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The existence of universal invariant measures on large sets

by

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Abstract. We consider countably additive, nonnegative, extended real-valued measures which vanish on singletons. Such a measure is *universal* on a set X iff it is defined on all subsets of X. We prove, in particular, that there exists a universal σ-finite measure on X which is invariant with respect to a given group G of bijections of X iff there exists a universal σ-finite measure on X such that for every subgroup H of G of cardinality ω_1 the set of all points of X with uncountable H-orbits has measure zero.

0. Terminology. Our set-theoretic notation and terminology are standard. Ordinals are identified with the sets of their predecessors and cardinals are defined as initial ordinals. If A is a set, then $P(A)$ denotes the family of all subsets of A, and $|A|$ is the cardinality of A. If $f\colon X \to Y$ is a function and $A \subset X$, then $f[A]$ denotes the image of A.

All measures considered in this paper are assumed to be:

— nonnegative extended real-valued;
— countably additive;
— vanishing on singletons;
— assuming at least one positive finite value.

A measure is called *universal* on a set X iff it is defined on $P(X)$. We adopt the convention that the phrase "measure on X" always means "universal measure on X".

Let $\varkappa$, λ be infinite cardinals. A measure μ on X is called:

— *$\varkappa$-additive* iff every union of less than $\varkappa$ sets of measure 0 has measure 0;
— *finite* iff $\mu(X) < +\infty$;
— *λ-finite* iff every set of positive measure is the union of less than λ pairwise disjoint subsets of positive finite measure.

Notice that if $\varkappa > \lambda$ and μ is $\varkappa$-additive, then it is λ-finite iff X is the union of less than λ sets of positive finite measure. Following traditional terminology, we write "σ-finite" instead of "ω_1-finite".

By an *ideal* on a set X we mean here a family $I \subset P(X)$ which contains all singletons, does not contain X and is closed under the operations of countable union and of taking subsets.

An ideal I on X is called:

— *$\varkappa$-complete* iff every union of less than $\varkappa$ sets from I belongs to I;

— *λ-saturated* iff for every family $\mathscr{A} \subset P(X) \setminus I$ of cardinality λ there exist two sets $A, B \in \mathscr{A}$ such that $A \cap B \notin I$.

We define $\mathrm{sat}(I)$ to be the least cardinal λ for which I is λ-saturated.

The most important (from our point of view) example of an ideal on X is the ideal consisting of all sets of measure 0 for a given measure μ on X. We call it the *ideal* of μ and denote it by I_μ. Notice that if $\varkappa > \lambda > \omega$ and μ is $\varkappa$-additive, then μ is λ-finite iff I_μ is λ-saturated.

We shall use some basic facts concerning the possibility of extending a given ideal I on X to the ideal of a measure on X.

PROPOSITION 0.1. *Assume that $\varkappa$ is real-valued measurable. If I is any $\varkappa$-complete ideal on a set X so that I is generated by $\leqslant \varkappa$ sets, then I can be extended to the ideal of a $\varkappa$-additive measure on X.*

Proof. Let $\{B_\xi : \xi < \varkappa\}$ be a family of $\leqslant \varkappa$ sets generating the ideal I. It is easy to construct a set $L \subset X$ such that $|L| = \varkappa$ and $|B_\xi \cap L| < \varkappa$ for each $\xi < \varkappa$. We take an arbitrary $\varkappa$-additive measure μ on L and define a measure m on X by

$$m(A) = \mu(A \cap L) \quad \text{for } A \subset X .$$

It follows immediately that $I \subset I_m$. ■

We say that an uncountable regular cardinal $\varkappa$ has *property* E iff, for any set X, every $\varkappa$-complete ideal on X can be extended to the ideal of a $\varkappa$-additive measure on X. It is easy to see that for $\varkappa > 2^\omega$ property E is equivalent to strong compactness. On the other hand, it follows from Fisher's Axiom (see [3] for its formulation) that 2^ω has property E. An unpublished result of Kunen states that if ZFC+ "There exists a strongly compact cardinal" is consistent, then so is ZFC+ Fisher's Axiom.

Finally, notice that if $\varkappa$ is any infinite cardinal, then every ideal of a $\varkappa$-additive measure on X can be extended to the ideal of a finite $\varkappa$-additive measure on X.

1. Preliminaries. Throughout this paper, G denotes a certain group of bijections of a given set X.

We say that a measure m on a set $Z \subset X$ is *G-invariant* (or simply: *invariant*) iff $m(g[A]) = m(A)$ whenever $A \subset Z$ and $g \in G$ are such that $g[A] \subset Z$. We say that an ideal I on a set $Z \subset X$ is *G-invariant* (or simply: *invariant*) iff $A \in I$ implies that $g[A] \in I$ whenever $A \subset Z$ and $g \in G$ are such that $g[A] \subset Z$.

These definitions still make sense in the case where $(X, \cdot)$ is a group and G is a subgroup of X. We only have to identify each element $g \in G$ with the associated left shift $\varphi_g : X \to X$ given by the formula $\varphi_g(h) = g \cdot h$.

Harazishvili [2] and, independently, Erdös and Mauldin [1] proved that there is no σ-finite X-invariant measure on any group X. In fact, their argument gives

the stronger result: there is no σ-finite G-invariant measure on X for any uncountable subgroup G of X. Moreover, it follows from a theorem of Ryll-Nardzewski and Telgarsky [5] that, under the above assumptions, there does not even exist an ω_1-saturated G-invariant ideal on X. On the other hand, Pelc [4] noticed that if G is an at most countable subgroup of X, then a σ-finite G-invariant measure always exists on X, provided that $|X| \geqslant$ the first real-valued measurable cardinal.

These results lead to the problem, formulated by Pelc (personal communication), of finding necessary and sufficient conditions for the existence of a σ-finite G-invariant measure on an arbitrary set X. The aim of the present paper is to give a solution to this problem. In Section 2, we prove in fact a stronger theorem on the existence of a $\varkappa$-additive λ-finite G-invariant measure on X, under the assumption that $\varkappa > \lambda > \omega$. As a tool we use the following auxiliary fact, which was proved in [6].

LEMMA 1.1. *If μ is a finite G-invariant measure on a set $Z \subset X$, then there exists a G-invariant measure m on X extending μ and such that*

$$I_m = \{V \subset X : \forall g \in G \; g[V] \cap Z \in I_\mu\} . \; ■$$

Our investigations reveal deep connection between the problem of existence of invariant measures on X and the problem of existence of invariant ideals on X. The latter is discussed in Section 3 where, assuming again that $\varkappa > \lambda > \omega$, we find necessary and sufficient conditions for the existence of a $\varkappa$-complete λ-saturated G-invariant ideal on X.

From now on we assume throughout the paper that $\varkappa$ and λ are fixed cardinals such that $\varkappa > \lambda > \omega$.

We use some special notation to describe the action of G on X. If $F \subset G$, then $[F]$ denotes the subgroup of G generated by F; if additionally $x \in X$, then Fx denotes the set $\{f(x) : f \in F\}$, called the *F-orbit of x*. If H is a subgroup of G, we write

$$O_\lambda(H) = \{x \in X : |Hx| = \lambda\} ;$$

$$O_{\geqslant\lambda}(H) = \{x \in X : |Hx| \geqslant \lambda\} .$$

Finally we define:

$$S_\lambda(G) = \{O_\lambda(H) : H \text{ — a subgroup of } G \text{ of cardinality } \lambda\} ;$$

$$S_{\lambda\varkappa}(G) = \{O_{\geqslant\lambda}(H) : H \text{ — a subgroup of } G \text{ of cardinality less than } \varkappa\} .$$

2. $\varkappa$-additive, $\varkappa$-finite measures. Our main result establishes a relationship between the existence of a $\varkappa$-additive λ-finite G-invariant measure on X and a certain property of the family $S_\lambda(G)$. The following observation is a first step in this direction.

LEMMA 2.1. *If m is a $\varkappa$-additive λ-finite G-invariant measure on X, then $S_\lambda(G) \subset I_m$.*

Proof. Let H be an arbitrary subgroup of G of cardinality λ. We partition the set $O_\lambda(H)$ into pairwise disjoint selectors $\{S_\alpha : \alpha < \lambda\}$ of the family of all H-orbits. Since the ideal I_m is λ-saturated, there exists $\alpha_0 < \lambda$ such that $S_{\alpha_0} \in I_m$.

Notice that $S_\alpha = \bigcup_{h\in H} (h[S_{\alpha_0}]\cap S_\alpha)$ for each $\alpha<\lambda$ and, furthermore, $O_\lambda(H) = \bigcup_{\alpha<\lambda} S_\alpha$. Hence, I_m being invariant and $\varkappa$-additive, we conclude that $O_\lambda(H)\in I_m$. ■

In view of this lemma, the existence of a $\varkappa$-additive λ-finite G-invariant measure on X implies the possibility of extending the family $S_\lambda(G)$ to the ideal of a $\varkappa$-additive measure on X. Surprisingly, the converse implication is also true, so the main result can be formulated as follows:

THEOREM 2.2. *The following statements are equivalent*:

(i) *There exists a $\varkappa$-additive λ-finite G-invariant measure on X.*

(ii) *The family $S_\lambda(G)$ can be extended to the ideal of a $\varkappa$-additive measure on X.*

Proof. In view of Lemma 2.1 only the implication (ii)→(i) requires a proof. It will be based on the following lemma.

LEMMA 2.3. *Suppose that there exists a set $Z\subset X$ and a G-invariant measure μ on Z with the properties*:

(i) *μ is $\varkappa$-additive and finite,*

(ii) *there exists a family $F\subset G$ such that $|F|<\lambda$ and*:

$$\forall A\subset Z\ \forall g\in G\ (g[A]\cap \bigcup_{f\in F} f[Z] = \emptyset \to \mu(A) = 0).$$

Then there exists a $\varkappa$-additive λ-finite G-invariant measure on X.

Proof of Lemma 2.3. By Lemma 1.1, μ extends to an invariant measure m on X such that

$$(0)\qquad I_m = \{V\subset X\colon \forall g\in G\ g[V]\cap Z\in I_\mu\}.$$

Obviously, m is $\varkappa$-additive.

To see that m is λ-finite, it suffices to represent X as a union of less than λ sets of positive finite measure. We have

$$X = \bigcup_{f\in F} f[Z]\cup Z\cup\big(X\setminus(Z\cup \bigcup_{f\in F} f[Z])\big)$$

where, for each $f\in F$,

$$m(f[Z]) = m(Z) = \mu(Z)\quad\text{and}\quad 0<\mu(Z)<+\infty .$$

Consequently, it is enough to prove that

$$m\big(X\setminus(Z\cup \bigcup_{f\in F} f[Z])\big) = 0 .$$

To see this we use equality (0). Let $g\in G$ be arbitrary and let

$$A = g\big[X\setminus(Z\cup \bigcup_{f\in F} f[Z])\big]\cap Z .$$

Then $g^{-1}[A]\cap \bigcup_{f\in F} f[Z] = \emptyset$; hence $\mu(A) = 0$ by assumption (ii). This concludes the proof of Lemma 2.3. ■

We continue the proof of Theorem 2.2. Let μ be a $\varkappa$-additive measure on X such that $S_\lambda(G)\subset I_\mu$. We may assume that μ is finite. We will prove that there exists a set $Z\subset X$ such that the measure $\mu|P(Z)$ satisfies the hypotheses of Lemma 2.3.

CLAIM 1. *There exists a set $Z_1\subset X$ such that*:

(i) $\mu(Z_1)>0$,

(ii) $\forall A\subset Z_1\ \forall g\in G\ (g[A]\subset Z_1\to \mu(A\Delta g[A]) = 0)$.

Proof of Claim 1. Suppose that there is no such set. Then clearly

(1) For every set $Y\subset X$ of positive measure there exists a set $V\subset Y$ and a function $g\in G$ such that

$$\mu(V)>0,\quad g[V]\subset Y\quad\text{and}\quad V\cap g[V] = \emptyset .$$

We claim that (1) implies:

(2) For every set $Y\subset X$ of positive measure there exists a subgroup $H(Y)\subset G$ and a set $W(Y)\subset Y$ with the following properties:

$$(2.1)\qquad |H(Y)|<\omega_1 ,$$

$$(2.2)\qquad \mu(Y\setminus W(Y)) = 0 ,$$

$$(2.3)\qquad \forall x\in W(Y)\ |Y\cap H(Y)x|\geqslant 2$$

To see this, let $R\subset P(Y)\times G$ be a maximal family such that

$$\langle V,g\rangle\in R\to \mu(V)>0 ,$$
$$\langle V_1,g_1\rangle, \langle V_2,g_2\rangle\in R\to V_1\cap V_2 = \emptyset ,$$
$$\langle V,g\rangle\in R\to g[V]\subset Y\quad\text{and}\quad V\cap g[V] = \emptyset$$

We define

$$H(Y) = [\{g\in G\colon \exists V\subset Y\ \langle V,g\rangle\in R\}] ,$$
$$W(Y) = \bigcup\{V\subset Y\colon \exists g\in G\ \langle V,g\rangle\in R\}$$

Easy verification of properties (2.1)–(2.3) completes the proof of (2).

Now we construct by induction two sequences

$$\langle X_\xi\colon \xi<\lambda\rangle,\ \text{a sequence of subsets of } X ,$$
$$\langle F_\xi\colon \xi<\lambda\rangle,\ \text{a sequence of subsets of } G ,$$

in such a way that putting

$$T = \bigcap_{\xi<\lambda} X_\xi\quad\text{and}\quad H = [\bigcup_{\xi<\lambda} F_\xi]$$

we get the following conditions satisfied

$$(3.1)\qquad \mu(T)>0 ,$$

$$(3.2)\qquad |H|\leqslant\lambda ,$$

$$(3.3)\qquad T\subset O_\lambda(H) .$$

In each step of the induction procedure we keep the condition

$$\forall_{\xi<\lambda}\ (\mu(X\setminus X_\xi) = 0 \text{ and } |F_\xi| \leqslant \lambda)$$

fulfilled

So, let $X_0 = X$, $F_0 = H(X)$ and assume that we have already constructed $\langle X_\xi\colon \xi<\alpha\rangle$ and $\langle F_\xi\colon \xi<\alpha\rangle$ for a certain $\alpha<\lambda$. We put $H_\alpha = [\bigcup_{\xi<\alpha} F]$; $|H_\alpha| \leqslant \lambda$. We partition the set X into pairwise disjoint (sub-) selectors $\{S^\alpha_\beta\colon \beta<\nu\}$ of the family of all H_α-orbits; $\nu \leqslant |H_\alpha|$. We let $P_\alpha = \{\beta<\nu\colon \mu(S^\alpha_\beta)>0\}$ and using (2) we define

$$X_\alpha = \bigcup_{\beta\in P_\alpha} W(S^\alpha_\beta),$$

$$F_\alpha = H_\alpha \cup \bigcup_{\beta\in P_\alpha} H(S^\alpha_\beta)$$

We have

$$X\setminus X_\alpha = \bigcup_{\beta\in P_\alpha} (S^\alpha_\beta\setminus W(S^\alpha_\beta)) \cup \bigcup_{\beta\notin P_\alpha} S^\alpha_\beta;$$

hence, by $\varkappa$-additivity of μ, $\mu(X\setminus X_\alpha) = 0$. Since it is also easy to see that $|F_\alpha| \leqslant \lambda$, the construction is completed

It now remains to verify that conditions (3.1)–(3.3) are satisfied. Only the last one requires a proof.

Let $x\in T$. It is enough to show that

$$\forall\alpha<\lambda\ F_\alpha x\setminus H_\alpha x \neq \emptyset.$$

Indeed, choosing $x_\alpha \in F_\alpha x\setminus H_\alpha x$ for each $\alpha<\lambda$, we obtain a set $\{x_\alpha\colon \alpha<\lambda\} \subset Hx$ of cardinality λ. Hence, by (3.2), it follows that $|Hx| = \lambda$.

So let $\alpha<\lambda$. Since $x\in X_\alpha$, there is an ordinal $\beta\in P_\alpha$ such that $x\in W(S^\alpha_\beta)$. Hence, by (2.3), there exists a function $g\in H(S^\alpha_\beta)$ such that $g(x)\in S^\alpha_\beta$ and $g(x)\neq x$. But then $g(x)\in F_\alpha x\setminus H_\alpha x$ since $x\in H_\alpha x$ and $|S^\alpha_\beta\cap H_\alpha x| = 1$. This concludes the proof of (3.3).

To complete the proof of Claim 1 it suffices to notice that conditions (3.1)–(3.3) imply $|H| = \lambda$ and $\mu(Q_\lambda(H))>0$. This contradicts the assumption that $S_\lambda(G)\subset I_\mu$.

Claim 2. *There exist a set $Z\subset Z_1$ and a family $F\subset G$ such that*

(i) $\mu(Z)>0$ *and* $|F|<\lambda$,

(ii) $\forall A\subset Z\ \forall g\in G(g[A]\cap \bigcup_{f\in F} f[Z] = \emptyset \to \mu(A) = 0)$.

Proof of Claim 2. Suppose that the claim is false, i.e.

(4) For every set $W\subset Z_1$ of positive measure and for any family $F\subset G$ of cardinality less than λ there exists a set $A\subset W$ and a function $g\in G$ such that

$$g[A]\cap \bigcup_{f\in F} f[W] = \emptyset \quad\text{and}\quad \mu(A)>0.$$

We construct by induction two sequences

$\langle X_\xi\colon \xi<\lambda\rangle$, a sequence of subsets of X,

$\langle F_\xi\colon \xi<\lambda\rangle$, a sequence of subsets of G.

Our goal is to obtain, as in the proof of Claim 1, the set $T = \bigcap_{\xi<\lambda} X_\xi$ and the subgroup $H = [\bigcup_{\xi<\lambda} F_\xi]$ of G with properties (3.1)–(3.3).

During the induction procedure we now control the condition

$$\forall\xi<\lambda\ (\mu(Z_1\setminus X_\xi) = 0 \text{ and } |F_\xi|<\omega_1).$$

So, let $X_0 = Z_1$, $F_0 = \{\mathrm{id}_X\}$ and assume that we have already constructed $\langle H_\xi\colon \xi<\alpha\rangle$ and $\langle F_\xi\colon \xi<\alpha\rangle$ for an ordinal $\alpha<\lambda$. Let $\bigcup_{\xi<\alpha} F_\xi = \{f_\beta\colon \beta<\nu\}$; $\nu<\lambda$. Let $R_\alpha\subset P(Z_1)\times G$ be a maximal family such that

$$\langle V, g\rangle\in R_\alpha \to \mu(V)>0,$$

$$\langle V_1, g_1\rangle, \langle V_2, g_2\rangle \in R_\alpha \to V_1\cap V_2 = \emptyset$$

$$\langle V, g\rangle\in R_\alpha \to g[V]\cap \bigcup_{\beta<\mu} f_\beta[Z_1] = \emptyset.$$

We define

$$X_\alpha = \bigcup\{V\subset Z_1\colon \exists g\in G\ \langle V, g\rangle\in R_\alpha\},$$

$$F_\alpha = \{g\in G\colon \exists V\subset Z_1\ \langle V, g\rangle\in R_\alpha\}.$$

Now suppose that, contrary to our intentions, $\mu(Z_1\setminus X)>0$. We set $W = Z_1\setminus X_\alpha$ and $F = \{f_\beta\colon \beta<\nu\}$. By (4) there exists a set $A\subset W$ and a function $g\in G$ such that

$$g[A]\subset X\setminus \bigcup_{\beta<\nu} f_\beta[W] \quad\text{and}\quad \mu(A)>0.$$

We have

$$X\setminus \bigcup_{\beta<\nu} f_\beta[W] \subset (X\setminus \bigcup_{\beta<\nu} f_\beta[Z_1]) \cup \bigcup_{\beta<\nu} f_\beta[Z_1\setminus W].$$

Hence $A = A_1\cup A_2$, where

$$g[A_1]\subset X\setminus \bigcup_{\beta<\nu} f_\beta[Z_1],$$

$$g[A_2]\subset \bigcup_{\beta<\nu} f_\beta[Z_1\setminus W].$$

For each $\beta<\nu$ we set $B_\beta = A_2\cap (g^{-1}\circ f_\beta)[Z_1\setminus W]$; then

$$B_\beta\subset Z_1 \quad\text{and}\quad (f_\beta^{-1}\circ g[B_\beta]\subset Z_1\setminus W\subset Z_1\setminus B_\beta;$$

hence $\mu(B_\beta) = 0$ by the properties of Z_1 (see Claim 1 (ii). Since $A_2 = \bigcup_{\beta<\nu} B_\beta$, we get $\mu(A_2) = 0$. But this implies that $\mu(A_1)>0$ and, consequently, the pair $\langle A_1, g\rangle$ violates the maximality of the family R_α. This contradiction proves that $\mu(Z_1\setminus X_\alpha) = 0$. Since E_μ is ω_1-saturated, we also have $|F_\alpha|<\omega_1$. The construction is completed.

We now verify conditions (3.1)–(3.3). Clearly, $\mu(T) = \mu(Z_1)$ and $|H|\leqslant\lambda$. To prove (3.3) let $x\in T$. It is enough to show that

$$\forall\alpha<\lambda\ F_\alpha x\setminus (\bigcup_{\xi<\alpha} F_\xi)x \neq \emptyset.$$

We fix an ordinal $\alpha < \lambda$. Since $x \in X_\alpha$, there exists a pair $\langle V, g\rangle \in R_\alpha$ such that $x \in V$. But then $g \in F_\alpha$ and $g(x) \in X\setminus(\bigcup_{\xi<\alpha} F_\xi)x$.

This completes the proof of Claim 2 since, as we have already seen, conditions (3.1)–(3.3) lead to a contradiction.

To conclude the proof of the theorem it suffices to notice that the set Z with the measure $\mu|P(Z)$ satisfies the hypotheses of Lemma 2.3. ■

There naturally arises the question whether condition (ii) of the above theorem may be replaced by a simpler one. We will show that this can be done in some cases.

First we need two technical Lemmas.

LEMMA 2.4. *Let μ be a finite $\varkappa$-additive measure on X. The following statements are equivalent*

(i) $S_\lambda(G) \subset I_\mu$,

(ii) $S_{\lambda\varkappa}(G) \subset I_\mu$.

Proof. It is obvious that (ii)→(i). For the converse, assume that $S_\lambda(G) \subset I_\mu$ and suppose, contrary to the claim, that $S_{\lambda\varkappa}(G)\setminus I_\mu \neq \emptyset$. Then there exists a subgroup $H \subset G$ and a cardinal ϱ such that $|H| < \varkappa$, $\lambda \leqslant \varrho < \varkappa$ and $\mu(O_\varrho(H)) > 0$.

Let $\{S_\xi\colon \xi < \varrho\}$ be a partition of $O_\varrho(H)$ into pairwise disjoint selectors of the family of all H-orbits of cardinality ϱ.

We may assume that $\mu(S_0) > 0$.

For every ξ, $0 < \xi < \lambda$, we consider a maximal family $R_\xi \subset P(S_0) \times G$ such that

$$\langle V, g\rangle \in R_\xi \to \mu(V) > 0\,,$$
$$\langle V_1, g_1\rangle, \langle V_2, g_2\rangle \in R_\xi \to V_1 \cap V_2 = \emptyset\,,$$
$$\langle V, g\rangle \in R_\xi \to g[V] \subset S_\xi,$$

and we set

$$Y_\xi = \bigcup\{V \subset S_0\colon \exists g \in G\ \langle V, g\rangle \in R_\xi\}\,,$$
$$F_\xi = \{g \in G\colon \exists V \subset S_0\ \langle V, g\rangle \in R_\xi\}\,.$$

Finally, we define

$$Y = \bigcap_{0<\xi<\lambda} Y_\xi \quad \text{and} \quad H_1 = [\bigcup_{0<\xi<\lambda} F_\xi]\,.$$

It follows easily that

$$|H_1| = \lambda, \quad \mu(Y) > 0 \quad \text{and} \quad Y \subset O_\lambda(H_1)\,;$$

but this is impossible. ■

LEMMA 2.5. *Assume that $\varkappa$ is regular. The following statements are equivalent:*

(i) *For every subgroup H of G, if $|H| < \varkappa$ then $|X\setminus O_{\geqslant\lambda}(H)| \geqslant \varkappa$.*

(ii) *The family $S_{\lambda\varkappa}(G)$ can be extended to a $\varkappa$-complete ideal on X.*

Proof. Implication (ii)→(i) is obvious. To see that the converse is true, it suffices to prove that $|X\setminus \bigcup_{\xi<\varrho} A_\xi| \geqslant \varkappa$ for every family $\{A_\xi\colon \xi < \varrho\} \subset S_{\lambda\varkappa}(G)$ of cardinality $\varrho < \varkappa$. So let, for each $\xi < \varrho$, $A_\xi = O_{\geqslant\lambda}(H_\xi)$, where H_ξ is a subgroup of G and $|H_\xi| < \varkappa$. We define $H = [\bigcup_{\xi<\varrho} H_\xi]$. It is enough to notice that $|H_\xi| < \varkappa$ and $\bigcup_{\xi<\varrho} O_{\geqslant\lambda}(H_\xi) \subset O_{\geqslant\lambda}(H)$. ■

If condition (i) of Lemma 2.5 holds, then the $\varkappa$-complete ideal generated by the family $S_{\lambda\varkappa}(G)$ will be denoted by $I_{\lambda\varkappa}(G)$. In view of Lemma 2.4 and Theorem 2.2 the existence of a $\varkappa$-additive λ-finite invariant measure on X is equivalent to the possibility of extending the ideal $I_{\lambda\varkappa}(G)$ to the ideal of a $\varkappa$-additive measure on X. This leads to the following refinement of Theorem 2.2.

THEOREM 2.6. *Assume that $\varkappa$ is real-valued measurable and $|G| \leqslant \varkappa$. The following statements are equivalent*:

(i) *There exists a $\varkappa$-additive λ-finite G-invariant measure on X.*

(ii) *For every subgroup H of G, if $|H| < \varkappa$ then $|X\setminus O_{\geqslant\lambda}(H)| \geqslant \varkappa$.*

Proof. Only the implication (ii)→(i) requires a proof. Let $G = \{g_\xi\colon \xi < \varkappa\}$. For each ordinal α, $\lambda \leqslant \alpha < \varkappa$, we set $H_\alpha = [\{g_\xi\colon \xi < \alpha\}]$. It is easy to see that the family $\{0_{\geqslant\lambda}(H_\alpha)\colon \lambda \leqslant \alpha < \varkappa\}$ generates the ideal $I_{\lambda\varkappa}(G)$. Hence, in view of Proposition 0.1, $I_{\lambda\varkappa}(G)$ extends to the ideal of a $\varkappa$-additive measure on X. ■

If we assume that $\varkappa$ has property E, then the above characterization is obviously true without any restriction on the cardinality of G.

THEOREM 2.7. *Assume that $\varkappa$ has property* E. *The following statements are equivalent*:

(i) *There exists a $\varkappa$-additive λ-finite G-invariant measure on X.*

(ii) *For every subgroup H of G, if $|H| < \varkappa$ then $|X\setminus O_{\geqslant\lambda}(H)| \geqslant \varkappa$.* ■

In particular, under Fisher's Axiom, the above characterization is true for $\varkappa = 2^\omega$.

The additional assumption on $\varkappa$ cannot be eliminated from the formulation of Theorem 2.7. In fact, if $\varkappa$ is a regular cardinal such that the implication (ii)→(i) holds for every group G, then $\varkappa$ has property E. This follows immediately from the next result.

PROPOSITION 2.8. *Assume that $\varkappa$ is regular. If I is a $\varkappa$-complete ideal on X, then there exists a group G of bijections of X such that $I = I_{\lambda\varkappa}(G)$*

Proof. Let I be a $\varkappa$-complete ideal on X generated by a family $\{B_\xi\colon \xi < \nu\}$ such that $|B_\xi| \geqslant \lambda$ for each $\xi < \nu$.

For every $\xi < \nu$, it is easy to construct a group G_ξ of bijections of X with the following properties:

(1) $|G_\xi| = \lambda$,

(2) $O_\lambda(G_\xi) = B_\xi$,

(3) $\forall x \notin B_\xi \quad G_\xi x = \{x\}$.

We define G to be the group of bijections of X generated by $\bigcup_{\xi<\nu} G_\xi$.

In order to see that $I_{\lambda\varkappa}(G) = I$, take H to be an arbitrary subgroup of G such that $|H| < \varkappa$. Then there exists a set $T \subset \nu$ with $|T| < \varkappa$ and $H \subset [\bigcup_{\xi \in T} G_\xi]$. It is easy to see that $O_{\geqslant\lambda}(H) \subset \bigcup_{\xi \in T} B_\xi$. Hence, by $\varkappa$-completeness of I, we have $O_{\geqslant\lambda}(H) \in I$. This shows that $I_{\lambda\varkappa}(G) \subset I$ and the converse inclusion follows from (1) and (2). ■

Remark 2.9. If we agree that the phrase "ω-finite measure" means "finite measure", then all results of this section remain true for $\lambda = \omega$. The proofs, however, must be changed (see [7] for details).

3. $\varkappa$-complete λ-saturated ideals. An examination of the reasonings in the previous section shows that the key role in our arguments is played by these properties of $\varkappa$-additive λ-finite measures which are expressible in terms of their ideals. For example, the proof of Lemma 2.1 gives the following more general result.

LEMMA 3.1. *If I is a $\varkappa$-complete λ-saturated G-invariant ideal on X, then* $S_\lambda(G) \subset I$. ■

This observation leads to an interesting corollary of Theorem 2.2.

PROPOSITION 3.2. *The following statements are equivalent:*

(i) *There exists a $\varkappa$-additive λ-finite G-invariant measure on X.*

(ii) *There exists a $\varkappa$-additive λ-finite measure on X whose ideal is G-invariant.* ■

Under some additional assumptions we may even obtain a much stronger fact.

PROPOSITION 3.3. *Assume that either $\varkappa$ is real-valued measurable and $|G| \leqslant \varkappa$ or that $\varkappa$ has property* E. *The following statements are equivalent:*

(i) *There exists a $\varkappa$-additive λ-finite G-invariant measure on X.*

(ii) *There exists a $\varkappa$-complete λ-saturated G-invariant ideal on X.*

Proof. Let I be a $\varkappa$-complete λ-saturated invariant ideal on X. Lemma 3.1 implies that $S_\lambda(G) \subset I$. By an easy modification of the proof of Lemma 2.4 we obtain the following:

CLAIM: *Let J be a $\varkappa$-complete λ-saturated ideal on X. Then*

$$S_\lambda(G) \subset J \quad \text{iff} \quad S_{\lambda\varkappa}(G) \subset J.$$

This implies that $S_{\lambda\varkappa}(G) \subset I$, and hence $|X \setminus O_{\geqslant\lambda}(H)| \geqslant \varkappa$ for every subgroup H of G such that $|H| < \varkappa$. The existence of a $\varkappa$-additive λ-finite invariant measure on X follows now from Theorem 2.6 or Theorem 2.7, respectively. ■

These results turn our attention to the problem of existence of an arbitrary $\varkappa$-complete λ-saturated invariant ideal on X. As might have been expected, the solution closely resembles the result obtained for invariant measures.

THEOREM 3.4. *The following statements are equivalent:*

(i) *There exists a $\varkappa$-complete λ-saturated G-invariant ideal on X.*

(ii) *The family $S_\lambda(G)$ can be extended to a $\varkappa$-complete λ-saturated ideal on X.*

Proof. Implication (i)→(ii) follows from Lemma 3.1. The proof of the converse is based on the following counterpart to Lemma 2.3:

LEMMA 3.5. *Suppose that there exist a set $Z \subset X$ and a G-invariant ideal J on Z with the properties:*

(i) *J is $\varkappa$-complete and λ-saturated,*

(ii) *there exists a family $F \subset G$ such that $|F| < \lambda$ and*

$$\forall A \subset Z \ \forall g \in G \ (g[A] \cap \bigcup_{f \in F} f[Z] = \emptyset \to A \in J).$$

Then there exists a $\varkappa$-complete λ-saturated G-invariant ideal on X

Proof of Lemma 3.5. We define an ideal I on X by

$$I = \{A \subset X\colon \forall g \in G \ g[A] \cap Z \in J\}.$$

It is easy to see that I is $\varkappa$-complete and G-invariant. It remains to verify that I is λ-saturated.

Let $H = [F]$, $|H| = \varrho < \lambda$. If we show that

$$I = \{A \subset X\colon \forall h \in H \ h[A] \cap Z \in J\}, \tag{1}$$

then it will easily follow that I is ν-saturated, where $\nu = \max(\varrho^+, \operatorname{sat}(J), \omega_1) \leqslant \lambda$ (see the proof of Proposition 3.1 from [6]).

Let $H = \{h_\xi\colon \xi < \varrho\}$. In order to prove (1) we take a set $A \subset X$ with

$$\forall h \in H \ h[A] \cap Z \in J \tag{2}$$

and we show that $A \in I$. Let $g \in G$ be arbitrary. For each $\xi < \varrho$, set $B_\xi = (g \circ h_\xi)[Z] \cap g[A] \cap Z$ and notice that, J being G-invariant

$$B_\xi \in J \quad \text{iff} \quad (h_\xi^{-1} \circ g^{-1})[B_\xi] \in J.$$

But, since $(h_\xi^{-1} \circ g^{-1})[B_\xi] \subset h_\xi^{-1}[A] \cap Z$ and $h_\xi^{-1}[A] \cap Z \in J$ by (2), this implies that

$$B_\xi \in J \quad \text{for each } \xi < \varrho. \tag{3}$$

We claim, moreover, that

$$(g[A] \cap Z) \setminus \bigcup_{\xi < \varrho} B_\xi \in J. \tag{4}$$

Indeed, we have

$$g^{-1}[(g[A] \cap Z/ \bigcup_{\xi < \varrho} B_\xi] \subset X \setminus \bigcup_{\xi < \varrho} h_\xi[Z],$$

so that (4) follows from assumption (ii).

Finally, (3) and (4) imply, by $\varkappa$-completeness of J, that $g[A] \cap Z \in J$. Since the choice of g was arbitrary, this completes the proof of (1), and hence of Lemma 3.5. ■

Now, in order to prove Theorem 3.4, we show that if J is a $\varkappa$-complete λ-saturated ideal on X extending the family $S_\lambda(G)$, then there exists a set $Z \subset X$ such that the ideal $J \cap P(Z)$ on Z satisfies the hypotheses of Lemma 3.5. We proceed as in the proof of Theorem 2.2. With a minor modification of the arguments used there we prove the following claims:

CLAIM 1. *There exists a set* $Z_1 \subset Z$ *such that*

(i) $Z_1 \notin J$,

(ii) $\forall A \subset Z_1\ \forall g \in G\ (g[A] \subset Z_1 \to A \triangle g[A] \in J)$:

CLAIM 2. *There exist a set* $Z \subset Z_1$ *and a family* $F \subset G$ *such that*

(i) $Z \notin J$ *and* $|F| < \lambda$,

(ii) $\forall A \subset Z\ \forall g \in G\ (g[A] \cap \bigcup_{f \in F} f[Z] = \emptyset \to A \in J)$.

This can be regarded as a satisfactory outline of the proof of Theorem 3.4.

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Locally connected curves viewed as inverse limits

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Abstract. Every locally connected curve is the limit of an inverse sequence of regular continua with monotone bonding surjections. Moreover, any space which is the limit of an inverse sequence of connected graphs with monotone bonding surjections is a rather small continuum.

1. Introduction. All spaces considered in this paper are assumed to be metric, all maps are continuous, d always denotes a distance function, and 'continuum' means 'compact connected (metric) space'.

We will say that a space X is:

(a) a *graph* provided X is a one-dimensional (compact) polyhedron;

(b) a *completely regular continuum* provided X is a continuum such that $\operatorname{int}(Y) \neq 0$ for each nondegenerate subcontinuum Y of X;

(c) a *regular continuum* if X is a continuum such that for any $\varepsilon > 0$ and each $x \in X$ there exists an open neighbourhood U of x in X such that $\operatorname{bd}(U)$ is finite and $\operatorname{diam} U < \varepsilon$ (regular continua are often called 'rim-finite continua');

(d) a *curve* provided X is a continuum of dimension 1.

Clearly, every connected graph is a completely regular continuum and every regular continuum is a locally connected curve. Moreover, each completely regular continuum is regular (see for example Proposition 3.2 below).

Recall that a (continuous) map $f\colon X \to Y$ is said to be *monotone* if $f^{-1}(y)$ is connected for each $y \in Y$.

It is well-known that every curve X is the limit of some inverse sequence (X_n, f_n) of (connected) graphs (see e.g. [2], Theorem 1.13.2, p. 145; it is not difficult to see that the sequence can be chosen in such a manner that all the bonding maps, $f_n\colon X_{n+1} \to X_n$, are surjections). If X is locally connected, one can use the general method of S. Mardešić to produce an inverse sequence (Y_n, g_n) of locally connected continua Y_n with monotone bonding surjections $g_n\colon Y_{n+1} \to Y_n$ such that $X = \liminv(Y_n, g_n)$ ((Y_n, g_n) is obtained as a 'modification' of (X_n, f_n); see [8], p. 164 — the proof of Theorem 2). However, in general, almost nothing can be proved about (Y_n, g_n). In particular, Y_n's need not (and often they can not) be graphs; they are simply locally connected continua. The only essential information on