\, d\alpha$$

$$\ll \sum_{\substack{r\leqslant H^{3/4}\\ (7.22)}} \sum_{a\leqslant rd} \int_{\mathfrak{R}(d,r,a)} \frac{YHP^{1+\delta}}{\{r(1+NY^{-3}|\alpha-a/rd|)\}^{2/3}}\, d\alpha.$$

We have already obtained the bound (7.36) for the expressions on the right-hand sides of (8.19) and (8.20), in the course of the proof of Lemma 19. Therefore the bound (8.5) follows on combining (8.9), (8.16), (8.17), (8.19) and (8.20). This completes the proof of Lemma 21.

Lemma 12 now follows on combining (6.10), (6.9), (6.18), (6.20), (7.13), (7.14) and (8.5). As explained in Section 5, with the completion of this step we have finished the proof of Theorem 2.

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Linear forms in two logarithms and Schneider's method, II

by

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Introduction. We consider an homogenous linear combination of two logarithms of algebraic numbers with integer coefficients

$$b_1 \log\alpha_1 - b_2 \log\alpha_2.$$

We refine the lower bound which was obtained in our previous paper [7] by using the assumption that b_1, b_2 are rational integers. Our result will be very sharp as far as the dependence on the heights of α_1 and α_2 is concerned. We pay also a special attention to the absolute constant, which is important in numerical applications (e.g. [4] and also [3]).

1. A lower bound for linear forms in two logarithms. Our main result is Theorem 5.11 in Section 5. The hypotheses are a bit technical, and we give here a simpler statement. However for concrete applications where the value of the constant is important, our estimates of Sections 5 and 6 below will give better numerical values than Corollary 1.1.

Here we consider the absolute logarithmic height $h(\alpha)$ of algebraic numbers. Namely, if α is algebraic of degree d over $\boldsymbol{Q}$, with conjugates $\sigma_1\alpha, \ldots, \sigma_d\alpha$, and minimal polynomial

$$c_0X^d+\ldots+c_d = c_0\prod_{i=1}^{d}(X-\sigma_i\alpha) \qquad (c_0>0)$$

then

$$h(\alpha) = d^{-1}\Big(\operatorname{Log} c_0 + \sum_{i=1}^{d} \operatorname{Log}\max(1, |\sigma_i\alpha|)\Big).$$

The measure of α is defined by

$$M(\alpha) = |c_0|\prod_{i=1}^{d}\max\{1, |\sigma_i\alpha|\} = \exp\{d\cdot h(\alpha)\}.$$

Let α_1, α_2 be two non-zero algebraic numbers of exact degrees D_1, D_2. Let D denote the degree over $\boldsymbol{Q}$ of the field $\boldsymbol{Q}(\alpha_1, \alpha_2)$. For $j = 1, 2$, let $\log\alpha_j$ be any non-zero determination of the logarithm of α_j.

Further, let b_1, b_2 be two positive rational integers such that

$$b_1 \log\alpha_1 \neq b_2 \log\alpha_2.$$

Define $B = \max\{b_1, b_2\}$ and choose two positive real numbers a_1, a_2 satisfying

$$a_j \geqslant 1, \quad a_j \geqslant h(\alpha_j) + \operatorname{Log} 2, \quad a_j \geqslant (2e/D)|\log\alpha_j|$$

for $j = 1$ and $j = 2$.

Then Theorem 5.11 implies the following result.

COROLLARY 1.1. *If α_1 and α_2 are multiplicatively independent, we have*

$$|b_1 \log\alpha_1 - b_2 \log\alpha_2| \geqslant \exp\{-500 D^4 a_1 a_2 (7.5 + \operatorname{Log} B)^2\}.$$

We shall deduce this result from Theorem 5.11 in Section 8.

Let us compare our estimate with the lower bound which is derived from Baker's method:

$$|\Lambda| > \exp\{-C_1 D^4 a_1 a_2 \operatorname{Log} B\},$$

where C_1 is an absolute constant. The best numerical value for C_1 which has been computed using Baker's method [2] is greater than $6 \cdot 10^9$. Hence our result is better for $B < \exp(10^7)$. In particular, for computational purposes (see [4], [3], [8] for instance), our estimate will be appropriate.

The fact that Schneider's method yields smaller numerical constants than Baker's was already pointed out in [7]. However, the constants in [7] are not less than $4 \cdot 10^6$ (but it works also for algebraic β).

The assumption that α_1, α_2 are multiplicatively independent is easy to remove, and we plan to do it in a further paper, where we study $|\beta \log\alpha - i\pi|$.

The constants $2e$ and 7.5 which appear in the statement of Corollary 1.1 could be changed without altering too much the main constant 500. In fact one should stress the point that, for any specific example where the actual numerical values of the constants are relevant, the best estimate is achieved by using our Theorem 5.11 below rather than Corollary 1.1.

It would be interesting to extend our estimate to linear forms in n logarithms for $n \geqslant 3$. However, the natural generalization of our method, involving Schneider's method in several variables, yields an estimate with $(\operatorname{Log} B)^N$, $N = n(n-1)$, while Baker's method gives $\operatorname{Log} B$ with the exponent 1. Even if the constants were less than 10^3 (which does not seem to be the case, partly because no interpolation formula is available yet), a result with the factor $(\operatorname{Log} B)^N$ would not be sharper than the results of [2] even for $n = 3$.

Our result could be translated in the p-adic case, but we did not compute the constant in this case (and neither the dependence in p). As far as Baker's method is concerned, lower bounds for linear forms in p-adic logarithms have been produced by Gel'fond, Schinzel, Kaufman, Sprindžuk, van der Poorten, and more recently by Yu Kunrui [10], but the best constants for linear forms in two logarithms are still bigger than 10^{11}.

The fact that we get a sharper estimate than in our previous work [7] comes from two facts. Firstly, we produce a new interpolation formula (§ 3 below) which we combine with sharp estimates for some finite products. Secondly, we use a zero estimate (§ 4 below) which was shown to us by D. W. Masser shortly after [7] was published. In the mean time, many other zero estimates have been proved, but none of them includes Masser's result (Proposition 4.1 below).

In Section 2, we collect several lemmas from different sources. The third section contains an interpolation formula. The above mentioned zero estimate due to David Masser is given in Section 4. We prove the main result in Section 5. The rest of the paper is devoted to applications of this result.

2. Auxiliary lemmas. We keep the lemmas used in [7] except for the following results. The next one will be used in place of Lemma 4 of [7].

LEMMA 2.1 (Siegel's lemma). *Let $\alpha_1, \ldots, \alpha_q$ be algebraic numbers of exact degrees $d_1, \ldots, d_q$, respectively. Define $D = [\mathbf{Q}(\alpha_1, \ldots, \alpha_q):\mathbf{Q}]$. Let*

$$P_{i,j} \in \mathbf{Z}[X_1, \ldots, X_q] \quad (1 \leqslant i \leqslant \nu,\ 1 \leqslant j \leqslant \mu)$$

be polynomials (not all zero) of degree at most $N_{j,h}$ in X_h (for $1 \leqslant h \leqslant q$). Define

$$L_j = \sum_{i=1}^{\nu} L(P_{i,j})$$

and

$$\gamma_{i,j} = P_{i,j}(\alpha_1, \ldots, \alpha_q) \quad (1 \leqslant i \leqslant \nu,\ 1 \leqslant j \leqslant \mu).$$

If $\nu > \mu D$, then there exist rational integers $x_1, \ldots, x_\nu$, not all of which are zero, such that

$$\sum_{i=1}^{\nu} \gamma_{i,j} x_i = 0 \quad (1 \leqslant j \leqslant \mu),$$

and

$$\max|x_i| \leqslant (2^\mu (V_1 \ldots V_\mu)^D)^{1/(\nu - \mu D)},$$

where

$$V_j = L_j \prod_{h=1}^{q} M(\alpha_h)^{N_{j,h}/d_h}.$$

Proof. Apply [5], Lemma 1.

The following lemma replaces Lemma 8 of [7].

LEMMA 2.2. *Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers of absolute heights at most $h_1, \ldots, h_n$ respectively. If $b_1, \ldots, b_n$ are rational integers such that the number*

$$\Lambda = b_1 \log\alpha_1 + \dots + b_n \log\alpha_n$$

(where $\log\alpha_i$ *is any determination of the logarithm of* α_i, $1 \leqslant i \leqslant n$*) is non-zero, then*

$$|\Lambda| > 2^{-D}\exp\bigl(-D(|b_1|h_1 + \dots + |b_n|h_n)\bigr)$$

where $D = [\boldsymbol{Q}(\alpha_1, \dots, \alpha_n):\boldsymbol{Q}]$.

Proof. We may suppose $|\Lambda| \leqslant 1/2$. Then $\Lambda \notin 2\pi i\boldsymbol{Z}$ and the number

$$\zeta = \alpha_1^{b_1} \dots \alpha_n^{b_n} - 1$$

is non-zero. Without loss of generality, we may suppose $b_j \geqslant 0$ for $1 \leqslant j \leqslant r$ and $b_j \leqslant 0$ for $r < j \leqslant n$. Liouville's estimate (see Lemma 2.3 below) applied to ζ considered as a polynomial in $\alpha_1, \dots, \alpha_r, \alpha_{r+1}^{-1}, \dots, \alpha_n^{-1}$ leads to the lower bound

$$|\zeta| \geqslant 2^{-D+1}\exp\bigl(-D(h_1|b_1| + \dots + h_n|b_n|)\bigr).$$

The lemma follows using the inequality $|e^z - 1| < 2|z|$ which is true for $0 < |z| \leqslant 1/2$.

Here is the Liouville estimate taken from [7], Lemma 3.

LEMMA 2.3. *Let* $\alpha_1, \dots, \alpha_q$ *be algebraic numbers of exact degree* $d_1, \dots, d_q$ *respectively. Define* $D = [\boldsymbol{Q}(\alpha_1, \dots, \alpha_q):\boldsymbol{Q}]$. *Let* $P \in \boldsymbol{Z}[X_1, \dots, X_q]$ *have degree at most* N_h *in* X_h $(1 \leqslant h \leqslant q)$, *and length* $L(P)$. *If* $P(\alpha_1, \dots, \alpha_q) \neq 0$ *then*

$$|P(\alpha_1, \dots, \alpha_q)| \geqslant L(P)^{1-D} \prod_{h=1}^{q} M(\alpha_h)^{-DN_h/d_h}.$$

LEMMA 2.4. *For a complex number* z, *and a rational integer* h, $0 \leqslant h \leqslant L_0$, *define a polynomial* Δ_h *of degree* h *by* $\Delta_0(z) = 1$ *and*

$$\Delta_h(z) = z(z-1)(z+1)\dots\bigl(z+(-1)^{h+1}[h/2]\bigr)/h!$$
$$= (1/h!)\prod_i (z+i), \quad -h/2 \leqslant i \leqslant (h-1)/2.$$

Then, for $|z| \leqslant R$,

$$|\Delta_h(z)| \leqslant R(R^2+1)(R^2+2^2)\dots\bigl(R^2+[(h-1)/2]^2\bigr)(R+[h/2])^{2[h/2]+1-h}/h!.$$

Moreover, when x *is real, then*

$$|\Delta_h(x)| \leqslant 2X^h/h!, \quad \text{where } X = \max\{|x|, h/2\}.$$

Proof. Put $h = 2h' + \varepsilon$, $\varepsilon \in \{0, 1\}$; so that $h' = [h/2]$. Then

$$h!\,|\Delta_h(z)| = \begin{cases} |z|\,|z^2-1|\dots|z^2-h'^2| & \text{if } \varepsilon = 1,\\ |z|\,|z^2-1|\dots|z^2-(h'-1)^2|\,|z-h'| & \text{if } \varepsilon = 0.\end{cases}$$

This leads at once to the first estimate. To get the second one, notice that, for x real and $k \in \boldsymbol{N}$, $|x^2 - k^2| \leqslant (\max\{|x|, k\})^2$.

COROLLARY 2.5. *For a complex number* z, $|z| \leqslant R$, *and* $h \in \boldsymbol{Z}$, $0 < h \leqslant L_0$,

$$|\Delta_h(z)| \leqslant \bigl(e(R+h/2)/h\bigr)^h \leqslant \bigl(e(R/L_0+1/2)\bigr)^{L_0}.$$

Since $h^h \leqslant e^h h!$, the first inequality is an immediate consequence of Lemma 2.3.

The second one is implied by the fact that $h \to h\bigl(1+\mathrm{Log}\bigl((R+h/2)/h\bigr)\bigr)$ is a non-decreasing function for $h > 0$.

We need also to estimate the denominator of the rational number $\Delta_h(a/b)$ for $a/b \in \boldsymbol{Q}$. It was pointed out to us by Dong Ping Ping that this denominator can be larger than b^h.

LEMMA 2.6. *Let* a *and* b *be non-zero rational integers. Put*

$$\Omega(b, h) = \prod_{p|b} p^{[h/(p-1)]} \quad (p \text{ prime}).$$

Then the number $b^h\Omega(b, h)\Delta_h(a/b)$ *is a rational integer, for any* $h \geqslant 0$.

Proof. By the definition of Δ_h, we have $\Delta_h(a/b) = b^{-h}.\,(\prod_i N_i)/h!$, where

$$N_i = a + ib, \quad -h/2 \leqslant i \leqslant (h-1)/2.$$

If m is a number relatively prime to b, then the function $i \bmod m \to (a+ib) \bmod m$ is one-to-one. This implies that if p is a prime number, and if k is a positive integer, then the number of integers N_i as above divisible by p^k is at least $[h/p^k]$. It follows that p does not divide the denominator of the rational number $\Delta_h(a/b)$. (Use the fact that $h!$ satisfies $v_p(h!) = \sum_{k\geqslant 1}[h/p^k]$, where — as usual — $v_p(x)$ is the highest exponent j such that p^j divides the integer x.)

If p is a prime number which divides b then clearly,

$$v_p\bigl(\Delta_h(a/b)\bigr) \geqslant -hv_p(b) - v_p(h!),$$

where $v_p(x/y) = v_p(x) - v_p(y)$, when x and y are non-zero integers.

The result follows, since $v_p(h!) = \sum_k [h/p^k] \leqslant [h/(p-1)]$.

LEMMA 2.7. *For a positive rational integer* b *put*

$$\omega(b) = \sum_{p|b} \frac{\mathrm{Log}\,p}{p-1}.$$

Then, for $b \geqslant 2$,

$$\omega(b) < \max\{2.21, 0.09 + \mathrm{Log}\,\mathrm{Log}\,b\}.$$

We will use the fact that, for $x \geqslant 19$, we have

$$\sum_{p\leqslant x} \frac{\mathrm{Log}\,p}{p-1} < \mathrm{Log}\Bigl(\sum_{p\leqslant x} \mathrm{Log}\,p\Bigr). \tag{2.8}$$

The proof of (2.8) follows easily (see below) from formulae (2.11), (3.23), and Theorem 10 of [9] together with the estimate

(2.9) $$F = \sum_{p \geqslant 2} \frac{\operatorname{Log} p}{p(p-1)} < 1.$$

Let $q_1, \ldots, q_k$ be the different prime divisors of b, and let $p_1, \ldots, p_k$ be the k first prime numbers. Then, since the function $(\operatorname{Log} x)/(x-1)$ is decreasing for $x \geqslant 2$, we have

(2.10) $$\omega(b) = \sum_{i=1}^{k} \frac{\operatorname{Log} q_i}{q_i - 1} \leqslant \sum_{i=1}^{k} \frac{\operatorname{Log} p_i}{p_i - 1} = u_k \quad \text{(say)}.$$

The inequalities (2.10) and (2.8) imply

(2.11) $$\omega(b) < \operatorname{Log}\operatorname{Log} b, \quad \text{if } k \geqslant 8.$$

Inequality (2.10) and the computation of u_k give the following estimates

(2.12) $$\omega(b) < 2.6 < 0.03 + \operatorname{Log}\operatorname{Log} b, \quad \text{if } k = 7,$$

(2.13) $$\omega(b) < 2.43 < 0.09 + \operatorname{Log}\operatorname{Log} b, \quad \text{if } k = 6,$$

(2.14) $$\omega(b) < 2.21, \quad \text{if } k < 6.$$

This proves the lemma.

Now we prove (2.8). By numerical computation one verifies that (2.8) is true for x in the range $19 \leqslant x \leqslant 349$. For $x \geqslant 349$, inequality (3.23) of [9] gives

$$\sum_{p \leqslant x} \frac{\operatorname{Log} p}{p} < \operatorname{Log} x + E + \frac{1}{2 \operatorname{Log} x},$$

with $E = -1.33258\ldots$ ([9], (2.11)), and also ([9], Theorem 10)

$$\operatorname{Log} \sum_{p \leqslant x} \operatorname{Log} p \geqslant \operatorname{Log} x + \operatorname{Log}(0.91) > \operatorname{Log} x - 0.095.$$

Thus, for $x \geqslant 349$:

$$\sum_{p \leqslant x} \frac{\operatorname{Log} p}{p-1} \leqslant \operatorname{Log} \sum_{p \leqslant x} \operatorname{Log} p + 0.095 + \frac{1}{2 \operatorname{Log} 349} + E + F,$$

and the result follows from (2.9).

To conclude, we give a quick proof of (2.9):

$$F < \sum_{p \leqslant 19} \frac{\operatorname{Log} p}{p(p-1)} + \int_{22}^{\infty} \frac{\operatorname{Log} x}{x(x-1)} dx < 0.714 + \int_{21}^{\infty} \frac{\operatorname{Log}(x+1)}{x^2} dx < 1,$$

since $\operatorname{Log}(x+1) < x^{0.4}$ for $x \geqslant 21$ and $21^{-0.6}/0.6 < 0.27$.

3. Interpolation formula.

This section contains some technical estimates which we shall use in the extrapolation step. We replace Lemma 6 of [7] by the following result.

LEMMA 3.1. *Let f be a function analytic on the disk $|z| \leqslant R$ and $z_0, \ldots, z_n$ points interior to this disk. Then*

$$|f(z_0)| \leqslant E_1 + E_2$$

where

$$E_1 = |f|_R (R/(R - |z_0|)) \prod_{j=1}^{n} (R|z_0 - z_j| / |R^2 - z_0 \bar{z}_j|)$$

and

$$E_2 = \sum_{j=1}^{n} |f(z_j)| \Big(\prod_{i=1}^{n} |(R^2 - z_j \bar{z}_i)/(R^2 - z_0 \bar{z}_i)|\Big)\Big(\prod_{i \neq j} |(z_0 - z_i)/(z_j - z_i)|\Big)$$

(as usual $|f|_R = \max\{|f(z)|;\ |z| \leqslant R\}$).

Proof. Consider the product of Blaschke factors

$$B(z) = \prod_{j=1}^{n} \frac{R^2 - z\bar{z}_j}{R(z - z_j)}.$$

By Cauchy's residue formula

$$B(z_0) f(z_0) = \frac{1}{2i\pi} \int_C \frac{B(\zeta) f(\zeta)}{\zeta - z_0} d\zeta + \sum_{j=1}^{n} \frac{B_j(z_j) f(z_j)}{z_0 - z_j},$$

where C denotes the circle $|\zeta| = R$, and

$$B_j(z) = (z - z_j) B(z), \quad 1 \leqslant j \leqslant n.$$

On this circle $|B(\zeta)| = 1$, and the lemma follows.

Lemma 7 of [7] is replaced by

LEMMA 3.2. *Let β be a rational number, $\beta = b_1/b_2$, $b_1, b_2 \in \mathbf{Z}$, $(b_1, b_2) = 1$. Let U and V be two positive integers. Put*

$$\Gamma = \{u + v\beta;\ (u, v) \in \mathbf{Z} \times \mathbf{Z},\ |u| \leqslant U,\ |v| \leqslant V\}$$

and

$$\Delta = \min_{\gamma \in \Gamma} \prod_{\gamma' \in \Gamma, \gamma' \neq \gamma} |\gamma' - \gamma|.$$

We suppose

(H) *the points $(u + v\beta)$, $|u| \leqslant 2U$ and $|v| \leqslant 2V$, are pairwise distinct.*

Then we have

$$\Delta \geqslant (V!)^2 b_2^{-2V} (U!)^{2(2V+1)} \exp\{-(7\pi^2/54)(V+1)^3 b_2^{-2}\}.$$

Proof. Let $\gamma_0 = u_0 + v_0\beta$ be a point of Γ where the minimum of Δ is attained. For v in $\mathbf{Z}$, $|v| \leqslant V$, put

$$\Delta_v = \begin{cases} \prod_{|u| \leqslant U} |u-u_0+\beta(v-v_0)| & \text{if } v \neq v_0, \\ \prod_{u \neq u_0} |u-u_0| & \text{if } v = v_0, \end{cases}$$

so that

$$\Delta = \prod_{|v| \leqslant V} \Delta_v.$$

Notice that if $v = v_0$ then

$$\Delta_v = (U-u_0)!(U+u_0)! \geqslant (U!)^2.$$

For $v \neq v_0$ fixed let $x = x_v$ be the minimum of $|u-u_0+\beta(v-v_0)|$, then Δ_v is a product of the form

$$\Delta_v = x(x+1)\dots(x+A)|1-x|\dots|A'-x|, \quad \text{where } A+A' = 2U.$$

Consider the two cases: (i) $A' = 0$, (ii) $A' > 0$. In the first case,

$$\Delta_v \geqslant x(2U!) \geqslant x(U!)^2.$$

Whereas, in case (ii) we have $0 < x_v \leqslant 1/2$ and it is easy to verify that the lowest value is when $A = A'$, so that

$$\Delta_v \geqslant x(1-x^2)(2^2-x^2)\dots(U^2-x^2).$$

Using the expansion of the function $\mathrm{Log}(1+y)$, we get

$$\mathrm{Log}(a^2-x^2) \geqslant \mathrm{Log}\, a^2 - (7/6)x^2a^{-2} \quad \text{for } a \geqslant 1;$$

this leads to

$$\Delta_v \geqslant x(U!)^2 \exp\{-x^2 7\pi^2/36\}.$$

Notice that the right-hand side is an increasing function of x.

Now, hypothesis (H) implies that each value of x_v can be obtained at most twice. Besides each value of x is equal to p/b_2 for some rational positive integer p. Thus,

$$\Delta \geqslant (V!)^2 b_2^{-2V} (U!)^{2(2V+1)} \exp\{-(14\pi^2/36)(1+2^2+\dots+V^2)b_2^{-2}\}.$$

And we obtain

$$\Delta \geqslant (V!)^2 b_2^{-2V} (U!)^{2(2V+1)} \exp\{-(7\pi^2/54)(V+1)^3 b_2^{-2}\}$$

because $1+2^2+\dots+n^2 < (n+1)^3/3$ for $n \geqslant 0$.

LEMMA 3.3. *Let x and y and a be positive numbers such that $x, y \leqslant X \leqslant a$, then*

$$\frac{|x-y|}{a^2-xy} \leqslant Xa^{-2},$$

if $xy \neq a^2$.

Proof. Without loss of generality, we may suppose $x \geqslant y$.
Let f be the left-hand side. Then

$$\frac{\partial f}{\partial x} = \frac{a^2-y^2}{(a^2-xy)^2} \geqslant 0, \quad \frac{\partial f}{\partial y} = \frac{x^2-a^2}{(a^2-xy)^2} \leqslant 0;$$

so that, in the domain considered,

$$\max f(x,y) = f(X,0) = Xa^{-2}.$$

COROLLARY 3.4. *Take again the notations of Lemma 3.2. Consider $R_1 \geqslant U+V|\beta|$, a real number x, $|x| \leqslant R_1 < R$. Then*

$$\prod_{\gamma \in \Gamma} \frac{R|x-\gamma|}{R^2-x\gamma} \leqslant (R_1/R)^{(2U+1)(2V+1)},$$

and

$$\sum_{\gamma \in \Gamma} \Big(\prod_{\gamma' \in \Gamma} \frac{R^2-\gamma\gamma'}{R^2-x\gamma'} \Big) \Big(\prod_{\gamma' \neq \gamma} \frac{|x-\gamma'|}{|\gamma-\gamma'|} \Big) \leqslant (2U+1)(2V+1)\Delta^{-1} \frac{2R^4}{R^4-R_1^4} R_1^{(2U+1)(2V+1)-1}.$$

Proof. Notice that $\gamma \in \Gamma$ implies $-\gamma \in \Gamma$. This remark shows that the first expression we want to estimate is equal to

$$\frac{|x|}{R} \prod_{\gamma \in \Gamma^*} \frac{R|x^2-\gamma^2|^{1/2}}{(R^4-x^2\gamma^2)^{1/2}}, \quad \text{where } \Gamma^* = \Gamma - \{0\}.$$

And the first inequality follows at once from Lemma 3.3.

The same remark shows that the second expression can be written

$$\Big(\prod_{\gamma' \in \Gamma^*} \frac{R^2}{R^2-x\gamma'} \Big) \Big(\prod_{\gamma' \in \Gamma^*} \frac{|x-\gamma'|}{|\gamma'|} \Big) + \sum_{\gamma \in \Gamma^*} \Big(\prod_{\gamma' \neq \gamma} |\gamma-\gamma'| \Big)^{-1} \frac{R^4-\gamma^4}{R^4-x^2\gamma^2} |x+\gamma| \Big(\prod_{\gamma' \neq \pm\gamma} \Big| \frac{(R^4-\gamma^2\gamma'^2)(x^2-\gamma'^2)}{R^4-x^2\gamma'^2} \Big| \Big)^{1/2}.$$

Applying again Lemma 3.3, we see that this expression is bounded above by

$$\Delta^{-1} R_1^{(2U+1)(2V+1)-1} + \sum_{\gamma' \in \Gamma^*} \Delta^{-1} \frac{2R^4}{R^4-R_1^4} R_1^{(2U+1)(2V+1)-1}.$$

And the second estimate follows easily.

4. Zero estimate. This section is essentially the content of a letter by D. W. Masser of June 19, 1979.

4.1. The result. This is a result intermediate between those of [1] and [6].

Let α, β, γ be complex numbers, $\alpha\gamma \neq 0$, and L, M, U, V be positive integers. We want to prove that, under suitable hypotheses, if a non-zero polynomial $P \in C[X, Y]$ is of degree at most L in X and at most M in Y, then at least one of the numbers

$$P(u+v\beta, \alpha^u\gamma^v), \qquad -U \leqslant u \leqslant U, \ -V \leqslant v \leqslant V, \ (u, v) \in Z^2,$$

is not zero.

For that it is necessary to add the following hypotheses

(i) the numbers L, M, U, V have to verify

$$(2U+1)(2V+1) \geqslant (L+1)(M+1),$$

and even we must have

$$\operatorname{Card}\{(u+v\beta, \alpha^u\gamma^v); \ -U \leqslant u \leqslant U, \ -V \leqslant v \leqslant V\} \geqslant (L+1)(M+1)$$

(if this second condition is not satisfied then a counterexample can be obtained using only linear algebra).

(ii) moreover the set of the first coordinates has to be rather big:

$$\operatorname{Card}\{u+v\beta; \ -U \leqslant u \leqslant U, \ -V \leqslant v \leqslant V\} \geqslant L+1,$$

(if not, then a polynomial P can be constructed which does not depend on Y).

(iii) the same must be true for the set of the second coordinates:

$$\operatorname{Card}\{\alpha^u\gamma^v; \ -U \leqslant u \leqslant U, \ -V \leqslant v \leqslant V\} \geqslant M+1.$$

PROPOSITION 4.1. *Let U_1, U_2, V_1, V_2 be positive integers. Put $U = U_1+U_2$ and $V = V_1+V_2$.*

We suppose that

(a) *the points*

$$u+v\beta \quad (-U_1 \leqslant u \leqslant U_1, \ -V_1 \leqslant v \leqslant V_1)$$

are pairwise distinct, and

$$(2U_1+1)(2V_1+1) > L;$$

(b)

$$\operatorname{Card}\{\alpha^u\gamma^v; \ -U_1 \leqslant u \leqslant U_1, \ -V_1 \leqslant v \leqslant V_1\} > M;$$

(c)

$$\operatorname{Card}\{u+v\beta; \ -U_2 \leqslant u \leqslant U_2, \ -V_2 \leqslant v \leqslant V_2\} > 2LM.$$

Then at least one of the numbers

$$P(u+v\beta, \alpha^u\gamma^v) \quad (-U \leqslant u \leqslant U, \ -V \leqslant v \leqslant V)$$

is non-zero.

4.2. Two preliminary lemmas. The first lemma is classical, it is obtained by the technique of Kronecker's U-resultant.

LEMMA 4.2. *Let $F_1, \ldots, F_r$ be polynomials in $C[X, Y]$, of degree at most L in X and at most M in Y, without any non trivial common divisor in the ring $C[X, Y]$. Let (ξ_i, η_i), $1 \leqslant i \leqslant N$, be common zeros to $F_1, \ldots, F_r$ in C^2, with $\xi_1, \ldots, \xi_N$ pairwise distinct. Then $N \leqslant 2LM$.*

Proof. Introduce the $2r$ new variables $U_1, \ldots, U_r, V_1, \ldots, V_r$. Then define the two polynomials G and H in the ring $A = C[U_1, \ldots, U_r, V_1, \ldots, V_r, X, Y]$ by

$$G = \sum_{j=1}^{r} U_j F_j(X, Y), \qquad H = \sum_{j=1}^{r} V_j F_j(X, Y).$$

Let $R \in C[U_1, \ldots, U_r, V_1, \ldots, V_r, X]$ be the resultant of the polynomials G and H with respect to the variable Y.

Then the following is true:

(α) $R \neq 0$. Indeed, suppose R is zero, then G and H have a common irreducible factor Q in the factorial ring A. Since G belongs to $C[U_1, \ldots, U_r, X, Y]$ the polynomial Q is one of the irreducible factors of G in this ring, hence it does not depend on $V_1, \ldots, V_r$. In the same way, the polynomial Q does not depend on $U_1, \ldots, U_r$. Then $Q \in C[X, Y]$ is a common factor of the polynomials $F_1, \ldots, F_r$. Contradiction.

(β) $\deg_X R \leqslant 2LM$.

(γ) $R(U_1, \ldots, U_r, V_1, \ldots, V_r, \xi_i) = 0$ for $i = 1, \ldots, N$, because R is a linear combination of G and H with coefficients in the ring A.

The lemma follows easily from these three properties.

LEMMA 4.3. *Let $Q \in C[X, Y]$ be a polynomial, and α, β, λ be three complex numbers, with $\alpha\beta \neq 0$. Assume that*

$$Q(X+\beta, \alpha Y) = \lambda Q(X, Y). \tag{4.4}$$

Then Q belongs to $C[Y]$.

Proof. We first notice that if Q belongs to $C[X]$ and satisfies $Q(X+\beta) = \lambda Q(X)$ with $\beta \neq 0$, then Q is a constant (otherwise it would have infinitely many zeros).

Write

$$Q(X, Y) = \sum_{i=0}^{d} a_i(X) Y^i.$$

From (4.4) we deduce

$$a_i(X+\beta)\alpha^i = \lambda a_i(X) \qquad (0 \leqslant i \leqslant d),$$

hence a_i is a constant for all i, $0 \leqslant i \leqslant d$, and Q belongs to $C[Y]$.

Remark. It is easy to find all the polynomials $Q \in C[Y]$ satisfying the relation

$$Q(\alpha Y) = \lambda Q(Y).$$

We need only to know that if Q satisfies this equation with $\alpha \neq 1$ and is irreducible, then $Q(Y) = aY$ for some non-zero complex number a. Indeed we can write

$$Q(Y) = aY + b, \quad a \neq 0.$$

From the relation $a\alpha Y + b = \lambda(aY + b)$, one deduces $\lambda = \alpha$ (because $a \neq 0$), and $b = 0$ (because $\lambda = \alpha \neq 1$).

4.3. Proof of Proposition 4.1. Obviously, we may suppose that Y does not divide P, and that $P \notin \boldsymbol{C}[X]$.

1° We prove that the polynomials

$$P(X + u + v\beta, \alpha^u \gamma^v Y) \quad (-U_1 \leqslant u \leqslant U_1, \ -V_1 \leqslant v \leqslant V_1)$$

do not have any non-trivial common divisor.

Let

$$P = c \prod_{i=1}^{k} Q_i^{r_i}$$

be a decomposition of P into irreducible factors, where $Q_i \in \boldsymbol{C}[X]$ for $i = 1, \ldots, h$ and $Q_i \notin \boldsymbol{C}[X]$ for $i = h+1, \ldots, k$ (notice that $h \leqslant L$ and $k - h \leqslant M$). Then

$$P(X + u + v\beta, \alpha^u \gamma^v Y) = c \prod_{i=1}^{k} Q_i(X + u + v\beta, \alpha^u \gamma^v Y)^{r_i}$$

is a decomposition into irreducible factors. If there exists an irreducible polynomial Q which divides all the polynomials $P(X + u + v\beta, \alpha^u \gamma^v Y)$ $(-U_1 \leqslant u \leqslant U_1, \ -V_1 \leqslant v \leqslant V_1)$, for each (u, v) there exists an index $i = i(u, v)$ (where $1 \leqslant i \leqslant h$ if $Q \in \boldsymbol{C}[X]$ and on the contrary $i > h$ if not), and a non-zero complex number $c_{u,v}$ such that

$$Q(X, Y) = c_{u,v} Q_i(X + u + v\beta, \alpha^u \gamma^v Y).$$

From hypotheses (a) and (b) we deduce that the number of (u, v) is

$$(2U_1 + 1)(2V_1 + 1) > \max\{L, M\} \geqslant \max\{h, k-h\}.$$

Thus, thanks to condition (b), there exist two different pairs of indices (u, v) and (u', v'), with either $Q \in \boldsymbol{C}[X]$ or $\alpha^{u-u'} \gamma^{v-v'} \neq 1$, for which the two indices $i(u, v)$ and $i(u', v')$ are equal. Then condition (a) gives $u + v\beta \neq u' + v'\beta$, and there exists a $\lambda \in \boldsymbol{C}$ such that

$$Q(X + (u-u') + (v-v')\beta, \ \alpha^{u-u'} \gamma^{v-v'} Y) = \lambda Q(X, Y).$$

Since Y does not divide Q, Lemma 4.3 gives the desired contradiction.

2° The application of Lemma 4.2 to the set of polynomials

$$\{F_1, \ldots, F_r\} = \{P(X + u_1 + v_1\beta, \alpha^{u_1}\gamma^{v_1} Y); \ -U_1 \leqslant u_1 \leqslant U_1, \ -V_1 \leqslant v_1 \leqslant V_1\}$$

and to the points

$$\{(\zeta_i, \eta_i); \ 1 \leqslant i \leqslant N\} = \{(u_2 + v_2\beta, \ \alpha^{u_2}\gamma^{v_2}); \ -U_2 \leqslant u_2 \leqslant U_2, \ -V_2 \leqslant v_2 \leqslant V_2\}$$

with $N \leqslant (2U_2 + 1)(2V_2 + 1)$ gives the conclusion.

Remarks. 1. It is easy to see that the assumption $\xi_1, \ldots, \xi_N$ pairwise distinct in Lemma 4.2 can be replaced by the weaker assumption $(\zeta_1, \eta_1), \ldots, (\zeta_N, \eta_N)$ pairwise distinct. Therefore one can replace condition (c) in Proposition 4.1 by

$$\operatorname{Card}\{(u + v\beta, \ \alpha^u\gamma^v); \ |u| \leqslant U_2, \ |v| \leqslant U_2\} > 2LM.$$

But we shall not use this remark.

2. It would be very interesting to know whether it is possible to improve the constant 2 on the right-hand side of condition (c).

5. The main result.

5.1. Common notations and hypotheses for Sections 5, 6 and 7. Let α_1, α_2 be two non-zero algebraic numbers of respective degrees equal to D_1 and D_2, the total degree of the field we are working in is $D = [\boldsymbol{Q}(\alpha_1, \alpha_2) : \boldsymbol{Q}]$, $\log \alpha_j$ is any non-zero determination of the logarithm of α_j, $l_j = |\log \alpha_j|$, $j = 1, 2$. Moreover let $\beta = b_1/b_2$ be a rational number, $b_1, b_2 \in \boldsymbol{Z}$, $0 < b_1, b_2$, $(b_1, b_2) = 1$, such that

$$\Lambda = \beta \log \alpha_1 - \log \alpha_2$$

does not vanish.

We put $B = \max\{b_1, b_2\}$.

We denote by $a_1, a_2, G, G', Z, \theta, f$ positive real numbers which satisfy the following relations:

$$a_1 b_1 \leqslant a_2 b_2,$$

$$f \geqslant 1, \quad \theta \geqslant 1,$$

$$a_j \geqslant 1, \quad a_j \geqslant h(\alpha_j) \quad \text{and} \quad a_j \geqslant f D^{-1} l_j, \quad j = 1, 2,$$

$$G' \geqslant \operatorname{Log}(e/2 + 2e l_1^{-1}),$$

$$G \geqslant \operatorname{Log} B + \operatorname{Log}\operatorname{Log} B + \max\{1, 0.59 + G'/D\},$$

$$Z \leqslant \min\{DG/\theta, \ Da_1, \ Da_2, \ \operatorname{Log}(2e a_1 a_2 D / f(a_2 l_1 + a_1 l_2))\};$$

furthermore we assume

(5.0) $$D^3 a_1 a_2 G^2 Z^{-3} \geqslant 2(D-1)(a_1 + a_2).$$

We put

$$\varepsilon = Z/\big(\mathrm{Log}(2ea_1a_2D/f(a_2l_1+a_1l_2))\big) \quad \text{(so that } \varepsilon \leqslant 1)$$

and

$$U = D^4a_1a_2G^2Z^{-3}.$$

Notice that (5.0) and the conditions on Z imply

$$U \geqslant \max\{\theta DG, 2D(D-1)(a_1+a_2), \theta^2Da_1, \theta^2Da_2\};$$

this inequality will be used implicitly in the sequel.

5.2. Notations and hypotheses for Sections 5 and 6. We assume $Z \geqslant 1$ and $\theta \geqslant 10$. Put $v = 1-\delta_{1D}$ (= if D equals 1 then 0 else 1). Let c_0, c_1, c, χ_1, χ_2, χ, C, η, μ, ϱ, p, ξ be positive real numbers. Assume

$$15 \leqslant c_0 \leqslant 290, \quad 1 \leqslant c_1 \leqslant 4.8, \quad 5.5c \leqslant c_0 - 1/\theta,$$

(5.1) $$2c_1 + c_0/c\theta + 2(1-1/\theta c_0) \leqslant 4c\xi,$$

$$3 \leqslant c \leqslant 17, \quad 1 \leqslant \chi \leqslant 2.5, \quad (2c-1/\theta)\xi \geqslant c_1,$$

(5.2) $$\left(c_0-\frac{1}{\theta}\right)\left(c_1-\frac{1}{\theta}\right) \geqslant 2\left(c+\frac{1}{\theta}\right)^2,$$

(5.3) $$\eta \geqslant \max\left\{\frac{(2c+1/\theta)^2}{c_0(2c_1-1/\theta)-(2c+1/\theta)^2}, \frac{1}{2}\right\}, \quad \varrho = \frac{(2c+1/\theta)^2}{2(c_0-1/\theta)(c_1-1/\theta)},$$

(Notice that (5.2) implies that η is positive and $0 < \varrho < 1$.)

(5.4) $$p \geqslant \eta\left\{\frac{c_1}{\theta}+c_0+2cc_1+0.6v\right\}+\frac{v}{2f}+0.1v,$$

(5.5) $$4c^2\left(1-\frac{1}{2\theta c}\right)^2\xi \geqslant p+4cc_1+c_0+\frac{v}{2},$$

(5.6) $$C \geqslant p+c_0+(4\chi^2c^2/Z)\,\mathrm{Log}(2e)+\chi c(2+5c_1+c_1/f)+1,$$

(Notice that (5.1) and (5.6) imply $C > c_0+5\chi cc_1 > 10c$.)

(5.7) $$0 < \xi \leqslant \varepsilon^{-1}+\frac{1}{Z}\mathrm{Log}\frac{2cf(1-1/2\theta c)^2}{Zc_1e^2}-\frac{e^{-C}}{Z},$$

(5.8) $$\chi_2 = \frac{\sqrt{c_0c_1}}{c}, \quad \chi = \chi_1+\chi_2,$$

(5.9) $$\chi > \frac{\sqrt{c_0c_1}}{c}+\frac{1+\sqrt{c_0\theta}}{2c\theta},$$

(5.10) either α_1 and α_2 are multiplicatively independent or $\chi > \dfrac{\sqrt{c_0c_1}}{c}+\dfrac{1}{\theta c}+\dfrac{c_1}{c}$.

Finally, we put $\mu = \omega(b_2)/\mathrm{Log}\,B$ and we suppose $\mathrm{Log}\,B \geqslant 10$. Then Lemma 2.7 implies

$$\mu\,\mathrm{Log}\,B < 0.09+\mathrm{Log}\,\mathrm{Log}\,B.$$

5.3. Statement of the main result.

THEOREM 5.11. *Under the above hypotheses, we have* $|\Lambda| > e^{-CU}$.

All the rest of Section 5 is devoted to the proof of this inequality. Therefore we assume $\mathrm{Log}|\Lambda| \leqslant -CU$ and we shall eventually reach a contradiction.

5.4. The parameters. We define L_0, L_1, M_1, M_2 by

$$L_0 = [c_0D^3a_1a_2GZ^{-3}], \quad L_1 = [c_1DGZ^{-1}],$$
$$M_1 = [cD^2Ga_2Z^{-2}], \quad M_2 = [cD^2Ga_1Z^{-2}].$$

We will often use the following inequalities

(5.12) $$L_0 \geqslant (c_0-1/\theta)D^3a_1a_2GZ^{-3}, \quad L_1 \geqslant (c_1-1/\theta)DGZ^{-1},$$
$$M_1 \geqslant (c-1/\theta)D^2Ga_2Z^{-2}, \quad M_2 \geqslant (c-1/\theta)D^2Ga_1Z^{-2},$$

which are all consequences of the definition of Z.

Notice also that

(5.13) $$2M_1+1 \leqslant (2c+1/\theta)D^2Ga_2Z^{-2}, \quad 2M_2+1 \leqslant (2c+1/\theta)D^2Ga_1Z^{-2}.$$

We claim that the numbers $u+v\beta$ ($|u| \leqslant 4M_1$, $|v| \leqslant 4M_2$) are pairwise distinct (here and in the sequel the letters u and v represent rational integers). Otherwise $b_1 < 8M_1$ and $b_2 < 8M_2$, hence by Lemma 2.2 and the definition of the a_i's

(*) $$|\Lambda| \geqslant 2^{-D}\exp(-b_1Da_1-b_2Da_2)B^{-1}$$
$$\geqslant \exp(-D(1+8M_1a_1+8M_2a_2)-G) \geqslant \exp(-(16c+2)U/10),$$

by the definition of the M_i's and the inequality at the end of § 5.1, which contradicts the assumption $|\Lambda| \leqslant e^{-CU}$, since $C > 2c$ and $c \geqslant 3$.

We also remark that $M_2 \leqslant b_2/33$: if not, since $b_1a_1 \leqslant b_2a_2$, (*) implies the estimate

$$|\Lambda| \geqslant \exp(-3b_2Da_2) \geqslant \exp(-99M_2a_2D),$$

which contradicts $|\Lambda| \leqslant e^{-CU}$, since $C > 10c$, $\theta \geqslant 10$ and $M_2a_2D \leqslant cU/\theta$. This remark will be used in the proof of Proposition 5.19.

5.5. The auxiliary function. Like in [7] we denote by $\{\xi_1, \ldots, \xi_D\}$ a basis of $\boldsymbol{Q}(\alpha_1, \alpha_2)$ over $\boldsymbol{Q}$, where $\xi_d = \alpha_1^{d_1}\alpha_2^{d_2}$, $0 \leqslant d_j < D_j$ ($j = 1, 2$).

For brevity we write α_1^z for $\exp(z\log\alpha_1)$. We shall construct an auxiliary function of the form

$$F(z) = \sum_{h=0}^{L_0}\sum_{k=-L_1}^{L_1} p_{h,k}\Delta_h(z)\alpha_1^{kz},$$

where

$$p_{h,k}=\sum_{d=1}^{D} p_{h,k,d}\xi_d, \qquad p_{h,k,d}\in \mathbf{Z}$$

and $\Delta_h(z)$ is defined in Lemma 2.4.

For rational integers u and v we put

$$\varphi(u,v)=\sum_{h=0}^{L_0}\sum_{k=-L_1}^{L_1} p_{h,k}\,\Delta_h(u+v\beta)\,\alpha_1^{ku}\alpha_2^{kv}.$$

Notice that

$$F(u+v\beta)-\varphi(u,v)=\sum_h\sum_k p_{h,k}\,\Delta_h(u+v\beta)\,\alpha_1^{ku}\alpha_2^{kv}\left((\alpha_1^{\beta}/\alpha_2)^{kv}-1\right).$$

With $\Omega(b,h)$ defined as in Lemma 2.6, put

$$\psi(u,v)=\varphi(u,v)\,\alpha_1^{L_1|u|}\alpha_2^{L_1|v|}\,b_2^{L_0}\,\Omega(b_2,L_0).$$

Notice that $\Omega(b_2,L_0)\leqslant \exp(L_0\omega(b_2))\leqslant \exp(\mu L_0 \operatorname{Log} B)$.

PROPOSITION 5.14. *There exist rational integers $p_{h,k,d}$, not all zero, such that*

$$\varphi(u,v)=\psi(u,v)=0 \quad \textit{for } -M_1\leqslant u\leqslant M_1,\ -M_2\leqslant v\leqslant M_2,$$

with

$$\operatorname{Log}\sum_h\sum_k |p_{h,k}|\leqslant p_1U/D \quad \textit{and} \quad \operatorname{Log}\sum_h\sum_k\sum_d |p_{h,k,d}|\leqslant p_2U/D,$$

where $p_1=p-\nu/2f$ and $p_2=p_1+\nu D/2f$.

Proof of Proposition 5.14. We have to solve in $\mathbf{Z}$ a linear system of $(2M_1+1)(2M_2+1)$ equations in the $D(L_0+1)(2L_1+1)$ unknowns $p_{h,k,d}$. We shall use Lemma 2.1.

We first check

$$(5.15)\quad ((2M_1+1)(2M_2+1))/((L_0+1)(2L_1+1)-(2M_1+1)(2M_2+1))\leqslant \eta.$$

This is an easy consequence of (5.3), (5.12) and (5.13).

With the notations of Lemma 2.1, we have

$$i\to(h,k,d),\quad j\to(u,v),\quad N_{j,1}=2L_1|u|+D_1-1,\quad N_{j,2}=2L_1|v|+D_2-1,$$

$$P_{i,j}=\Delta_h(u+v\beta)\,b_2^{L_0}\Omega(b_2,L_0)\,X_1^{L_1|u|+ku+d_1}X_2^{L_1|v|+kv+d_2}.$$

By Lemma 2.4, $\beta a_1\leqslant a_2$ and (5.1):

$$L(P_{i,j})\leqslant 2(X^h/h!)\,b_2^{L_0}\Omega(b_2,L_0),\qquad X=\max\{|u+v\beta|,\ L_0/2\}=L_0/2.$$

Notice that

$$\sum_{h=0}^{L_0}\frac{X^h}{h!}\leqslant e^X,$$

so that

$$\sum_i L(P_{i,j})\leqslant 2D(2L_1+1)(\sqrt{e}\,b_2)^{L_0}\,\Omega(b_2,L_0),$$

and

$$V_{u,v}\leqslant 2D(2L_1+1)(\sqrt{e}\,b_2)^{L_0}\,\Omega(b_2,L_0)$$
$$\times\exp\{(D_1-1+2L_1|u|)\,h(\alpha_1)+(D_2-1+2L_1|v|)\,h(\alpha_2)\}.$$

Now we have

$$\sum_{u=-M_1}^{M_1}(D_1-1+2L_1|u|)\,h(\alpha_1)=((2M_1+1)(D_1-1)+2L_1M_1(M_1+1))\,h(\alpha_1)$$

and

$$2M_1(M_1+1)\,h(\alpha_1)\leqslant (1/2)(2M_1+1)^2a_1.$$

Hence, since a similar result holds for the summation over v,

$$\sum_{u,v}\operatorname{Log}V_{u,v}\leqslant (2M_1+1)(2M_2+1)\{\operatorname{Log}(2D(2L_1+1)(\sqrt{e}\,b_2)^{L_0}\Omega(b_2,L_0))$$
$$+a_1L_1(M_1+1/2)+(D-1)a_1+a_2L_1(M_2+1/2)+(D-1)a_2\}.$$

We notice that

$$(5.16)\qquad (D-1+L_1/2)(a_1+a_2)\leqslant c_1U/(\theta D)+\nu U/2D.$$

Next we show

$$(5.17)\qquad \operatorname{Log}(2(2L_1+1))<0.53U/DG \quad\text{and}\quad D\operatorname{Log}D<\nu U/10.$$

Put $Y=U/DG$. We first notice that $2L_1+1\leqslant 9.7\,Y$ by (5.1). The first inequality comes from the estimates $Y\geqslant\theta\geqslant 10$ and $\operatorname{Log}(19.4\,x)\leqslant 0.53\,x$ for $x\geqslant 10$.

For $D\leqslant 5$ we have $D\operatorname{Log}D\leqslant 10\leqslant U/10$ because $U\geqslant\theta^2\geqslant 100$, while for $D\geqslant 6$

$$\operatorname{Log}D<0.4(D-1)\quad\text{and}\quad U\geqslant 4D(D-1).$$

This completes the proof of (5.17).

Next we have $L_1(a_1M_1+a_2M_2)\leqslant 2cc_1(U/D)$, and, by Lemma 2.7 and the definition of G,

$$\operatorname{Log}((\sqrt{e}\,b_2)^{L_0}\Omega(b_2,L_0))\leqslant L_0(0.5+(1+\mu)\operatorname{Log}B)\leqslant c_0(U/D)-4c_0U/10DG.$$

Hence from (5.16) and (5.17) we deduce

$$\sum_{u,v}\operatorname{Log}V_{u,v}\leqslant (2M_1+1)(2M_2+1)\left\{\frac{c_1}{\theta}+c_0+2c_1c+\frac{0.53}{G}-\frac{4c_0}{10G}+0.6\nu\right\}\frac{U}{D}.$$

Then, by Lemma 2.1, there exists a solution $p_{h,k,d}$ in $\mathbf{Z}$,

$$0 \leqslant \max \operatorname{Log}|p_{h,k,d}| < \eta\left\{\frac{c_1}{\theta}+c_0+2c_1c+\frac{0.53}{G}-\frac{4c_0}{10G}+0.6\nu\right\}\frac{U}{D}+\frac{\eta \operatorname{Log} 2}{D}.$$

Remark that the conditions $\eta \geqslant 0.5$ and $c_0 \geqslant 15$ imply that the sum of the terms in (5.4) containing G^{-1} is negative.

We bound $(\operatorname{Log} 2)/D$ by $0.7\, Y/\theta D$. Finally, we show

$$\operatorname{Log}\big(D(L_0+1)(2L_1+1)\big) \leqslant 1.26U/DG+\nu U/10D. \tag{5.18}$$

We have only to bound $(L_0+1)(2L_1+1) < (291Y)(9.7Y) < 2823Y^2$ and

$$\operatorname{Log}(2823Y^2) \leqslant 1.26Y \quad \text{for } Y \geqslant 10.$$

Using (5.17), this proves (5.18).

Using the upper bounds

$$(D_1-1)l_1+(D_2-1)l_2 \leqslant (D-1)(l_1+l_2) \leqslant f^{-1}D(D-1)(a_1+a_2) \leqslant \nu U/2f,$$

we deduce Proposition 5.14 from (5.4) and (5.18).

5.6. The extrapolation. Put

$$M_1^* = [\chi c D^2 a_2 G Z^{-2}] \quad \text{and} \quad M_2^* = [\chi c D^2 a_1 G Z^{-2}].$$

In this section we prove that

$$(*) \qquad \varphi(u, v) = 0 \quad \text{for } -M_1^* \leqslant u \leqslant M_1^*, \quad -M_2^* \leqslant v \leqslant M_2^*.$$

By construction, this is true for $-M_1 \leqslant u \leqslant M_1$ and $-M_2 \leqslant v \leqslant M_2$.

We plan to use an inductive argument [indeed, we could prove (*) in one single step, but an induction yields slightly smaller constants]. Define

$$N = M_1^*+M_2^*-M_1-M_2.$$

For each integer n in the range $1 \leqslant n \leqslant N$, choose

$$\varepsilon_n^{(1)} = 1 \quad \text{and} \quad \varepsilon_n^{(2)} = 0 \quad \text{for } 1 \leqslant n \leqslant M_1^*-M_1,$$

$$\varepsilon_n^{(1)} = 0 \quad \text{and} \quad \varepsilon_n^{(2)} = 1 \quad \text{for } M_1^*-M_1 < n \leqslant N.$$

Then define, for $0 \leqslant n \leqslant N$,

$$M_j^{(n)} = M_j+\sum_{h=1}^{n}\varepsilon_h^{(j)} \quad \text{for } j = 1, 2.$$

For $1 \leqslant n \leqslant N$, define χ_n by

$$M_1^{(n)} = \chi_n c a_2 D^2 G Z^{-2}, \quad \text{so that} \quad 1 \leqslant \chi_n.$$

We shall prove, by induction on n $(0 \leqslant n \leqslant N)$, that

$$(\mathrm{P})_n \qquad \varphi(u, v) = 0 \quad \text{for } |u| \leqslant M_1^{(n)} \text{ and } |v| \leqslant M_2^{(n)}.$$

As already seen, this is true for $n = 0$, while $(\mathrm{P})_N$ is nothing else than (*).

We suppose that $(\mathrm{P})_{n-1}$ is true for some n, $1 \leqslant n \leqslant N$, and we shall prove $(\mathrm{P})_n$. We consider the set

$$\Gamma_{n-1} = \{z_1, \ldots, z_m\} = \{u+v\beta;\ |u| \leqslant M_1^{(n-1)},\ |v| \leqslant M_2^{(n-1)}\},$$

$$m = (2M_1^{(n-1)}+1)(2M_2^{(n-1)}+1),$$

and a point $z_0 \in \Gamma_n$, $z_0 \notin \Gamma_{n-1}$.

From our assumption $\beta a_1 \leqslant a_2$ we get

$$M_1^{(n-1)}+1 \geqslant \beta M_2^{(n-1)}.$$

Define

$$R_1 = M_1^{(n)}+M_2^{(n)}\beta, \quad R = m/(L_1 l_1).$$

PROPOSITION 5.19. *We have*

$$|F(z_0)| \leqslant E_1+E_2$$

where

$$\operatorname{Log} E_1 \leqslant p_2U/D+L_0G'+1+L_0\operatorname{Log}\big(m/(2M_1+1)(2M_2+1)\big)-m\xi Z \tag{5.20}$$

and

$$\operatorname{Log} E_2 \leqslant \operatorname{Log}\max_{\gamma\in\Gamma_{n-1}}|F(\gamma)|+m\operatorname{Log}2e+2M_2^{(n-1)}\operatorname{Log}(1.3b_2/M_2^{(n-1)}). \tag{5.21}$$

Proof of Proposition 5.19. Using Lemmas 3.1, 3.2 and Corollary 3.4 we have

$$|F(z_0)| \leqslant E_1+E_2$$

where

$$\operatorname{Log} E_1 \leqslant -m\operatorname{Log}(R/R_1)+\operatorname{Log}\big(R/(R-R_1)\big)+\operatorname{Log}|F|_R$$

and

$$\begin{aligned}\operatorname{Log} E_2 \leqslant{}& \operatorname{Log}\max_{\gamma\in\Gamma_{n-1}}|F(\gamma)|-(4M_2^{(n-1)}+2)\operatorname{Log}(M_1^{(n-1)}!)\\ &+1.28(M_2^{(n-1)}+1)^3b_2^{-2}+2M_2^{(n-1)}\operatorname{Log}b_2-2\operatorname{Log}(M_2^{(n-1)}!)\\ &+\operatorname{Log}m+(m-1)\operatorname{Log}R_1+\operatorname{Log}\big(2R^4/(R^4-R_1^4)\big).\end{aligned}$$

From Corollary 2.5 we deduce

$$\operatorname{Log}\max_{0\leqslant h\leqslant L_0}|\Delta_h|_R \leqslant L_0\operatorname{Log}\left(\frac{R}{L_0}+\frac{1}{2}\right)+L_0,$$

hence

$$\operatorname{Log}|F|_R \leqslant p_2\frac{U}{D}+L_0\operatorname{Log}\left(\frac{R}{L_0}+\frac{1}{2}\right)+L_0+L_1Rl_1.$$

We first show the following four inequalities, and then we use them to complete the proof of Proposition 5.19:

$$\frac{R}{L_0} \leqslant \frac{m(e^{G'-1}-1/2)}{(2M_1+1)(2M_2+1)} \cdot \frac{(2c+1/\theta)^2}{2(c_0-1/\theta)(c_1-1/\theta)}, \tag{5.22}$$

$$\operatorname{Log} R/(e R_1) \geqslant \xi Z, \tag{5.23}$$

$$\operatorname{Log}\big((M_1^{(n-1)})!\big) \geqslant (M_1^{(n-1)}+1)\operatorname{Log}\big((M_1^{(n-1)}+1)/e\big) - (1/2)\operatorname{Log}(M_1^{(n-1)}+1)+(1/2)\operatorname{Log} 2\pi, \tag{5.24}$$

$$R_1/(M_1^{(n-1)}+1) \leqslant 2. \tag{5.25}$$

Proof of (5.22): Recall that $R = m/L_1 l_1$. The choice of G' implies

$$1/l_1 \leqslant (e^{G'-1}-1/2)/2.$$

Hence, using the definitions of M_1, M_2, L_0 and L_1, we get

$$\frac{R}{L_0} = \frac{m}{L_0 L_1 l_1} \leqslant \frac{m}{(2M_1+1)(2M_2+1)\, l_1} \cdot \frac{(2c+1/\theta)^2}{(c_0-1/\theta)(c_1-1/\theta)}.$$

And (5.22) follows from these two inequalities.

Proof of (5.23): We have

$$\frac{R}{R_1} = \frac{(2M_1^{(n-1)}+1)(2M_2^{(n-1)}+1)}{L_1 l_1 (M_1^{(n)}+\beta M_2^{(n)})}.$$

According to (5.7), the inequality (5.23) we are checking reads

$$\frac{R}{R_1} \geqslant \frac{2fc\, e^{Z/\varepsilon}}{ec_1 \, Z}\left(1-\frac{1}{2\theta c}\right)^2 e^{-e^{-c}}. \tag{5.26}$$

We consider two cases

(a) $1 \leqslant n \leqslant M_1^* - M_1$. In this case $M_2^{(n-1)} = M_2$ while $M_1^{(n-1)} = M_1^{(n)}-1$, thus

$$\frac{2M_1^{(n-1)}+1}{M_1^{(n)}+\beta M_2} \geqslant \frac{(2c-1/\theta)\,\chi_n D^2 G Z^{-2} a_2}{c\chi_n D^2 G Z^{-2} a_2 + \beta M_2} \geqslant \frac{(2c-1/\theta)\, D^2 G Z^{-2} a_2}{cD^2 G Z^{-2} a_2 + \beta M_2}$$

and

$$\frac{2M_2+1}{cD^2GZ^{-2}a_2+\beta M_2} \geqslant \frac{2(c-1/2\theta)\, D^2 G a_1 Z^{-2}}{cD^2GZ^{-2}a_2+\beta c D^2 G a_1 Z^{-2}} \geqslant 2\left(1-\frac{1}{2\theta c}\right)\frac{a_1}{a_2+\beta a_1}.$$

From the upper bound $L_1 \leqslant c_1 DGZ^{-1}$ we get

$$\frac{R}{R_1} \geqslant 4c\left(1-\frac{1}{2\theta c}\right)^2 \frac{D a_1 a_2}{c_1 l_1 Z (a_2+\beta a_1)}.$$

The condition $|\Lambda| < e^{-CU}$ implies

$$\beta l_1 \leqslant l_2 + e^{-CU},$$

and therefore

$$(a_2+\beta a_1) l_1 \leqslant a_2 l_1 + a_1 l_2 + a_1 e^{-CU} \leqslant (a_2 l_1 + a_1 l_2)(1+e^{-CU+DG})$$

because $l_1 > e^{-G'}$ and $l_2 > e^{-DG}$.

Now $U \geqslant \theta DG$, hence

$$CU - DG \geqslant C(1-1/\theta)\, U \geqslant C(\theta-1) \geqslant C,$$

and $\operatorname{Log}(1+e^{-C}) \leqslant e^{-C}$. This completes the proof of (5.26) in case (a).

(b) $M_1^* - M_1 < n \leqslant N$. In this case $M_2^{(n-1)} = M_2^{(n)}-1$, while $M_1^{(n-1)} = M_1^{(n)} = M_1^*$. We have

$$\frac{R}{R_1} \geqslant \frac{2M_1^*(2M_2^{(n)}-1)}{L_1 l_1 (M_1^* + \beta M_2^{(n)})}.$$

Thanks to the inequalities

$$M_2^{(n)} \geqslant M_2 + 1 > 2cD^2 G a_1 Z^{-2},$$

$$M_1^* \geqslant M_1 + 1 > 2cD^2 G a_2 Z^{-2},$$

we obtain

$$\frac{R}{R_1} \geqslant 2\left(2c-\frac{1}{\theta}\right)\frac{D^2 G a_1 a_2}{L_1 l_1 (a_2+\beta a_1) Z^2}.$$

We have already seen that

$$\operatorname{Log}\big((a_2+\beta a_1)\, l_1\big) \leqslant \operatorname{Log}(a_2 l_1 + a_1 l_2) + e^{-C}.$$

Finally (5.26) easily follows. This completes the proof of (5.23).

Proof of (5.24): For any integer $A \geqslant 1$ it is known that we have $A! \geqslant (A/e)^A \sqrt{(2\pi A)}$. Hence for any integer $A \geqslant 2$ we have

$$(A-1)! \geqslant \sqrt{2\pi}\, (A/e)^A A^{-1/2}.$$

We apply this inequality for $A = M_1^{(n-1)}+1$.

Proof of (5.25). For $1 \leqslant n \leqslant M_1^* - M_1$, we have

$$\frac{R_1}{M_1^{(n-1)}+1} \leqslant \frac{M_1^{(n-1)}+1+\beta M_2}{M_1^{(n-1)}+1} = 1 + \frac{\beta M_2}{M_1^{(n-1)}+1} \leqslant 2.$$

Whereas, for $M_1^* - M_1 < n \leqslant N$, we have

$$\frac{R_1}{M_1^{(n-1)}+1} \leqslant \frac{M_1^* + \beta M_2^*}{M_1^{(n-1)}+1} = 1 + \frac{\beta M_2^*}{M_1^{(n-1)}+1} \leqslant 2.$$

End of the proof of Proposition 5.19. We first check (5.20). We already know

$$\operatorname{Log} E_1 \leqslant -m \operatorname{Log}\frac{R}{R_1} + \operatorname{Log}\frac{R}{R-R_1} + p_2\frac{U}{D} + L_0 \operatorname{Log}\left(\frac{R}{L_0}+\frac{1}{2}\right) + L_0 + L_1 R l_1.$$

We substitute m to $L_1 R l_1$ and use (5.23):

$$-m \operatorname{Log}(R/R_1) + L_1 R l_1 \leqslant -m\xi Z.$$

Now we use (5.22) together with the inequalities $m \geqslant (2M_1+1)(2M_2+1)$ and $0 < \varrho < 1$; we get

$$L_0 \operatorname{Log}\left(e\left(\frac{R}{L_0}+\frac{1}{2}\right)\right) \leqslant L_0\left(G' + \operatorname{Log}\frac{m}{(2M_1+1)(2M_2+1)}\right).$$

Finally we bound $R/(R-R_1)$ by e, because from (5.23) and (5.7) we deduce $R > eR_1$. This completes the proof of (5.20).

We now check (5.21). We prove

(5.27) $$(m-1)\operatorname{Log} R_1 + \operatorname{Log}\frac{2R^4}{R^4-R_1^4} + \operatorname{Log} m + 1.28\,(M_2^{(n-1)}+1)^3\, b_2^{-2}$$

$$\leqslant (4\,M_2^{(n-1)}+2)\operatorname{Log}(M_1^{(n-1)}!) + m \operatorname{Log} 2e - (2\,M_2^{(n-1)}+1)(-0.1 + \operatorname{Log}(2\pi/e)).$$

We already know that $R/R_1 \geqslant e$, hence $2R^4/(R^4-R_1^4) < 2.04$. Let us show:

(5.28) $$\operatorname{Log} 2.04\, m < \operatorname{Log} R_1 + 0.19\, M_2^{(n-1)}.$$

Because of $c \geqslant 3$, we have $M_1^{(n-1)} \geqslant 30$ and $M_2^{(n-1)} \geqslant 30$. Thus, we have

$$m \leqslant (2+1/30)^2\, M_1^{(n)} M_2^{(n-1)},$$

$$2.04\, m/R_1 \leqslant 2.04\,(2+1/30)^2\, M_2^{(n-1)} < 8.5\, M_2^{(n-1)},$$

$$\operatorname{Log}(8.5\, M_2^{(n-1)}) \leqslant 0.19\, M_2^{(n-1)}.$$

Since $R_1 \geqslant M_1^{(n)}$, this proves (5.28).

In order to complete the proof of (5.27), we first notice that (5.24) implies

$$\begin{aligned} 2\,(2\,M_2^{(n-1)}+1)\operatorname{Log}(M_1^{(n-1)}!) &\geqslant m \operatorname{Log}((M_1^{(n-1)}+1)/e) + (2\,M_2^{(n-1)}+1)\operatorname{Log}((M_1^{(n-1)}+1)/e) \\ &\quad - (2\,M_2^{(n-1)}+1)\operatorname{Log}((M_1^{(n-1)}+1)/2\pi) \\ &= m \operatorname{Log}((M_1^{(n-1)}+1)/e) + (2\,M_2^{(n-1)}+1)\operatorname{Log}(2\pi/e). \end{aligned}$$

Thus, from (5.25) we get

$$m \operatorname{Log} R_1 \leqslant 2\,(2\,M_2^{(n-1)}+1)\operatorname{Log}(M_1^{(n-1)}!) + m\operatorname{Log} 2e - (2\,M_2^{(n-1)}+1)\operatorname{Log}(2\pi/e).$$

In § 5.4 we proved that $M_2 \leqslant b_2/33$. From $M_2 \geqslant 30$ and $\chi \leqslant 2.5$, we deduce

$$M_2^{(n-1)} \leqslant M_2^* - 1 \leqslant \chi c D^2 a_2 Z^{-2} - 1 \leqslant \chi(M_2+1) - 1 \leqslant 2.6\, M_2.$$

Hence, $M_2^{(n-1)} \leqslant (2.6/33)\, b_2$ and

$$1.28\,(M_2^{(n-1)}+1)^3\, b_2^{-2} \leqslant 1.28\,(31/30)^3 (M_2^{(n-1)})^3\, b_2^{-2} \leqslant 0.01\, M_2^{(n-1)}.$$

We deduce (5.27).

Remarking that $\operatorname{Log}(M_2^{(n-1)}!) \geqslant M_2^{(n-1)} \operatorname{Log}(M_2^{(n-1)}/e)$ and using (5.27), we get (5.21). This completes the proof of Proposition 5.19.

Proposition 5.29. *Put* $\lambda_n = \max\{1/2,\ 2\chi_n c/(c_0 - 1/\theta)\}$. *For* $\gamma = u + v\beta \in \Gamma_n$, *we have*

$$|F(\gamma) - \varphi(u, v)| \leqslant E_3$$

where

$$\begin{aligned} \operatorname{Log} E_3 \leqslant{} & -CU + p_2 U/D + \lambda_n L_0 + \operatorname{Log}(L_1 M_2^*) + L_1 M_1^* l_1 \\ & + 2\, D a_2/f + e^{-CU} + (L_1 M_2^* + 1)\, D a_2 + 1. \end{aligned}$$

Proof. We first show that for $-L_1 \leqslant k \leqslant L_1$ and $-M_2^* \leqslant v \leqslant M_2^*$, we have

(5.30) $$|\alpha_1^{\beta k v} - \alpha_2^{k v}|$$

$$\leqslant \exp\{-CU + 2\, D a_2/f + e^{-CU} + \operatorname{Log} L_1 M_2^* + (L_1 M_2^* + 1)\, D a_2 + L_1 M_2^* D a_2/CU\}.$$

We start from the inequality $|e^z - e^{z'}| \leqslant |z - z'|\, e^{|z| + |z'|}$ for $z, z' \in \mathbf{C}$. We deduce

$$|\alpha_1^\beta - \alpha_2| \leqslant |\beta \log \alpha_1 - \log \alpha_2| \exp(\beta l_1 + l_2) \leqslant \exp\{-CU + 2\, D a_2/f + e^{-CU}\}.$$

Now we use the upper bound

$$|x^n - y^n| \leqslant n\,|x - y| \max\{1, |x|, |y|\}^{n-1} \quad \text{if } x, y \in \mathbf{C} \text{ and } n \geqslant 1,$$

so that, for any $x, y \in \mathbf{C}^*$ and $n \in \mathbf{Z}$,

$$|x^n - y^n| \leqslant |n|\,|x - y| \max\{1, |xy|^{-1}\} \max\{|x|, |y|, |x|^{-1}, |y|^{-1}\}^{|n|-1}.$$

In our case we get

$$\begin{aligned} |\alpha_1^{\beta k v} - \alpha_2^{k v}| \leqslant{} & |\alpha_1^\beta - \alpha_2|\, L_1 M_2^* \max\{1, |\alpha_1^\beta \alpha_2|^{-1}\} \\ & \times \exp((L_1 M_2^* - 1)\operatorname{Log}(\max\{|\alpha_1^\beta|, |\alpha_2|, |\alpha_1^\beta|^{-1}, |\alpha_2|^{-1}\})). \end{aligned}$$

We have $\operatorname{Log}|\alpha_1| \leqslant l_1$, $\operatorname{Log}|\alpha_2| \leqslant l_2 \leqslant D a_2$ and $|\alpha_1^\beta| \leqslant |\alpha_2| + e^{-CU/2}$ so that

$$\operatorname{Log}|\alpha_1^\beta| \leqslant D a_2 + e^{-CU/2}.$$

From Liouville's inequality (Lemma 2.3) we deduce $\operatorname{Log}|\alpha_2| \geqslant -D h(\alpha_2) \geqslant -D a_2$. We now show

(5.31) $$\operatorname{Log}|\alpha_1^\beta| \geqslant -Da_2(1+1/(CU)).$$

Clearly we have

$$|\alpha_1^\beta| \geqslant \exp(-Da_2)-\exp(-CU+2\,Da_2+e^{-CU}).$$

But $\exp\{Da_2/CU\}-1 \geqslant Da_2/(CU)$ and

$$CU > \frac{Da_2}{CU}+3\,Da_2+e^{-CU}-\operatorname{Log}\left(\frac{Da_2}{CU}\right).$$

Hence

$$\exp\{Da_2/CU\}-\exp\{(3+1/CU)\,Da_2+e^{-CU}-CU\} \geqslant 1,$$

multiplying each side by $\exp\{-Da_2/CU-Da_2\}$ we get (5.31). These lower bounds give

(5.32) $$\operatorname{Log}(|\alpha_1^\beta\alpha_2|^{-1}) \leqslant Da_2(2+1/CU)$$

and

$$\operatorname{Log}(\max\{|\alpha_1^\beta|, |\alpha_2|, |\alpha_1^\beta|^{-1}, |\alpha_2|^{-1}\}) \leqslant Da_2(1+1/CU),$$

and (5.30) follows.

Notice also that

$$\operatorname{Log}2+(1/CU)\,L_1M_2^*\,Da_2 \leqslant \operatorname{Log}2+\chi c_1 c/C < 1/5+\operatorname{Log}2 < 1.$$

Now Proposition 5.29 follows from the relation

$$F(\gamma)-\varphi(u,v) = \sum_h\sum_k p_{h,k}\,\Delta_h(u+v\beta)\,\alpha_1^{ku}(\alpha_1^{\beta kv}-\alpha_2^{kv})$$

provided that we prove

(5.33) $$|\Delta_h(u+v\beta)| \leqslant \exp\{\lambda_n L_0\}$$

for $0 \leqslant h \leqslant L_0$, $|u| \leqslant M_1^{(n)}$, $|v| \leqslant M_2^{(n)}$, where $\lambda_n = \max\{1/2, 2\chi_n c/(c_0-1/\theta)\}$.

We deduce (5.33) from Lemma 2.3 as follows: if $X = \max\{|u+v\beta|, h/2\}$ then

$$|\Delta_h(u+v\beta)| \leqslant 2\,X^h/h! \leqslant (2/e)(Xe/h)^h \quad (\text{use } h! \geqslant h^h e^{1-h}, \text{ true for } h \geqslant 1).$$

This implies

$$|\Delta_h(u+v\beta)| \leqslant (2/e)\,e^X.$$

We have

$$|u+v\beta| \leqslant M_1^{(n)}+M_2^{(n)}\beta \leqslant 2\,\chi_n cD^2GZ^{-2}a_2,$$

while

$$L_0 \geqslant c_0(1-1/\theta c_0)\,D^3a_1a_2GZ^{-3};$$

hence

$$|u+v\beta| \leqslant (2\chi_n c/(c_0-1/\theta))\,L_0.$$

This completes the proof of (5.33), and also of Proposition 5.29.

Remark. Thanks to (5.1), we always have $\lambda_n \leqslant 0.91$.

Proposition 5.34. *For $(u, v) \in \mathbf{Z}\times\mathbf{Z}$ with $|u| \leqslant M_1^{(n)}$ and $|v| \leqslant M_2^{(n)}$, either $\varphi(u, v)$ is equal to zero or*

$$|\varphi(u, v)| \geqslant E_4$$

with

$$-\operatorname{Log}E_4 \leqslant (1-1/D)p_1U+(D-1)\,L_0\lambda_n+(1+\mu)\,D\,L_0\operatorname{Log}B$$
$$+2\,DL_1(a_1M_1^{(n)}+a_2M_2^{(n)})+D(D-1)(a_1+a_2).$$

Proof. By Lemma 2.6, the number

$$b_2^{L_0}\Omega(b_2, L_0)\,\varphi(u, v) = \sum_h\sum_k\sum_d p_{h,k,d}\,\Delta_h(u+v\beta)\,b_2^{L_0}\,\Omega(b_2, L_0)\,\alpha_1^{ku+d_1}\alpha_2^{kv+d_2}$$

is the value of a polynomial in α_1, α_2, α_1^{-1}, α_2^{-1}, with integer coefficients. The length of this polynomial is at most

$$\exp\{p_1U/D+(1+\mu)\,L_0\operatorname{Log}B+\lambda_n L_0\},$$

as shown by (5.33) and Lemma 2.7. Proposition 5.34 immediately follows from Liouville estimate (see Lemma 2.3).

Proposition 5.35. *Assume that in (5.21) we have*

$$\max_{\gamma\in\Gamma_{n-1}} |F(\gamma)| \leqslant E_3;$$

then

$$E_1+E_2+E_3 < E_4.$$

Proof. We use (5.5) to check

(5.36) $$E_1 < E_4/3,$$

and then we shall use (5.6) to check

(5.37) $$\max\{E_2, E_3\} < E_4/3.$$

The first inequality can be written

(5.38) $$m\xi Z \geqslant pU+(D-1)\,L_0\lambda_n+(1+\mu)\,DL_0\operatorname{Log}B+2\,DL_1(a_1M_1^{(n)}+a_2M_2^{(n)})$$
$$+D(D-1)(a_1+a_2)+1+\operatorname{Log}3+L_0G'+L_0\operatorname{Log}(m/(2M_1+1)(2M_2+1)).$$

Let us prove that it is sufficient to check (5.38) for $n = 1$. If we replace n by $n+1$, then the left-hand side of (5.38) increases by $2(2M_2^{(n)}+1)\,\xi Z$ and the right-hand side at most by (notice that $(\chi_n-\chi_{n-1})\,cD^2a_2GZ^{-2} \leqslant 1$)

$$2DL_1a_1+L_0\operatorname{Log}\frac{2M_1^{(n)}+3}{2M_1^{(n)}+1}+2\left(1-\frac{1}{\theta c_0}\right)^{-1}D^2a_1Z^{-1}$$

if $n<M_1^*-M_1$, while if $n\geqslant M_1^*-M_1$ the LHS increases by

$$2(2M_1^{(n)}+1)\xi Z$$

and the RHS by

$$2DL_1a_2+L_0\operatorname{Log}\frac{2M_2^{(n)}+3}{2M_2^{(n)}+1}.$$

Therefore our claim will follow from the upper bounds

(5.39) $$2DL_1a_1+L_0\operatorname{Log}\frac{2M_1^{(n)}+3}{2M_1^{(n)}+1}+2\left(1-\frac{1}{\theta c_0}\right)^{-1}D^2a_1Z^{-1}\leqslant 2(2M_2^{(n)}+1)\xi Z$$

and

(5.40) $$2DL_1a_2+L_0\operatorname{Log}\frac{2M_2^{(n)}+3}{2M_2^{(n)}+1}\leqslant 2(2M_1^{(n)}+1)\xi Z.$$

Now we have $2DL_1a_1\leqslant 2c_1D^2Ga_1Z^{-1}$ and

$$L_0\operatorname{Log}\frac{2M_1^{(n)}+3}{2M_1^{(n)}+1}\leqslant\frac{2L_0}{2M_1^{(n)}+1}\leqslant\frac{2L_0}{2M_1+3}\leqslant\frac{c_0}{c}Da_1Z^{-1},$$

and the left-hand side in (5.39) is at most

$$(2c_1+c_0/cDG+2/(1-1/\theta c_0)G)D^2Ga_1Z^{-1}$$

while the right-hand side in (5.39) is at least

$$2(2M_2+3)\xi Z\geqslant 4cD^2Ga_1Z^{-1}\xi.$$

We have $DG\geqslant\theta$, moreover from (5.1) c_0 satisfies $2c_1+c_0/c\theta+2/(1-1/\theta c_0)\leqslant 4c\xi$, hence this proves (5.39).

The proof of (5.40) is almost the same (but simpler). Now the proof of (5.36) reduces to (notice that $\lambda_1=1/2$ by (5.1))

$$(2M_1+1)(2M_2+1)\xi Z-(pU+(D-1)L_0/2+(1+\mu)DL_0\operatorname{Log}B+L_0G'$$
$$+D(D-1)(a_1+a_2)+2DL_1((M_1+1)a_1+M_2a_2)+1+\operatorname{Log}3)\geqslant 0.$$

Since $(2c-1/\theta)\xi\geqslant c_1$ it is easy to verify that the LHS is an increasing function of M_1 and M_2 (in the real intervals $M_1\geqslant cD^2a_2GZ^{-2}-1$ and $M_2\geqslant cD^2a_1GZ^{-2}-1$) so that (5.36) is true if

$$(2c-1/\theta)^2\xi U\geqslant pU+(D-1)L_0/2+(1+\mu)DL_0\operatorname{Log}B+L_0G'$$
$$D(D-1)(a_1+a_2)+4cc_1U-2DL_1a_2+1+\operatorname{Log}3.$$

We have

$$D(D-1)(a_1+a_2)\leqslant\nu U/2,\qquad 1+\operatorname{Log}3\leqslant 2DL_1$$

and (by the choice of G)

$$(1+\mu)DL_0\operatorname{Log}B+(D-1)L_0/2+L_0G'\leqslant c_0U.$$

Therefore (5.36) is a consequence of (5.5).

We now prove (5.37). It is sufficient to check (as already remarked $\lambda_n\leqslant 0.91$)

(5.41) $$pU+D(a_1+a_2)(D-1)+2Da_2/f+e^{-cU}+\operatorname{Log}9+(1+\mu)DL_0\operatorname{Log}B$$
$$+\lambda_nL_0+\operatorname{Log}L_1M_2^*+L_1M_1^*l_1+2DL_1(M_1^{(n)}a_1+M_2^{(n)}a_2)$$
$$+2M_2^{(n-1)}\operatorname{Log}(1.3B/M_2^{(n-1)})+(L_1M_2^{(n-1)}+1)Da_2+m\operatorname{Log}2e<CU.$$

We have

$$L_1M_1^*l_1\leqslant\chi cc_1U/f,$$
$$2DL_1(M_1^{(n)}a_1+M_2^{(n)}a_2)\leqslant 4\chi cc_1U,$$
$$L_1M_2^{(n-1)}Da_2\leqslant\chi cc_1U,$$
$$2M_2^{(n-1)}\operatorname{Log}(1.3B/M_2^{(n-1)})\leqslant 2M_2^{(n-1)}G-4M_2^{(n-1)}\leqslant 2\chi cU-4M_2^{(n-1)}.$$

We now bound m:

$$m\leqslant(2M_1^*+1)(2M_2^{(n-1)}-1)\leqslant 4M_1^*M_2^{(n-1)}+2M_2^{(n-1)}\leqslant 4\chi^2c^2U/Z+2M_2^{(n-1)}.$$

So that (5.41) is implied by the condition

$$p+c_0+(4\chi^2c^2/Z)\operatorname{Log}2e+\chi c(2+5c_1+c_1/f)+\nu/2+0.15/D\leqslant C,$$

provided we prove the upper bound

$$D(a_1+a_2)(D-1)+\operatorname{Log}9+\operatorname{Log}L_1M_2^*+Da_2(1+2/f)+e^{-cU}<(\nu/2+0.15/D)U.$$

We have (since $f\geqslant 1$ and $\theta\geqslant 10$)

$$D(a_1+a_2)(D-1)+Da_2(1+2/f)\leqslant(\nu/2+(1+2/f)/\theta^2)U/D<(\nu/2+0.03/D)U,$$

and, thanks to the bounds of c, c_1, χ given in (5.1),

$$\operatorname{Log}(9L_1M_2^*)<\operatorname{Log}(9cc_1\chi U/D)<\operatorname{Log}(1836U/D),\qquad\text{where } U/D\geqslant\theta^2,$$

hence

$$\operatorname{Log}(9L_1M_2^*)+e^{-cU}\leqslant 0.13U/D$$

so that the upper bound above is true. This shows that (5.41) is a consequence of (5.6), because $0.15<(1-\nu/2)D$.

We now complete the extrapolation argument.

Proposition 5.42. *For $(u,v)\in\mathbf{Z}\times\mathbf{Z}$, with $|u|\leqslant M_1^*$ and $|v|\leqslant M_2^*$, we have*

$$\varphi(u,v)=0.$$

Proof. We prove by induction on n, $0\leqslant n\leqslant N$, that

$$\varphi(u,v)=0\quad\text{for } |u|\leqslant M_1^{(n)}\text{ and }|v|\leqslant M_2^{(n)}.$$

This is true for $n=0$ by Proposition 5.14.

We assume that this is true for $n-1$, with $1 \leqslant n \leqslant N$; we choose the point

$$z_0 = u+v\beta \in \Gamma_n, \quad z_0 \notin \Gamma_{n-1}.$$

From Propositions 5.19 and 5.29 we deduce

$$|\varphi(u,v)| \leqslant E_1+E_2+E_3.$$

Now Propositions 5.34 and 5.35 give $\varphi(u,v)=0$.

5.7. End of the proof. The non-zero polynomial $\sum_{h,k} p_{h,k} \Delta_h(X)\, Y^{L_1+k}$ vanishes at the points

$$(u+v\beta,\ \alpha_1^u\alpha_2^v), \quad (u,v)\in \mathbf{Z}\times\mathbf{Z}, \quad |u| \leqslant M_1^*, \quad |v| \leqslant M_2^*.$$

According to Proposition 4.1 (zero estimate), we will obtain a contradiction with Proposition 5.42 if we prove the following result.

PROPOSITION 5.43. *There exist positive integers U_1, U_2, V_1, V_2 satisfying*

$$U_1+U_2 \leqslant M_1^*, \quad V_1+V_2 \leqslant M_2^*; \tag{5.44}$$

$$(2U_1+1)(2V_1+1) > L_0, \tag{5.45}$$

$$\operatorname{Card}\{(\alpha_1^u\alpha_2^v);\ |u| \leqslant U_1,\ |v| \leqslant V_1\} > 2L_1, \tag{5.46}$$

$$(2U_2+1)(2V_2+1) > 4L_0L_1. \tag{5.47}$$

Proof. We define U_2 and V_2 by the conditions

$$\chi_2 cD^2Ga_2Z^{-2}-1/2 \leqslant U_2 < \chi_2 cD^2Ga_2Z^{-2}+1/2,$$

$$\chi_2 cD^2Ga_1Z^{-2}-1/2 \leqslant V_2 < \chi_2 cD^2Ga_1Z^{-2}+1/2.$$

Since $\chi_2^2 = c_0c_1c^{-2}$, we obtain (5.47).

We define now

$$U_1 = M_1^*-U_2, \quad V_1 = M_2^*-V_2.$$

Therefore (5.44) is clear. We now deduce (5.45) from (5.8) and (5.9). We have

$$M_1^* > \chi cD^2Ga_2Z^{-2}-1,$$

hence

$$2U_1+1 > 2\chi_1 cD^2Ga_2Z^{-2}-2 \geqslant 2\chi_1 c(1-1/\chi_1\theta c)\,D^2Ga_2Z^{-2}.$$

Similarly

$$2V_1+1 > 2\chi_1 c(1-1/\chi_1\theta c)\,D^2Ga_1Z^{-2}.$$

On the other hand $L_0 \leqslant c_0D^3Ga_1a_2Z^{-3}$, and therefore (5.45) will be a consequence of

$$4\chi_1^2c^2\left(1-\frac{1}{\chi_1\theta c}\right)^2\frac{DG}{Z} \geqslant c_0.$$

Indeed from (5.8) and (5.9) one deduces

$$2\chi_1 c\theta \geqslant 1+\sqrt{(c_0\theta)}. \tag{5.48}$$

This completes the proof of (5.45).

Finally we prove (5.46). We consider two cases:

(i) The points $\alpha_1^u\alpha_2^v$ ($|u| \leqslant U_1$, $|v| \leqslant V_1$) are pairwise distinct. This will be the case for instance when α_1, α_2 are multiplicatively independent. In this case, we show that (5.46) is a consequence of (5.9). We have to check

$$(2U_1+1)(2V_1+1) > 2L_1.$$

This is implied by (5.45) and the inequality $L_0 \geqslant 2L_1$.

(ii) We assume

$$\chi > \frac{\sqrt{c_0c_1}}{c}+\frac{1}{\theta c}+\frac{c_1}{c}.$$

If α_1 is not a root of unity, then

$$\operatorname{Card}\{\alpha_1^u\alpha_2^v;\ |u| \leqslant U_1,\ |v| \leqslant V_1\} \geqslant 2U_1+1 \geqslant 2\chi_1 c\left(1-\frac{1}{\chi_1\theta c}\right)D^2Ga_2Z^{-2}.$$

It is sufficient to notice that

$$\chi_1 c\left(1-\frac{1}{\chi_1\theta c}\right) > c_1.$$

We have proved (5.46).

If α_2 is not a root of unity, the argument is the same.

Finally if both α_1 and α_2 are roots of unity, we write $\alpha_1^m = 1$ and $\alpha_2^n = 1$, m and n positive and minimal. Then

$$|\Lambda| > 2\pi/\mathrm{l.c.m.}(m,n) \geqslant \pi/D^2 > \exp(-CU).$$

This completes the proof of Theorem 5.11.

6. Numerical examples. We use the notation and hypotheses of Sections 5.1 and 5.2, and we produce suitable values for the constant C, so that the assumptions (5.1) to (5.10) have been checked. Therefore the conclusion $|\Lambda| > \exp(-CU)$ of Theorem 5.11 holds.

From the assumptions of Section 5.1 we deduce $\varepsilon \leqslant 1$ and $D \geqslant 1$. In the computations which follow, we shall check (5.4), (5.5), (5.6) and (5.7) with ε replaced by 1 and D by 2, which is plainly sufficient to deduce the general case. If either $\varepsilon < 1$ or $D \neq 2$, then better numerical values for C can actually be obtained. Also, since the coefficients of $1/G$ in the right-hand side of (5.4) and (5.5) are negative, we can omit these terms.

We also choose $f = 2e$. We proceed as follows. We fix $\theta \geqslant 10$ and $Z \geqslant 1$.

We choose a finite subset E of the rectangle $E_0 = \{(c, c_1) \in \mathbf{R}^2;\ 1 \leqslant c \leqslant 16,\ 1 \leqslant c_1 \leqslant 4.8\}$. In practice we take $E \subset \{(c, c_1) \in E_0;\ 100c \in \mathbf{Z},\ 100c_1 \in \mathbf{Z}\}$.

For each (c, c_1) in E, we compute the numbers

$\xi = 1 + Z^{-1}\,\mathrm{Log}(4c(1-1/2c\theta)^2/Zc_1e) - Z^{-1}e^{-30}$,

$\xi_1 = 4c^2\xi(1-1/2\theta c)^2 - 4cc_1 - v/2$,

$\eta_1 = (2c+1/\theta)^2$,

$\eta_2 = 2c_1 - 1/\theta$,

$p_1 = 2cc_1 + c_1/\theta + 0.6v$,

$p_2 = v/4e + 0.1v$,

$\xi_2 = p_2 - \xi_1$,

$\xi_3 = \eta_1(p_1 - p_2 + \xi_1)/\eta_2$.

We consider the quadratic equation

$$x^2 + \xi_2 x + \xi_3 = 0$$

which arises from replacing (5.4) and (5.5) by equalities and solving the corresponding equation in c_0. For a suitable choice of E as above, it turns out that there exists a non-empty subset E' of E such that, for each (c, c_1) in E', this quadratic equation has real solutions x' and x'', $x' < x''$.

We take for c_0 the smallest point of the interval $[x', x'']$ which satisfies the conditions (5.1) and (5.2). Again, for a non-empty subset E'' of E', we can check these conditions.

From (5.6), with the value of χ given by (5.9) and (5.10), we deduce a suitable value for C. Finally we choose (c, c_1) in E'' so that the corresponding value for C is minimal.

This result is given in Tables 1 and 2 (recall that we consider only the case when the numbers α_1 and α_2 are multiplicatively independent).

Table 1. Numbers multiplicatively independent, $Z = 1$

θ	10	11	12	13	14	15	16	17	18
C	558	570	554	541	530	521	512	505	498
c_0	33.2	32.58	32.32	31.98	31.69	31.48	31.33	31.23	30.92
c_1	1.46	1.45	1.43	1.42	1.41	1.4	1.39	1.38	1.38
c	3.47	3.44	3.41	3.38	3.36	3.34	3.32	3.3	3.29

θ	19	20	21	22	23	24	25	26	27
C	492	487	482	478	473	470	466	463	460
c_0	30.85	30.62	30.61	30.42	30.45	30.31	30.18	30.2	30.1
c_1	1.37	1.37	1.36	1.36	1.35	1.35	1.35	1.34	1.34
c	3.28	3.27	3.26	3.25	3.24	3.23	3.22	3.22	3.21

In Table 1 we fix $Z = 1$, θ varies, and we display the optimal value of C together with the corresponding choices of c, c_1 and c_0.

Table 2. Numbers multiplicatively independent, values of C/Z^3

$Z \backslash \theta$	12	13	14	15	20	30	50	100	c_1	c
1	554	541	530	521	487	452	424	400	1.28 1.43	3.09 3.41
1.1	477	466	457	450	422	395	371	352	1.38 1.53	3.5 3.82
1.2	414	405	398	392	369	347	327	311	1.47 1.62	3.89 4.23
1.3	362	355	349	344	325	306	290	277	1.56 1.71	4.3 4.63
1.4	318	312	307	303	288	272	258	247	1.65 1.8	4.69 5.03
1.5	282	277	272	269	256	242	231	221	1.74 1.88	5.07 5.41
2	162	159	157	156	149	143	137	132	2.1 2.24	6.81 7.17
3	65	64	63	63	61	59	57	55	2.65 2.78	9.39 9.76
5	16.2	16	15.9	15.8	15.4	14.9	14.5	14.2	3.25 3.38	12.02 12.39
c_1	1.43 3.38	1.42 3.37	1.41 3.36	1.4 3.35	1.37 3.32	1.33 3.3	1.3 3.27	1.28 3.25		
c	3.41 12.39	3.38 12.36	3.36 12.33	3.34 12.31	3.27 12.23	3.2 12.14	3.14 12.07	3.09 12.02		

In Table 2, both Z and θ vary, and we display the optimal value of CZ^{-3}. At the end of each row (resp. each column) we display the range for (c, c_1) corresponding to the given row (resp. column). For instance, at the end of the first row in Table 2 the indication

$$3.09 \leqslant c \leqslant 3.41, \qquad 1.28 \leqslant c_1 \leqslant 1.43$$

means that for $Z = 1$ and for the given values of θ (with $10 \leqslant \theta \leqslant 100$), we took for set E:

$$E = \{(c, c_1);\ c = n/100,\ c_1 = m/100,\ 309 \leqslant n \leqslant 341,\ 128 \leqslant m \leqslant 143\},$$

with $(m, n) \in \mathbf{Z} \times \mathbf{Z}$.

7. A consequence of the main result. With the notation and hypotheses of Section 5.1, we take $f = 2e$, $\theta = 10$ and we shall deduce from Theorem 5.11:

COROLLARY 7.1. *Suppose that α_1 and α_2 are multiplicatively independent, then*

$$|\Lambda| > \exp\{-2200\,U\}.$$

Proof. We first consider the case $\operatorname{Log} B \leqslant 10$, and in this case we prove the result with 2200 replaced by 441.

We use Liouville estimate (Lemma 2.2):

$$|\Lambda| > 2^{-D} B^{-1} \exp\{-Db_1 h(\alpha_1) - Db_2 h(\alpha_2)\}.$$

We notice that

$$h(\alpha_i) \leqslant a_i, \quad i = 1, 2, \quad b_1 a_1 \leqslant b_2 a_2$$

and

$$2\,DBa_2 + \operatorname{Log} B + D \operatorname{Log} 2 \leqslant (2\,e^{10} + 10 + \operatorname{Log} 2)\,Da_2 < 44100\,Da_2.$$

Since

$$\frac{Da_2}{U} = \frac{Z^3}{D^3 a_1 G^2} \leqslant \frac{Z^2}{D^2 G^2} \leqslant \theta^{-2},$$

we get $|\Lambda| > \exp\{-441\,U\}$ in that case.

From now on, we assume $\operatorname{Log} B \geqslant 10$ (as in Sections 5 and 6), then $DG \geqslant 11$.

We get the result with the constant 2200 by dividing the interval $[1, \infty[$ in 14 intervals. On each of them, say $[Z_{\min}, Z_{\max}[$, we choose c_1, c and c_0 so that (5.5) is valid for all Z in the range $Z_{\min} \leqslant Z \leqslant Z_{\max}$ and with ε replaced by 1 in the definition (5.7) of ξ, and we compute the value of the number C by replacing Z by $Z_{\min}$ in (5.6).

The numerical values we obtain are displayed in Table 3 below.

For instance, in the range $1 \leqslant Z < 1.5$, one can choose

$$c_1 = 1.91, \quad c = 5.5, \quad c_0 = 63.1,$$

and one gets $C = 1327$.

Remarks. 1. By choosing smaller intervals, one can reduce slightly the constant 2200 in Corollary 7.1. However for $Z = 6$ one gets $C = 2058$; notice that in this case we have $C/Z^3 < 10$.

2. One can prove that Corollary 7.1 holds also when α_1 and α_2 are multiplicatively dependent.

Table 3

Z	1	1.5	2	2.5	3	3.5	4	4.5	5	5.5	6	6.5	7	8
C	1327	1626	1849	2000	2094	2146	2170	2174	2165	2147	2123	2097	2100	209[illegible]
c_0	63.1	92.84	119.46	141.4	158.83	172.79	183.94	192.55	199.34	205.06	208.75	212.23	216.12	216[illegible]
c_1	1.91	2.28	2.57	2.81	3.01	3.17	3.3	3.41	3.5	3.57	3.64	3.69	3.78	4.[illegible]
c	5.5	7.26	8.7	9.85	10.76	11.48	12.04	12.48	12.83	13.1	13.32	13.49	13.74	14[illegible]

8. Proof of Corollary 1.1. We assume that the hypotheses of Corollary 1.1 are fulfilled, and we shall prove the conclusion by considering several cases. Without loss of generality, we may assume

$$a_1 b_1 \leqslant a_2 b_2.$$

(a) Assume $\operatorname{Log} B \leqslant 10.64$. Then we prove the estimate in Corollary 1.1 with the constant 254 instead of 500. For this we use Lemma 2.2:

$$|\Lambda| \geqslant b_2^{-1} |b_1 \log\alpha_1 - b_2 \log\alpha_2| \geqslant \exp\{-D \operatorname{Log} 2 - 2\,DBa_2 - \operatorname{Log} B\}.$$

Since $\operatorname{Log} B \leqslant 10.64$ we have $2B + \operatorname{Log} 2 + \operatorname{Log} B < 254(7.5 + \operatorname{Log} B)^2$, hence

$$2\,DBa_2 + D \operatorname{Log} 2 + \operatorname{Log} B \leqslant 254\,D^4 a_1 a_2 (7.5 + \operatorname{Log} B)^2,$$

which proves our claim.

(b) From now on we assume $\operatorname{Log} B \geqslant 10.64$. There is no loss of generality to assume that b_1 and b_2 are relatively prime. We are going to use Theorem 5.11 with

$$f = 2e \quad \text{and} \quad G = \operatorname{Log} B + \operatorname{Log}\operatorname{Log} B + \max\{1,\, 0.59 + G'\},$$

where $G' \geqslant \operatorname{Log}(e/2 + 2e/l_1)$, $l_j = |\log\alpha_j|$ $(j = 1, 2)$.

Let us prove

$$l_i \geqslant e^{-Da_i} \quad (i = 1, 2) \quad \text{and} \quad 2\,l_1 \geqslant e^{-Da_2}. \tag{8.1}$$

For $i = 1, 2$, from Liouville's inequality (Lemma 2.5) we have, if $\alpha_i \neq 1$,

$$|\alpha_i - 1| \geqslant 2^{-D+1} \exp\{-Dh(\alpha_i)\},$$

so that $|\log\alpha_i| \geqslant 2^{-D} \exp\{-Dh(\alpha_i)\} \geqslant \exp\{-Da_i\}$, because $\log\alpha_i \neq 0$ and $a_j \geqslant h(\alpha_j) + \operatorname{Log} 2$.

If $a_1 \leqslant a_2 + (1/D)\operatorname{Log} 2$ then $1/2\,l_1 \leqslant (1/2)\exp\{Da_1\} \leqslant \exp\{Da_2\}$. On the other hand if $a_1 > a_2 + (1/D)\operatorname{Log} 2$ then $b_1/b_2 \leqslant a_2/a_1 \leqslant 1 - (\operatorname{Log} 2)/Da_1$; in this case, from the inequalities $l_1 \geqslant \exp\{-Da_1\}$ and $Da_2 \exp\{-CU + Da_2\} < \operatorname{Log} 2$ we deduce

$$l_2 \leqslant (b_1/b_2)\,l_1 + e^{-CU} \leqslant l_1 - l_1 (\operatorname{Log} 2)/Da_1 + e^{-CU} < l_1,$$

which completes the proof of (8.1).

(c) Assume $l_1 \geqslant 1/2\,e^2$. In this case we take $Z = 1$, $G' = 4.41$, and then $G = 5 + \operatorname{Log} B + \operatorname{Log}\operatorname{Log} B$. Obviously $Da_1 a_2 G^2 \geqslant a_1 + a_2$. We prove the inequality of Corollary 1.1 with the constant 496 instead of 500.

We use the estimates of Section 6 with admissible choices of θ.

We notice that the function $F(x) = (5 + x + \operatorname{Log} x)/(7.5 + x)$ is increasing for $3 \leqslant x \leqslant x_0 = 39.953\ldots$ and decreasing for $x \geqslant x_0$, with $F(x_0) < 1.026$.

To prove our claim we consider the following five cases (we put $F = F(\operatorname{Log} B)$):

$10.64 \leqslant \operatorname{Log} B < 11.56$, then $\theta = 18$, $C = 498$, $F < 0.9973$ and $F^2 C < 496$,
$11.56 \leqslant \operatorname{Log} B < 12.48$, then $\theta = 19$, $C = 492$, $F < 1.002$ and $F^2 C < 494$,
$12.48 \leqslant \operatorname{Log} B < 14.34$, then $\theta = 20$, $C = 487$, $F < 1.008$ and $F^2 C < 495$,
$14.34 \leqslant \operatorname{Log} B < 19.1$, then $\theta = 22$, $C = 478$, $F < 1.017$ and $F^2 C < 495$,
$19.1 \leqslant \operatorname{Log} B$, then $\theta = 27$, $C = 460$, $F < 1.026$ and $F^2 C < 485$.

(d) From now on we assume $l_1 < 1/2e^2$. Hence one may choose

$$G' = 2.41 + Z_0, \quad G = 3 + \operatorname{Log} B + \operatorname{Log}\operatorname{Log} B + Z_0 \quad \text{where } Z_0 = \operatorname{Log}(1/2l_1).$$

Let us check

$$2 \leqslant Z_0 \leqslant \min\left\{Da_1,\ Da_2,\ \operatorname{Log}\frac{Da_1a_2}{a_2l_1 + a_1l_2}\right\}. \tag{8.2}$$

The inequalities $Z_0 \leqslant Da_j$ $(j = 1,2)$ follow from (8.1). Now $a_1b_1 \leqslant a_2b_2$ hence $a_1l_2 + a_2l_1 \leqslant 2a_2l_1 + e^{-cU}$. From our hypothesis $l_1 < 1/2e^2$ we deduce $Da_1 \geqslant 2$, hence $a_1l_2 + a_2l_1 \leqslant 2Da_1a_2l_1$, which completes the proof of (8.2).

(e) Assume $DG \geqslant 10Z_0$ and $G \geqslant \sqrt{2}Z_0$. We prove the estimate with 500 replaced by 278. In this case we take $Z = Z_0$. We obviously have

$$1 \leqslant Z \leqslant \min\left\{DG/10,\ Da_1,\ Da_2,\ \operatorname{Log}\frac{Da_1a_2}{a_2l_1 + a_1l_2}\right\}. \tag{8.3}$$

Before we can apply Theorem 5.11 we have to check

$$Z^3(a_1 + a_2) \leqslant Da_1a_2G^2. \tag{8.4}$$

We know that $Z \leqslant Da_j$ $(j = 1,2)$, hence $Z(a_1 + a_2) \leqslant 2Da_1a_2$. Our assumption $Z \leqslant G/\sqrt{2}$ yields (8.4). Now, from Corollary 7.1 we deduce

$$|b_1 \log\alpha_1 - b_2\log\alpha_2| \geqslant \exp\{-2200\,D^4a_1a_2G^2Z^{-3}\}. \tag{8.5}$$

Since $Z \geqslant 2$ and

$$G^2/Z^3 \leqslant (5.5 + \operatorname{Log} B + \operatorname{Log}\operatorname{Log} B)^2/8 < (1.04^2/8)(7.5 + \operatorname{Log} B)^2$$

we have proved our claim.

(f) Assume $G \leqslant \sqrt{2}Z_0$. We prove the estimate in Corollary 1.1 with the constant 335.

We take $Z = G/10$. In view of the inequalities $D \geqslant 1$ and $Z \leqslant Z_0$, we easily obtain inequality (8.3). It remains to check (8.4): since $G \leqslant \sqrt{2}Z_0 \leqslant \sqrt{2}Da_j$ $(j = 1,2)$, we have $G(a_1 + a_2) \leqslant 2\sqrt{2}Da_1a_2$, hence

$$Z^3(a_1 + a_2) \leqslant 10^{-3}(a_1 + a_2)G^3 \leqslant 2\sqrt{2}\,10^{-3}Da_1a_2G^2 \leqslant Da_1a_2G^2.$$

We conclude from Corollary 7.1 that the lower bound (8.5) is still valid. However

$$G^2Z^{-3} = 10^3G^{-1} \leqslant (1000/20)(7.5 + 10.64)^{-2}(7.5 + \operatorname{Log} B)^2$$

because $\operatorname{Log} B \geqslant 10.64$ and $G \geqslant 20$. Our claim follows from the inequality

$$2200(1000/20)(7.5 + 10.64)^{-2} < 335.$$

(g) Finally we assume $\sqrt{2}Z_0 \leqslant G \leqslant (10/D)Z_0$.

Again in this case we shall prove the conclusion of Corollary 1.1 with the constant 335 instead of 500.

We take $Z = DG/10$.

Since $Z \leqslant Z_0$, inequality (8.3) follows from (8.2). We now check (8.4): we have

$$D \leqslant 10/\sqrt{2} \quad \text{and} \quad G \leqslant (10/D)Z_0 \leqslant 10a_j \quad (j = 1,2),$$

hence

$$D^2(a_1 + a_2)G \leqslant 2(10/\sqrt{2})^2\,10a_1a_2 = 10^3a_1a_2$$

and

$$Z^3(a_1 + a_2) \leqslant (DG/10)^3(a_1 + a_2) \leqslant DG^2a_1a_2.$$

We conclude from Corollary 7.1 that (8.5) holds. Now $G^2Z^{-3} = 10^3/D^3G$ and we conclude as in case (f).

Now the proof of Corollary 1.1 is complete.

9. An example. In a paper by J. M. Cherubini and R. V. Wallisser [3], our previous bound [7] was applied to compute all the imaginary quadratic fields of class number one.

The linear form which is used by these authors is

$$\Lambda = p\operatorname{Log}(5 + 2\sqrt{6}) - 2q\operatorname{Log}(2 + \sqrt{3}), \qquad p, q \in \mathbf{Z}.$$

By arguments of analytic number theory, they prove the estimate

$$|\Lambda| < 50\exp\{-\pi\Delta/24\}, \qquad \Delta = \sqrt{|d|},$$

where d is the discriminant of the considered quadratic field.

Moreover $|p|, |q| \leqslant 2\Delta$. We suppose $\Delta \geqslant e^{18.9}$.

Take $\alpha_i = 5 + 2\sqrt{6}$, $\alpha_j = 2 + \sqrt{3}$, where $\{i, j\} = \{1,2\}$ (because of the condition $a_1b_1 \leqslant a_2b_2$, we do not know the right choice of the indices); then $D = 4$.

Put $l_i = \operatorname{Log}\alpha_i = 2.29243\ldots$ and $l_j = \operatorname{Log}\alpha_j = 1.31695\ldots$ We can choose $a_i = l_i/2$, $a_j = 1$, $f = 1$. We notice that

$$\frac{2ea_1a_2D}{f(a_2l_1 + a_1l_2)} = \frac{4e}{1 + l_j/2} > 6.556.$$

This shows that we can take $Z = 1.88$, $\varepsilon = 1$.

We can take $G' = \operatorname{Log}(e/2 + 2e/l_j)$, so that $G'/D < 0.43$.

The estimates (2.11) to (2.14) imply that $\omega(b) < \max\{2.6, \operatorname{Log}\operatorname{Log} b\}$ for any integer $b \geqslant 2$, this allows us to choose

$$G = 0.93 + \operatorname{Log} 2\Delta + \operatorname{Log}\operatorname{Log} 2\Delta.$$

Thus $G > 23.5$, and we can take $\theta = 50$.

If we choose $c_0 = 81.5$, $c_1 = 2.03$ and $c = 6.47$ then the same computation as in Section 6 gives $C = 1049$. So $C/Z^3 < 158$, and we get

$$158 \cdot 4^4 \cdot a_1 a_2 (0.93 + \operatorname{Log} 2\Delta + \operatorname{Log}\operatorname{Log} 2\Delta)^2 \geqslant \pi \Delta/24 - 4,$$

or

$$\Delta \leqslant 3.09 \cdot 10^5 (0.93 + \operatorname{Log} 2\Delta + \operatorname{Log}\operatorname{Log} 2\Delta)^2 + 31.$$

This gives $\Delta < 1.73 \cdot 10^8$ $(< e^{18.97})$, so that

$$d > -3 \cdot 10^{16},$$

whereas the lower bound of our previous paper gave only $d > -10^{34}$.

10. Another example. Let x, y, p, q be positive rational integers with $x^p \neq y^q$. Let X, Y, B be positive real numbers satisfying

$$X \geqslant \max\{x, 3\}, \quad Y \geqslant \max\{y, 3\}, \quad B \geqslant \max\{p, q\}.$$

COROLLARY 10.1. *We have*

$$|x^p y^{-q} - 1| > \exp\{-500 \operatorname{Log} X \operatorname{Log} Y (8 + \operatorname{Log} B)^2\}.$$

Proof. We consider three cases.

(a) If x and y are multiplicatively dependent, we can write $x = z^u$, $y = z^v$ where z, u, v are positive integers. Put $m = up - vq$. Then

$$|x^p y^{-q} - 1| = |z^m - 1| \geqslant 1/z,$$

and the result is obvious.

(b) Assume $\operatorname{Log} B \leqslant 12.33$. We have

$$|x^p y^{-q} - 1| \geqslant y^{-q} \geqslant \exp\{-B \operatorname{Log} Y\},$$

and the assumption $\operatorname{Log} B \leqslant 12.33$ implies $B < 548 (8 + \operatorname{Log} B)^2$.

Now we have $\operatorname{Log} X \geqslant \operatorname{Log} 3$ and $548/\operatorname{Log} 3 < 499$. Therefore, we get the conclusion.

(c) Now we assume $\operatorname{Log} B > 12.33$ and x, y multiplicatively independent. We shall use Theorem 5.11 with $Z = \varepsilon = f = D = 1$. We choose $G = 1.002 (8 + \operatorname{Log} B)$. Notice that

$$\operatorname{Log}(e/2 + 2e/\operatorname{Log} x) \leqslant \operatorname{Log}(e/2 + 2e/\operatorname{Log} 2) \leqslant 2.22$$

and

$$\operatorname{Log} B + \operatorname{Log}\operatorname{Log} B + 2.82 < 1.003 (8 + \operatorname{Log} B).$$

We take $\theta = 20.33$, $c_0 = 31.18$, $c_1 = 1.36$, $c = 3.28$ and find $C = 497$. Therefore

$$|p \operatorname{Log} x - q \operatorname{Log} y| > \exp\{-C' \operatorname{Log} X \operatorname{Log} Y (8 + \operatorname{Log} B)^2\}$$

where $C' = (1.003)^2 C$.

Now, if $x^p > y^q$, then $x^p y^{-q} - 1 \geqslant p \operatorname{Log} x - q \operatorname{Log} y$, while if $x^p < y^q$, then

$$1 - x^p y^{-q} \geqslant (p \operatorname{Log} x - q \operatorname{Log} y)/2,$$

because $e^{-x} < 1 - x/2$ for $0 < x \leqslant 1$. Finally we have

$$C' + (\operatorname{Log} 2)/(16 \operatorname{Log} 3)^2 < C' + 0.003 = C'' \text{ (say)},$$

hence

$$C' \operatorname{Log} X \operatorname{Log} Y (8 + \operatorname{Log} B)^2 + \operatorname{Log} 2 < C'' \operatorname{Log} X \operatorname{Log} Y (8 + \operatorname{Log} B)^2,$$

where $C'' < 500$.

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