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DRUKARNIA UNIWERSYTETU JAGIELLOŃSKIEGO W KRAKOWIE

Definitions of finite

by

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Abstract. We consider the statement which asserts the equivalence of two definitions of finite to be a weak form of the axiom of choice. The relationships between several statements of this type are considered.

The assertion that two definitions of finite are equivalent can be considered to be a weak form of the axiom of choice. See for example [4], [8], [9], [14] and [15].

ZFA will denote Zermelo–Fraenkel set theory weakened to permit the existence of atoms without the axiom of choice and we will use the notation $P(A)$ to denote the power set of any set A. We will consider the following definitions of finite taken from [8]:

DEFINITION I. A is *finite*(I) if every nonempty family of subsets of A has an element maximal under set inclusion.

Ia. A is *finite*(Ia) if it is not th on of two disjoint sets neither of which is finite(I).

II. A is *finite*(II) if every non-empty monotone (i.e. linearly ordered by inclusion) family of subsets of A has a maximal element.

III. A is *finite*(III) if the power set of A is Dedekind finite.

IV. A is *finite*(IV) if A is Dedekind finite.

V. A is *finite*(V) if $|A| = 0$ or $2|A| > |A|$.

VI. A is *finite*(VI) if $|A| = 0$ or 1 or $|A|^2 > |A|$.

VII. A is *finite*(VII) if it is not the case that (A is well orderable and $\omega \leqslant |A|$).

In addition we will say that a set is *finite*(D) if A has at most one element or $A = C \cup D$ where $|C| < |A|$ and $|D| < |A|$. We note that finite(I) is equivalent to the usual definition of finite. Therefore a set which is finite(I) will simply be called *finite*. We also note that a set is finite(II) if and only if $P(A)$ has no infinite $\subseteq$-chain. The proof is left to the reader. We refer the reader to [8] for other historical information and a proof of

THEOREM 1. *If a set is finite according to any of the above definitions then it is finite according to any definition which follows.*

A straightforward argument also gives

THEOREM 2. *Finite*(IV) *implies finite*(D) *and finite*(D) *implies finite*(VII).

Theorems 1 and 2 are proved in ZFA. In the theory ZFA+AC all of the above definitions are equivalent. If K and J are any two of I, Ia, II, III, IV, V, VI or VII we let $E(J, K)$ be the sentence in the language of ZFA which says: For every set x, x is finite(K) if and only if x is finite(J).

We will observe the convention that when we write $E(J, K)$, J occurs before K, in the above definition so that by Theorem 1, A is finite(J) implies A is finite(K). In what follows we will abbreviate this $J < K$. We first note: If $M \leqslant K$ and $J \leqslant L$ then $E(J, K)$ implies $E(L, M)$. Combining this with the results of [8], [9] and [15] and several well-known independence results gives us diagram 1.

Many of the arrows in diagram 1 can be reversed. We leave the proofs to the reader.

THEOREM 3. (*In* ZFA) E(VI, VII) *implies* AC. (*And therefore every arrow in the upper right hand diagonal of diagram* 1 *can be reversed.*)

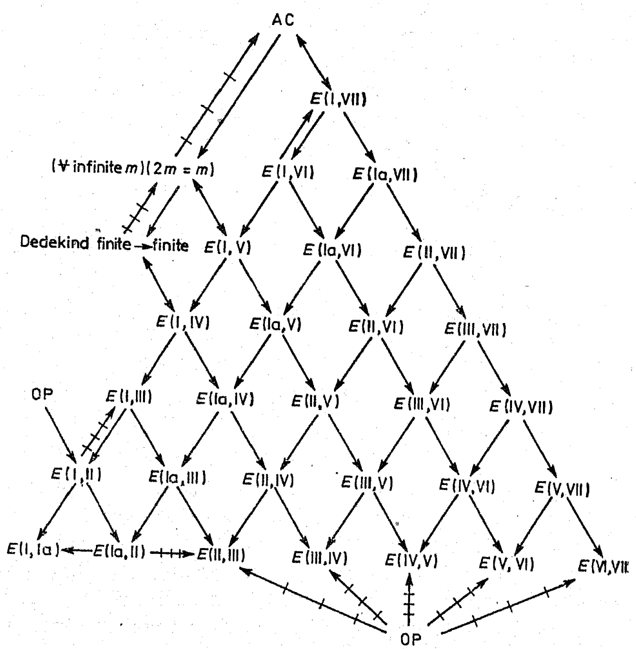

Diagram 1

THEOREM 4. E(V, VI) *implies* AC.

THEOREM 5. E(IV, V) *implies* (*for every infinite cardinal number* m, $2m = m$).

THEOREM 6. E(III, IV) *implies* (*every Dedekind finite set is finite*).

THEOREM 7. E(Ia, III) *implies* E(I, III).

THEOREM 8. E(Ia, II) *implies* E(I, II).

We now construct Fraenkel–Mostowski models of ZFA to show that several of the implications in diagram 1 cannot be reversed and to show the independence results mentioned above regarding finite (D).

Given a model M' of ZFA which has A as its set of atoms, a permutation model M of ZFA is determined by a group G of permutations of A and a filter Γ of subgroups of G which satisfies

$$(\forall a \in A)(\exists H \in \Gamma)(\forall \psi \in H)(\psi(a) = a)$$

and

$$(\forall \psi \in G)(\forall H \in \Gamma)(\psi H \psi^{-1} \in \Gamma).$$

Each permutation of A extends uniquely to a permutation of M' by $\in$ induction and for any $\psi \in G$ we identify ψ with its extension.

If H is a subgroup of G and $x \in M'$ and $(\forall \psi \in H)(\psi(x) = x)$ we say H *fixes* x. If it is also the case that $(\forall \psi \in H)(\forall y \in x)(\psi(y) = y)$ we say H *fixes* x *pointwise*. The permutation model determined by M, G and Γ consists of all those $x \in M'$ such that for every y in the transitive closure of x, there is some $H \in \Gamma$ such that H fixes y. We refer the reader to [7, p. 46] for a proof that M is a model of ZFA.

We will construct six permutation models $M1$ through $M6$. In what follows if G is a group of permutations of a set A and $E \subseteq A$, then $\mathrm{fix}_G(E)$ will denote $\{\psi \in G\colon (\forall x \in E)(\psi(x) = x)\}$.

$M1$, $M2$ and $M4$ are respectively the basic Fraenkel model, the ordered Mostowski model and the second Fraenkel model found in [7, pp. 48–49]. To construct $M3$, let $M3'$ be a model of ZFA+AC with a countable set A of atoms and an order $<$ of A with order type of the rationals. $G3$ is the group of all permutations ψ of A with the property that for every bounded subset B of A, $\psi(B)$ is bounded and $\Gamma 3$ is generated by the groups $\mathrm{fix}_{G3}(E)$ such that E is a bounded subset of A. $M3$ is the model determined by $M3'$, $G3$ and $\Gamma 3$. Similarly, $M5$ is the model determined by $M5'$, $G5$ and $\Gamma 5$ where $M5'$ is $M3'$ and $G5 = \{\psi\colon \psi$ is a permutation of A and $\{t \in A\colon \psi(t) \neq t\}$ is finite$\}$. For each finite subset E of A, let

$$G(E) = \{\psi \in G5\colon (\forall a \in E)[\psi(a) = a \text{ and } (\forall t \in A)((t < a \text{ implies } \psi(t) < a) \text{ and } (a < t \text{ implies } a < \psi(t)))]\}$$

$\Gamma 5$ is the filter of subgroups generated by the groups $G(E)$ such that E is a finite subset of A. To construct $M6$ we let $M6'$ be $M3'$, $G6 = G5$ and $\Gamma 6$ is the filter generated by the subgroups $\mathrm{fix}_{G6}(E)$ for E a bounded subset (under $<$) of A.

THEOREM 9. *In* $M1$ *there is a set* B *which is finite*(D) *and not finite*(VI).

Proof. We will need the following lemma, the proof of which we omit:

LEMMA. *If* C *is any set such that* $\omega \leqslant C$, *then* C^* (*all finite sequences of elements of* C) *satisfies* $|C^* \times C^*| = |C^*|$ *and therefore* C^* *is not finite*(VI).

Let $C = A \cup \omega$ (in $M1$). By the Lemma $B = C^*$ is not finite(VI). We show B is finite(D) by constructing two sets X and $Y \in M1$ such that $B = X \cup Y$, $|X| < |B|$ and $|Y| < |B|$.

Define a sequence of natural numbers $(n(k)\colon k \in \omega)$ by

$$n(0) = 0 \quad \text{and} \quad n(k) = n(k-1)+2k\,.$$

We note that $n(k) = k(k+1)$ but the recursive definition will be more useful to us. Partition B into a countable number of pieces $\{B(k)\colon k \in \omega\}$ where

$$B(k) = \{s \in B\colon |\mathrm{range}(s) \cap A| = k\}$$

and finally let

$$X = \bigcup \{B(j)\colon (\exists i \in \omega)(n(2i) \leqslant j < n(2i+1))\}\,,$$

$$Y = \bigcup \{B(j)\colon (\exists i \in \omega)(n(2i+1) \leqslant j < n(2i+2))\}\,.$$

Since $\varnothing$ is a support of $B(k)$ for each $k \in \omega$, X and Y are in $M2$. It is also clear that $B = X \cup Y$ and that X and Y are disjoint. We complete the proof of Theorem 9 by showing that $|X| = |B|$ is false in $M2$. The proof that $|Y| \neq |B|$ in $M2$ is similar.

Suppose that $|X| = |B|$ then there is a one to one function f in $M1$ from Y into X. Suppose f has support $E \subseteq A$ and that $|E| = r \in \omega$. Choose an odd integer $2i+1$ greater than r, then $n(2i+2) = n(2i+1)+2(2i+1)$ so $B(j) \subseteq Y$ for any j satisfying $n(2i+1) \leqslant j < n(2i+1)+2(2i+1)$. Therefore if we let $j_0 = n(2i+1)+(2i+2)$ and choose $s \in B(j_0)$ then $s \in Y$ and $f(s) \in X$. It follows that $f(s) \in B(k)$ where $k < n(2i+1)$ or $n(2i+1)+2(2i+1) \leqslant k$. In the first case

$$j_0 - k = n(2i+1)+2i+2-k > n(2i+1)+2i+2-n(2i+1) = 2i+1 > r\,.$$

So $k+r < j_0$. Since $|\mathrm{range}(s) \cap A| = j_0$ and

$$\big|((\mathrm{range}\, f(s)) \cap A) \cup E\big| \leqslant |\mathrm{range}\, f(s) \cap A| + |E| = k+r$$

we conclude that

$$(\mathrm{range}(s) \cap A) - \big[((\mathrm{range}\, f(s)) \cap A) \cup E\big] \neq \varnothing$$

which implies that for some $\psi \in \mathrm{fix}_{G1}(E)$, $\psi(f(s)) = f(s)$ and $\psi(s) \neq s$. Since f is one to one this means E is not a support of f, a contradiction.

Similarly in the second case

$$k - j_0 = k - (n(2i+1)+2i+1)$$
$$\geqslant n(2i+1)+2(2i+1)-(n(2i+1)+2i+2) = 2i+2 > r$$

therefore $k > j_0 + r$. This implies the existence of a $\psi \in \mathrm{fix}_{G1}(E)$ such that $\psi(s) = s$ and $\psi(f(s)) \neq f(s)$ again contradicting the choice of E as a support of f. Theorem 9 is therefore proved.

THEOREM 10. *In* $M1$, $E(\mathrm{II}, \mathrm{III})$ *is true and* $E(\mathrm{I}, \mathrm{Ia})$ *is false.*

Proof. The proof that $E(\mathrm{I}, \mathrm{Ia})$ is false in $M1$ can be found in [8].

That $E(\mathrm{II}, \mathrm{III})$ holds in $M1$ follows from the fact that every set which can be linearly ordered in $M1$ can be well ordered in $M1$. (This fact can be proved using the following property: For every finite $E \subseteq A$ and $\psi \in G1$, there is a $\psi' \in G1$ such that $(\forall a \in E)(\psi'(a) = \psi(a))$ and for some positive integer n, $(\psi')^n =$ the identity permutation. We leave the details to the reader.)

COROLLARY. $E(\mathrm{Ia}, \mathrm{II})$, $E(\mathrm{Ia}, \mathrm{III})$ *and* OP *are false in* $M1$.

THEOREM 11. *In* $M2$ $E(\mathrm{I}, \mathrm{II})$ *is true and* $E(\mathrm{I}, \mathrm{III})$ *and* $E(\mathrm{II}, \mathrm{III})$ *are false.*

Proof. This theorem follows from the results of [8].

THEOREM 12. *In* $M3$, (*every Dedekind finite set is finite*) *and* OP *is false.*

Proof. We first assume x is an infinite set in $M3$ and show x has an infinite subset which is well orderable in $M3$. Suppose X has no such subset and that E is a bounded subset of A such that for every $\psi \in \mathrm{fix}_{G3}(E)$, $\psi(x) = x$. (That is, E is a support of x).

Since x is infinite and has no infinite well orderable subset, there must be a $t \in x$ and $\psi_0 \in \mathrm{fix}_{G3}(E)$ such that $\psi_0(t) \neq t$. Assume $E \cup F$ is a bounded support of t where $F \cap E$ is empty. Then by our assumption $F \neq \varnothing$.

CLAIM. If ψ and ψ' are chosen in $\mathrm{fix}_{G3}(E)$ so that

$$\psi(F \cup \psi_0(F)) \cap \psi'(F \cup \psi_0(F)) = \varnothing$$

then $\psi(t) \neq \psi'(t)$.

For suppose the hypotheses of the claim are satisfied, then

$$(\psi')^{-1}[\psi(F \cup \psi_0(F)) \cap \psi'(F \cup \psi_0(F))] = \varnothing$$

so $(\psi')^{-1}[\psi(F \cup \psi_0(F))] \cap (F \cup \psi_0(F)) = \varnothing$.

We can therefore choose an $\sigma \in \mathrm{fix}_{G3}(E)$ such that $\sigma \in \mathrm{fix}_{G3}((\psi')^{-1}\psi(F \cup \psi_0(F)))$ and σ agrees with ψ_0 on F. Therefore $\sigma(t) = \psi_0(t) \neq t$. Further, since $(\psi')^{-1}\psi(F) \cup E$ is a support of $(\psi')^{-1}\psi(t)$, $\sigma((\psi')^{-1}\psi(t)) = (\psi')^{-1}\psi(t)$. It follows that $(\psi')^{-1}\psi(t) \neq t$ and therefore that $\psi(t) \neq \psi'(t)$. This completes the proof of the claim.

Since $E \cup F \cup \psi_0(F)$ is bounded (in the ordering on A), there is an $a \in A$ such that

$$(\forall t \in E \cup F \cup \psi_0(F))\ (t < a).$$

Choose an element $b \in A$ so that $a < b$ and choose a countable number of pairwise disjoint intervals $\{I(j): j \in \omega\}$ such that for all $j \in \omega$, $I(j) \subseteq [a, b]$.

For each $j \in \omega - \{0\}$ let $\psi_j \in \mathrm{fix}_{G3}(E)$ be chosen so that $\psi_j(F \cup \psi_0(F)) \subseteq I(j)$. Then for j as above, $I(j)$ will be a bounded support of $\psi_j(t)$. By the claim, $\psi_j(t) \neq \psi_i(t)$ if $i \neq j$. Further $\psi_j(t) \in x$ and, since $E \cup [a, b]$ is a bounded support of $\psi_j(t)$ for all $j \in \omega - \{0\}$, the set

$$\{\psi_j(t): j \in \omega - \{0\}\}$$

in addition to being infinite, is well ordered in $M3$. Therefore x has an infinite well-ordered subset in $M3$.

Since A has no linear ordering in $M3$, OP fails in $M3$. Thus Theorem 12 is proved.

THEOREM 13. *In $M4$, $E(\mathrm{I}, \mathrm{III})$ is true and $E(\mathrm{I}, \mathrm{IV})$ is false.*

The proof is left to the reader.

We now turn to $M5$. Facts about $M5$ will appear in Theorem 14 through 16.

For $x \in M5$ and $E \subseteq A$, we say E is a *support of* x if E is finite and

$$(\forall \psi \in G(E))\ (\psi(x) = x).$$

If $a, b \in A$, we will use (a, b) and $[a, b]$ to denote the open and closed intervals respectively relative to the linear ordering $<$ of A. $(a; b)$ will be used to denote the transposition in $G5$.

THEOREM 14 (support lemma). *If E and F are supports of $x \in M5$, then $E \cap F$ is a support of x.*

Proof. Since every element of $G(E \cap F)$ can be written as a finite product of transpositions in $G(E \cap F)$, it suffices to prove:

LEMMA. *If $\psi \in G(E \cap F)$ is a transposition $(d_1; d_2)$ then $\psi(x) = x$.*

The proof of the lemma which we omit is by induction on the number of elements in $(E \cup F) \cap (d_1, d_2)$.

We note that from Theorem 12 we can conclude that each element t of $M5$ has a minimum support E with the property that for all $\psi \in G5$, $\psi(t) = t$ implies $\psi(E) = E$.

THEOREM 15. *$E(\mathrm{I}, \mathrm{Ia})$ is true in $M5$.*

Proof. Let C be an infinite set in $M5$ with support E. If C is fixed pointwise by $G(E)$, then C is finite(Ia) and we are done. Otherwise choose $t \in C$ with minimum support F where F is not a subset of E and choose $a \in F - E$. Let $E' = (E \cup F) - \{a\}$ and let $C(1) = \{\psi(t): \psi \in G(E')$ and $a < \psi(a)\}$. $C(1)$ has support $E \cup F$ and therefore is in $M5$. Similarly $C(2) = \{\psi(t): \psi \in G(E')$ and $\psi(a) < a\}$ is a subset of C and in $M5$. $C(1)$ and $C(2)$ are disjoint since a is an element of the minimum support of t. We also claim that $C(1)$ and $C(2)$ are infinite, for suppose $E' = \{b(1), \ldots, b(n)\}$ where $b(1) < \ldots < b(n)$ and $b(j) < a < b(j+1)$. For $d \in (b(j), b(j+1))$ define $\psi(d)$ (a permutation of A) by $\psi(d) = (d; a)$, then $\psi(d) \in G(E')$. Further if $d \neq d'$ then $\psi(d)(t) \neq \psi(d')(t)$. (If $\psi(d)(t) = \psi(d')(t)$ then $\sigma = \psi(d')^{-1}\psi(d)$ fixes t and therefore must fix F, the least support of t. But $\sigma(a) = d$ which is not in F.) This proves the claim.

Since we can write $C = C(1) \cup (C - C(1))$ the theorem is proved.

THEOREM 16. *$E(\mathrm{Ia}, \mathrm{II})$ is false in $M5$.*

Proof. We show that A is the union of two infinite disjoint sets in $M5$, $(A = \{a: a < a_0\} \cup \{a: a_0 \leqslant a\}$ for any fixed a_0 in $A)$ but that every non-empty monotone family of subsets of A has a maximal element.

Suppose Z is such a family with no maximal element and that Z has support E. E partitions $A - E$ into open intervals and for at least one of these intervals I and some $x \in Z$, $x \cap I$ and $(A - x) \cap I$ are non-empty (otherwise Z is finite). Choose $d \in x \cap I$ and $e \in (A - x) \cap I$, then $(d; e) \in G(E)$ so $(d; e)(x) \in Z$. But $x \nsubseteq (d; e)(x)$ and $(d; e)(x) \nsubseteq x$ contradicts the monotonicity of Z.

THEOREM 17. *In $M6$ A is finite(V) but not finite(D).*

Proof. The fact that A is not finite(V) can be seen by supposing f is a bijection from $2 \times A$ to A in $M6$ with support $E \subseteq A$ and considering $f(0, a)$ and $f(1, a)$ where a is not in E.

The fact that A is finite(D) in $M6$ follows from the following two lemmas:

LEMMA. *If $A = C \cup D$ where $C, D \in M6$ and C and D are disjoint, then C or D is bounded.*

LEMMA. *If C is a subset of A and $A - C$ is bounded then C and A have the same cardinal number in $M6$.*

Combining our results with diagram 1 gives diagram 2. Numbers on the implication edges of diagram 2 refer to theorems in this paper or references where the results may be found. Similarly, numbers on the non-implication edges refer to model numbers in this paper or are reference numbers.

Several other definitions of finite have appeared in the literature. We refer the reader to [3] for two definitions due to Joseph Diel which lie between finite(I) and finite(II) in strength: A set A is "almost finite" if there is no function from A onto ω and is "strongly Dedekind finite" if there is no function from a proper subset of A onto A. The relationship between these and Ia, II and III is an open problem.

In [15] Tarski has given a series of definitions of finite. We list those which are not known to be equivalent to any of the definitions given so far. Following Tarski's notation: A set A is finite$(T(n))$ if for every $S \subset P(A)$, (if every collection of n non-empty, pairwise disjoint subsets of A contains an element of S, then there are $2n+1$ elements $x(i)$, $1 \leqslant i \leqslant 2n+1$ of S such that $A \subset \bigcup \{x(i)\colon 1 \leqslant i \leqslant 2n+1\}$). We pose the question of how these definitions fit into the scheme described in this paper.

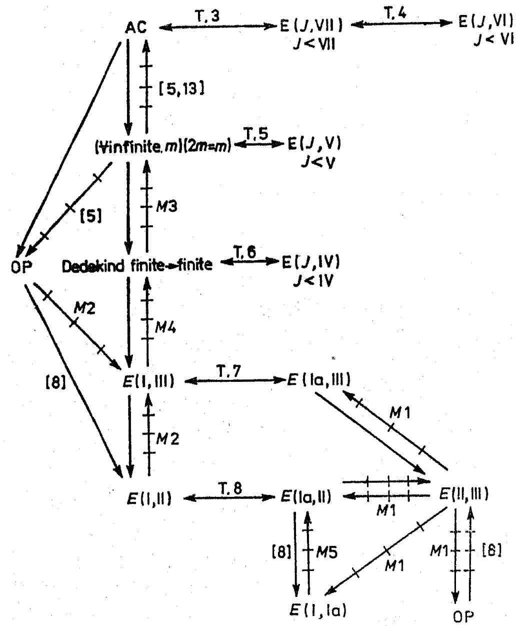

Diagram 2

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