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Functions provably total in $I^-\Sigma_n$

by

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Abstract. The main theorem of the paper can be formulated as follows. *Let* $n \geq 2$. *If, in the proof of the totality of a recursive function* $f\colon N \to N$ *we use only* m *different axioms of* Σ_n*-induction without parameters and additionally only axioms of* PA^-, *then* f *can be bounded (almost everywhere) by a function* H_β *in Hardy's hierarchy, where* $\beta = \omega_{n-1}^{\omega^m \cdot k}$ *for some* $k \in \omega$.

§ 1. Introduction. The aim of this paper is a generalization to the case of $I^-\Sigma_n$ of the following theorem proved in [A–B], where $I^-\Sigma_n$ denotes the theory of Σ_n-induction without parameters.

1.1. THEOREM [A–B]. *If, in the proof of the totality of a recursive function* $f\colon N \to N$ *in the theory* $I^-\Sigma_1$ *we use only* m *different axioms of* Σ_1*-induction, then* f *can be bounded by the* $2m$*-th function in the Grzegorczyk hierarchy starting from* 2^x.

Let F_m denote the mth function in Grzegorczyk hierarchy. Thus $F_0(x) = 2^x$, $F_{m+1}(x) = F_m^{x+1}(x)$ for $m \in \omega$. Let us remark that a slight modification of estimation of growth in the proof of the theorem just quoted shows that f can be bounded by an iterate F_m^k of the function F_m for some $k \in N$. This estimation can be formalized in the theory $I\Delta_0 + \exp + \forall x \exists y F_m(x) = y$.

We say that a function $f\colon N \to N$ is *provably total* and *of the class* Σ_r in a theory T iff there exists $\varphi(x, y) \in \Sigma_r$ such that φ defines f in N and $T \vdash \forall x \exists! y \varphi(x, y)$. We will speak shortly: f is *total* and Σ_r in T. Functions which are total and Σ_1 in T are often called functions *provably recursive* in T.

Before we formulate our theorem, which generalizes Theorem 1.1, we need some notation. By H_α, for $\alpha < \varepsilon_0$, let us denote the Hardy's function with index α (see [W]); compare the definition of G_α below. Ordinal numbers ω_n^α, where $n \in \omega$, are defined by the following inductive conditions: $\omega_0^\alpha = \alpha$, $\omega_{n+1}^\alpha = \omega^{\omega_n^\alpha}$ for $n \in \omega$. If $f, g\colon N \to N$ then $f \leqslant g$ means that $\exists n \forall m \geqslant n\, f(m) \leqslant g(m)$; we say that g *dominates* f. If φ is a formula of the language of arithmetic with one free variable then let $\mathrm{Ind}\,\varphi$ denote the formula $\varphi(0) \wedge \forall x(\varphi(x) \to \varphi(x+1)) \to \forall x \varphi(x)$. The theory $I\Delta_0 + \exp + \mathrm{Ind}\,\varphi_1 + \ldots + \mathrm{Ind}\,\varphi_m$ will be denoted by $I^-[\varphi_1, \ldots, \varphi_m]$.

1.2. THEOREM. *Let* $n \geqslant 1$ *and let* $\varphi_1, \ldots, \varphi_m$ *be* Σ_n*-formulas. If* $f\colon N \to N$ *is provably total and* Σ_1 *in* $I^-[\varphi_1, \ldots, \varphi_m]$ *then there exists a* $k \in \omega$ *such that* f *is do-*

minated by the Hardy's function H_α *with* $\alpha = \omega_{n-1}^{\omega^m\cdot k}$ *when* $n \geqslant 1$ *and* $\alpha = \omega^{m+2}\cdot k$ *when* $n = 1$.

For $n = 1$ we obtain Theorem 1.1. In fact, $H_{\omega^2} \leqslant F_0^2$, which implies that $H_{\omega^{m+2}\cdot k} \leqslant F_m^{2k}$. The proof of Theorem 1.2 and proofs of all the remaining theorems are divided in to two parts: the semantical part, § 2, where we give a proof of the main lemma, and the combinatorial part, § 3 (the last section).

The idea of the proof of Theorem 1.2 consists in a fusion of two notions: the notion of K-closure from Adamowicz and Bigorajska [A–B] and the notion of approximation, in the sense of Pudlak and Friedman, of functions by finite sets. We use here the idea of approximations which was used in [R] (see also [K–R] for the estimation of growth of functions provably total and Σ_1 in $I\Sigma_n$.

The method used in the proof of Theorem 2.1 can be applied without difficulties to the estimation of functions provably total and Σ_r in $I^-[\varphi_1, \ldots, \varphi_m]$, where $\varphi_1, \ldots, \varphi_m \in \Sigma_n$, $2 \leqslant r \leqslant n$. Assume that $r \geqslant 1$. To make this result precise let us notice that there exists a function H^{r+1} of class Π_r in N, which dominates all functions of class Σ_r.

Let φ_{r-1} denote an universal formula for Π_{r-1} formulas in $I\Delta_0+\exp$, Δ_r in $I\Delta_0+\exp$. A formula ψ_{r+1} defining H^{r+1} is the following:

$$\forall u, x < a[(\exists y \varphi_{r-1}(u, x, y) \to \exists y < b\, \varphi_{r-1}(u, x, y)) \wedge \\ \wedge \forall z < b \exists u, x < a \forall y < b(\varphi_{r-1}(u, x, y \to z \leqslant y)] .$$

It is easy to observe that ψ_{r+1} is of class Π_r in $I\Sigma_r$. Since the theory $I\Sigma_r$ is equivalent to strong Σ_r-collection (see [H–P]), $I\Sigma_n$ is equivalent to the theory $I\Delta_0+\exp+\forall a \exists b \psi_{r+1}(a, b)$ (for $r \geqslant 1$).

In particular, H^{r+1} is provably total and Π_r in $I\Sigma_r$. It follows that for every $k \in \omega$ the usual iterate $(H^{r+1})^k$ is provably total and Σ_{r+1} in $I\Sigma_r$. If $r < n$ then, in view of $I\Sigma_r \subseteq I^-\Sigma_n$, it follows that the functions $(H^{r+1})^k$ are provably total and Σ_{r+1} in $I^-\Sigma_n$.

Before we formulate the next theorem, we must make it clear what we mean by iteration in the sense of Hardy of any function $G\colon N \to N$ (denoted in the sequel by G_α). We define G_α by induction on $\alpha < \varepsilon_0$: $G_0 = \mathrm{id}$, $G_{\alpha+1}(n) = G_\alpha(G(n))$ and $G_\alpha(n) = G_{\{\alpha\}(n)}(n)$ if α is limit; the symbol $\{\alpha\}(n)$ denotes the nth term of the standard fundamental sequence converging to α. Let us mention that H_α is the α-fold iterate of the function $G(x) = x+1$ let us also notice that this definition of iteration makes sense for a partial function $G\colon \subseteq \omega \to \omega$.

Let $H^1(x) = 2^x$. In view of Lemma 3.1, which says, in particular, that H^1 with index $\omega^m\cdot k$ is dominated by H with index $\omega^{m+2}\cdot k$, it follows that the next theorem can be treated as a generalization of Theorem 1.2.

1.3. THEOREM. *Let* $1 \leqslant r \leqslant n$ *and let* $\varphi_1, \ldots, \varphi_m$ *be* Σ_n*-formulas. If* $f\colon N \to N$ *is provably total and* Σ_r *in* $I\Sigma_{r-1}+I^-[\varphi_1, \ldots, \varphi_m]$ *then there exists a* $k \in \omega$ *such that* f *is dominated by* H_α^r, *where* $\alpha = \omega_{n-r}^{\omega^m\cdot k}$.

At the end of this paper we discuss the problem of formalization of the combinatorial estimates from 1.3 in theories of the form $I\Delta_0+\{\forall x H_\alpha^r(x)\downarrow : \alpha < \beta\}$.

§ 2. **The semantical part.** The main scheme of the proof of Theorem 1.3 is the same as that of Theorem 1.2. The basic idea in these proofs is the same as for 1.1 in [A–B]. The semantical part is presented in the form of separate lemmas, Lemma 2.10 for 1.2 and Lemma 2.3 for the general situation. The rest of the proof consists in combinatorial estimations.

To prevent the idea of the proofs from getting lost we first introduce a set of notions which are necessary to formulate Lemma 2.10, and then we describe changes — necessary for generalization.

At first we define the notion of a witness function; K-fold iteration of a witness function corresponds to the notion of K-closure from [A–B]. Definition of a witness function for ψ (which is a finite approximation of the Skolem function for ψ) depends on a distinguished sequence of quantifiers at the beginning of ψ. To exhibit a distinguished sequence of quantifiers, we write ψ e.g. in the form $\forall \vec{x} \exists \vec{y} \psi_1(\vec{x}, \vec{y})$; this denotes that the quantifiers $\forall \vec{x} \exists \vec{y}$ constitute the sequence in question. The sequence $\vec{x}$ can be empty.

2.1. DEFINITION ($I\Delta_0+\exp$). We say that a unary function f with a finite domain is a *witness function* for the formula $\forall \vec{x} \exists \vec{y} \varphi(x, \vec{y})$ iff

(1) $\forall a \in dm(f)\, a < f(a)$,

(2) $\forall a \in dm(f) \forall \vec{x} \leqslant a \exists \vec{y} \leqslant f(a) \varphi(\vec{x}, \vec{y})$, where the formula $\exists \vec{y} \leqslant f(a) \varphi(\vec{x}, \vec{y})$ is an abbreviation for $\forall b[f(a) = b \to \exists \vec{y} \leqslant b\, \varphi(\vec{x}, \vec{y})]$; if the sequence $\vec{x}$ is empty then the quantifier $\forall \vec{x} \leqslant a$ does not appear.

Next we extend the notion of a witness function to the case of formulas to which distinguished formulas logically equivalent to them and having a distinguished sequence of quantifiers are mapped.

2.2. DEFINITION. Let us assume that $\forall \vec{x}_1 \exists \vec{y}_1 \varphi_1$ is the distinguished formula logically equivalent to ψ_1 and that $\forall \vec{x}_2 \exists \vec{y}_2 \varphi_2$ is the distinguished formula mapped to ψ_2. Then the *distinguished formula mapped to* $\psi_1 \vee \psi_2$ is defined us

$$\forall \vec{x}_1 \forall \vec{x}_2' \exists \vec{y}_1 \exists \vec{y}_2' (\varphi_1(\vec{x}_1, \vec{y}_1) \vee \varphi_2(\vec{x}_2', \vec{y}_2')),$$

where $\vec{x}_2'$, $\vec{y}_2'$ are new variables substituted for $\vec{x}_2$ and for $\vec{y}_2$.

2.3. FACT ($I\Delta_0+\exp$). *f is a witness function for $\psi_1 \vee \psi_2$ iff for each $a \in dm(f)$ $\{a, f(a)\}$ is a witness function for ψ_1 or is a witness function for ψ_2.*

The symbol Indstφ always denotes the formula:

$$\forall x(\varphi(x) \to \varphi(x+1)).$$

2.4. DEFINITION. The distinguished formula logically equivalent to Indst$\exists \vec{y} \varphi(x, \vec{y})$ is

$$\forall x \forall \vec{y} \exists \vec{y}_1[\varphi(x, \vec{y}) \to \varphi(x+1, \vec{y}_1)],$$

where $\vec{y}_1$ are new variables substituted for $\vec{y}$.

The next important notion is that of approximation to functions in the sense of Friedman and Pudlak. We say that a finite set $S \neq \emptyset$ is an *approximation to* $f\colon \subseteq N \to N$ iff

$$\forall a \in S \setminus \{\max S\}\, \forall x < a-2\big(f(x) < a^{+} \vee f(x) \geqslant \max S\big),$$

where $a^{+} = \min(S \setminus \{x\colon x \leqslant a\})$; in the original formulation [S] the quantifier $\forall x$ is bounded by a. The above definition can be formulated on the ground of $I\Delta_0+\exp$.

Let $\psi := \exists \bar{y}\varphi(\bar{x}, \bar{y})$, where $\varphi \in \Delta_0$.

2.5. DEFINITION ($I\Delta_0+\exp$). We say that a finite set $S \neq \emptyset$ is an *approximation to* ψ iff

$$\forall a \in S \setminus \{\max S\}\, \forall \bar{x} \leqslant \lg_2 a [\exists \bar{y} < a^{+} \varphi(\bar{x}, \bar{y}) \to \exists \bar{y} < \max S \varphi(\bar{x}, \bar{y})].$$

The relation of this notion to that of approximation to a function is made precise by the following:

2.6. LEMMA. *Let $\psi_0, \ldots, \psi_{m-1}$ be Σ_1-formulas. Then for a certain $l \in N$, which depends only of the number of unbounded quantifiers in $\psi_0, \ldots, \psi_m$, the theory $I\Delta_0+\exp$ proves: for every a there exists an f with a finite domain such that every approximation S to f satisfying $S \subseteq [l, a]$ is also a common approximation to $\psi_0, \ldots, \psi_{m-1}$.*

Proof. Let us assume that $\psi_i := \exists \bar{y}_i \varphi_i(\bar{x}_i, \bar{y}_i)$ for $i = 0, \ldots, m-1$, where $\varphi_i \in \Delta_0$. Let $k_i = \operatorname{length}(\bar{x}_i)$ and let $\langle \bar{x}_i \rangle$ denote a polynomial code of the sequence $\bar{x}_i$. We take $l_0 \in N$ such that $I\Delta_0+\exp \vdash \langle \bar{x}_i \rangle < (\max(2, \bar{x}_i))^{l_0}$ for $i = 0, \ldots, m-1$.

Now we work in $I\Delta_0+\exp$. We take a and define

$$g(i, x) = \min_{z \leqslant a} \exists \bar{y}_i \leqslant z \varphi_i((x)_0, \ldots, (x)_{k_i-1}, \bar{y}_i),$$

where $(x)_i$ is the decoding function and if $\neg \exists z \leqslant a\, R(z)$ then $\min_{z \leqslant a} R(z) = a$. Next we define

$$f(x) = g(x - m[x/m], [x/m]).$$

Let us put $l = 2^{2^{m+l_0+5}}$. It is easy to check that for every $y \geqslant l$, $m(\lg_2 y)^{l_0} < y-2$. Assume that $S \subseteq [l, a]$ is an approximation to f and $0 \leqslant i < m$. We show that S is an approximation to ψ_i.

Let $y \in S \setminus \{\max S\}$. Take a sequence $x_i \leqslant \lg_2 y$. Hence $\langle x_i \rangle < (\lg_2 y)^{l_0}$. Let $x = m\langle x_i \rangle + i$. Hence $x \leqslant m(\lg_2 y)^{l_0} < y-2$ and thus $f(x) = g(i, \langle x_i \rangle) < y^{+}$ or $g(i, \langle x_i \rangle) \geqslant \max S$, and this means that $\exists \bar{y}_i < y^{+} \varphi_i(\bar{x}_i, \bar{y}_i)$ or $\neg \exists \bar{y}_i < \max S \varphi(\bar{x}_i, \bar{y}_i)$, since $\max S \leqslant a$. ∎

We now state a lemma which will allow us to better understand the notion of approximation. This lemma will be used only in §3 as the main tool of combinatorial estimations. To make it precise we define $H^S(a) = a^{+}$ for $a \in S \setminus \{\max S\}$, and $dm H^S = S \setminus \{\max S\}$. Let $\alpha < \varepsilon_0$. We say that a set S is α-*large* iff $H^S_\alpha(\min S)\downarrow$, where the index α refers to the α-fold iteration, in the sense of Hardy, of the partial function H^S.

2.7. LEMMA [R]. *Let $\alpha < \varepsilon_0$. If S is ω^α-large then for every function f there exists an α-large set $S_1 \subseteq S \setminus \{\max S\}$, which is an approximation to f.*

Another (shorter) proof of this lemma appears in [K–R]. One can show that the proof of 4.6 from [K–R], which implies our 2.7, is formalizable in $I\Sigma_1$. One can use the observation that

$$I\Sigma_1 \vdash \forall n, b \text{ “}\{\beta < \omega^1_n\colon \exists S(S \subseteq [2, b] \wedge H^S_\beta(\min S)\downarrow)\} \text{ is finite”}.$$

Therefore, transfinite induction which was used in the proof can be replaced by usual (finite) Δ_0-induction on the set of all $\beta < \omega^1_n$ such that

$$\exists S(S \subseteq [2, b] \wedge H^S_\beta(\min S)\downarrow).$$

We use this observation in the discussion of the problem of formalization of combinatorial estimations at the end of the paper.

Now we generalize the notion of approximation to the case of class Σ_n and we prove a lemma, which throws some light on the role played by approximations in the construction of models.

If S is an approximation to a Σ_1-formula ψ: $\exists \bar{y}\varphi(\bar{x}, \bar{y})$ where $\varphi \in \Delta_0$, then we denote the formula $\exists \bar{y} < \max S \varphi(\bar{x}, \bar{y})$ by ψ^S. It will be called an *approximation formula to* ψ. For convenience we assume that every $S \neq \emptyset$ is an approximation to every $\varphi \in \Delta_0$ and that $\varphi^S := \varphi$.

If $\psi \in \Pi_n$ then $\sim\psi$ will denote the formula of class Σ_n obtained from ψ by changing all unbounded universal quantifiers to existential quantifiers (and conversely) and by substituting the negation sign before the maximal subformula of class Δ_0.

If $|S| \geqslant n$ then $l_n S$ will denote the nth element of S, counting from the end.

2.8. DEFINITION. We proceed by induction on $n \geqslant 1$. Assume that the notion: S is an approximation to φ and the formula φ^S have been defined for $\varphi \in \Sigma_n$. We say (in $I\Delta_0+\exp$) that S is an *approximation to a* $\varphi \in \Pi_n$ iff S is an approximation to $\sim\varphi$. We put $\varphi^S := \neg(\sim\varphi)^S$.

Let $\varphi \in \Sigma_{n+1}$, $\varphi := \exists \bar{y}\psi(\bar{x}, \bar{y})$, where $\psi \in \Pi_n$. We say (in $I\Delta_0+\exp$) that S is an *approximation to* φ iff $|S| \geqslant n+1$ and

(1) S is an approximation to ψ,

(2) $S \setminus \{l_1 S, \ldots, l_n S\}$ is an approximation to $\exists \bar{y}\psi^S(\bar{x}, \bar{y})$.

We put $\varphi^S := \exists \bar{y} < l_{n+1} S\, \psi^S(\bar{x}, \bar{y})$.

If $M \models \mathrm{PA}^{-}$ then Σ^*_n will denote the class of all Σ_n-formulas with parameters from M. If $I \subset_e M$ and $\varphi \in \Sigma^*_0$, we define: $I \models \varphi[\bar{a}] \Leftrightarrow N \models \varphi[\bar{a}]$. Starting from the relation $I \models \varphi[\bar{a}]$ for $\varphi \in \Sigma^*_0$ we define $I \models \varphi[\bar{a}]$ for $\varphi \in \Sigma^*_n$ by Tarski's inductive conditions for the model I.

2.9. LEMMA. *Assume that $M \models I\Delta_0+\exp$, $\varphi(\bar{x}) \in \Sigma_n$ and $M \models$ “S is an approximation to φ” $\wedge \forall a \in S \setminus \{\max S\}\, 2^a \leqslant a^{+}$. It follows that for every cut $I \subseteq_e M$ such that $I \cap S$ is unbounded in I we have $I \models \varphi[\bar{a}] \Leftrightarrow M \models \varphi^S[\bar{a}]$ for all $\bar{a} \in I$.*

Proof. Let us fix $M \models I\Delta_0+\exp$. We prove the lemma by induction on $n \in N$.

For Σ_0^*-formulas the lemma is obviously true. Assume that the lemma is true for Σ_{n-1}^*-formulas. Let $\varphi(x) \in \Sigma_n^*$ and assume that $M \models$ "S is an approximation to φ". Let $\varphi(\bar{x}) := \exists \bar{y} \varphi_0(\bar{x}, \bar{y})$, where $\varphi_0 \in \Pi_{n-1}^*$. Let $I \subset_e M$ such that $S \cap I$ is cofinal in I.

By Definition 2.8, S is also an approximation to φ_0. Hence, by the inductive assumption and by the fact: $\varphi_0^S := \neg(\sim\varphi_0)^S$, we have

$$(*) \qquad \forall \bar{a}, \bar{b} \in I \quad I \models \varphi_0[\bar{a}, \bar{b}] \Leftrightarrow M \models \varphi_0^S[\bar{a}, \bar{b}].$$

Take $\bar{a} \in I$ and assume that $I \models \exists \bar{y} \varphi_0[\bar{a}, \bar{y}]$. Then

$$\exists \bar{b} < l_n S \quad I \models \varphi_0[\bar{a}, \bar{b}],$$

whence by $(*)$ and the definition of S we have $M \models \varphi^S[\bar{a}]$.

Now assume, contrary to the claim, that $\exists \bar{b} < l_n S \; M \models \varphi^S[\bar{a}, \bar{b}]$. Since $S \cap I$ is unbounded in I, there exists $a_0 \in S \cap I$ such that $\bar{a} < a_0$. Let $a = a_0^+$. Then $\bar{a} \leqslant \lg_2 a$ and $a \in S \setminus \{l_1 S, \ldots, l_n S\}$. Since $S \setminus \{l_1 S, \ldots, l_{n-1} S\}$ is an approximation to $\exists \bar{y}\, \varphi_0^S(\bar{x}, \bar{y})$ in M, then $M \models \exists \bar{y} < a^+ \; \varphi_0^S[\bar{a}, \bar{y}]$; it follows that $\exists \bar{b} \in I \; M \models \varphi_0^S[\bar{a}, \bar{b}]$, because of $a^+ = a_0^{++} \in I$. By $(*)$ we infer that $I \models \exists \bar{y} \; \varphi_0[\bar{a}, \bar{y}]$. ■

2.10. Main Lemma. *Let $n \geqslant 1$ and let $\varphi_1(x, \bar{y}), \ldots, \varphi_m(x, \bar{y}) \in \Pi_{n-1}$, $\varphi(x, y) \in \Sigma_0$. If $I^-[\exists \bar{y}\varphi_1, \ldots, \exists \bar{y}\varphi_m] \vdash \forall x \exists y \varphi(x, y)$ then there exists a $k \in N$ such that for every $J \subseteq [1, m]$ the theory $I\Delta_0 + \exp$ proves:*

for all x_0, f, S, if S is an approximation to $\varphi_1, \ldots, \varphi_m$ and $x_0 \leqslant dm\, f$, $f\colon \subseteq S \to S$, $\forall a \in dm\, f \; a < f(a)$ and $\forall a \in S \setminus \{\max S\} \; 2^a \leqslant a^+$, then:

f is a witness function to $\forall x \exists \bar{y} \varphi_j^S$: for $j \in J \to$
f^k is a witness function to $\displaystyle\bigvee_{j \in [1,m] \setminus J} \mathrm{Indst}\, \exists \bar{y}\, \varphi_j^S(x, \bar{y}) \vee \exists y \varphi(x_0, y)$.

We will in fact prove a result more general than Lemma 2.10. But we first show how Theorem 1.3 can be reduced to a relativized version of Theorem 1.2 and then we formulate and prove a generalization being a relativized version of 2.10.

Let the symbol $L_{\mathrm{PA}} \cup \{G\}$ denote the language L_{PA} extended by the unary function constant G. We say that φ is of the class $\Delta_0(G)$ iff all quantifiers in φ are bounded in the usual way or by terms $G^k(x)$: $k \in N$. We can speak of $\Sigma_n(G)$ and $\Pi_n(G)$ formulas.

2.11. Definition. We say that the formula $\psi(x, y)$ is *conditionally absolute* (shortly *c-absolute*) iff for every $M \models I\Delta_0 + \exp$ and $I \subset_e M$ the fact $\forall x \in I \; \exists y \in I \; M \models \psi(x, y)$ implies that ψ is absolute with respect to I and M.

We will show that formulas ψ_r are absolute; but first we formulate:

2.12. Theorem (a relativized version of 1.2). *Let $\psi(x, y)$ be c-absolute and assume that ψ defines in N a total function $\bar{G}$ such that $\forall a \, \bar{G}(a) \geqslant 2^a$. Let $n \geqslant 1$. Assume that $\varphi_1(x), \ldots, \varphi_m(x) \in \Sigma_n(G)$ and $\varphi(x, y) \in \Sigma_1(G)$.*

If $I^-[\varphi_1, \ldots, \varphi_m] + \forall x, y(G(x) = y \Leftrightarrow \psi(x, y)) \vdash \forall x \exists y \varphi(x, y)$ then there exists $k \in N$ such that the function

$$f(a) = \min_b (N, \bar{G}) \models \varphi(a, b)$$

is dominated by the function $\bar{G}_\alpha$ with $\alpha = \omega_{n-1}^{\omega m k}$.

A proof of this theorem is given in § 3. Now we only show how Theorem 1.3 can be reduced to 2.2.

To begin with, let $r = 1$. The formula ψ_1 of the class Δ_0 defining the exponential function H^1 is obviously c-absolute. Hence, Theorem 2.2 for $r = 1$ and $\psi = \psi_1$ immediately yields the case $r = 1$ of Theorem 1.3.

Let $r \geqslant 2$. We first show that ψ_r is c-absolute. Assume that $M \models I\Delta_0 + \exp$, $I \subset_e M$ and $\forall a \in I \; \exists b \in I \; M \models \psi_r(a, b)$. By definition of ψ_r in § 1 it follows that

$$\forall u, x \in I\big(M \models \exists y \varphi_{r-2}(u, x, y) \to \exists y \in I M \models \varphi_{r-2}(u, x, y)\big).$$

Since φ_{r-2} is universal for Π_{r-2}-formulas, then for each $\varphi(x, y) \in \Sigma_{r-1}$, we have

$$\forall a \in I\big(M \models \exists y \varphi(a, y) \to \exists b \in I M \models \varphi(a, b)\big).$$

Hence $I <_{\Sigma_{r-1}} M$. Since the formula ψ_r can be written in the $\Delta_0(\Sigma_{r-1})$-form, it is absolute in the usual sense which finishes the proof.

Using 2.2 we now deduce the remaining cases of 1.3. Let $2 \leqslant r \leqslant n$ and $\varphi_1, \ldots, \varphi_m \in \Sigma_n$. Let f be a function provably total and Σ_r in $I\Sigma_{r-1} + I^-[\varphi_1, \ldots, \varphi_m]$; i.e. we have $\varphi(x, y) \in \Sigma_r$ defining f such that $I\Sigma_{r-1} + I^-[\varphi_1, \ldots, \varphi_m] \vdash \forall x \exists y \varphi(x, y)$.

Since, as was observed in § 1, $I\Sigma_{r-1} \equiv I\Delta_0 + \exp + \forall a \exists b \psi_r(a, b)$, then $I\Sigma_{r-1}$ is included in

$$I\Delta_0 + \exp + \forall x, y(G(x) = y \Leftrightarrow \psi_r(x, y));$$

denote this theory by T_0. Let the theory

$$I^-[\varphi_1, \ldots, \varphi_m] + \forall x, y(G(x) = y \Leftrightarrow \psi_r(x, y))$$

be denoted by T. Hence $T_0 \subseteq T$ and $T \vdash \forall x \exists y \varphi(x, y)$.

Now we show that, in the theory T_0, formulas of the class Σ_n are equivalent to $\Sigma_{n-r+1}(G)$-formulas. To see this, it is enough to show that $\Sigma_{r-1} \subseteq \Delta_0(G)$ in T_0, which in particular implies that the formula φ is $\Sigma_1(G)$ in T. We show by induction on p, $1 \leqslant p \leqslant r-1$, that $\Sigma_p \subseteq \Delta_0(G)$ in T_0. Let us take $\chi(\bar{x}) := \exists \bar{y} \chi_1(\bar{x}, \bar{y}) \in \Sigma_p$, where $\chi_1 \in \Pi_{p-1}$ and assume that χ_1 is $\Delta_0(G)$ in T_0. By the definition of ψ_r it follows that $\chi(\bar{x}) \Leftrightarrow \exists \bar{y} < G(\max(\ulcorner \chi \urcorner, \bar{x}))\chi_1(\bar{x}, \bar{y})$ in T_0.

Let us choose formulas $\varphi_1', \ldots, \varphi_m' \in \Sigma_{n-r+1}(G)$ such that $T_0 \vdash \varphi_i \Leftrightarrow \varphi_i'$ for $i = 1, \ldots, m$ and a formula $\varphi' \in \Sigma_1(G)$ such that $T \vdash \varphi \Leftrightarrow \varphi'$. Hence $T \equiv I^-[\varphi_1', \ldots, \varphi_m'] + \forall x, y(G(x) = y \Leftrightarrow \psi_r(x, y))$ and $T \vdash \forall x \exists y \varphi'(x, y)$. Thus by 2.2 there exists $k \in \omega$ such that the function $f(a) = \min_b (N, \bar{G}) \models \varphi'(a, b)$ is dominated by $\bar{G}_\alpha$ with $\alpha = \omega_{n-r}^{\omega m \cdot k}$, where $\bar{G} = \{(a, b)\colon N \models \psi_r(a, b)\} = H^r$. This finishes the proof of 1.2, on account of 2.12. ■

Now we can formulate a relativized form of Lemma 2.10. To this end, observe that the notion of a witness function can be referred without changes to formulas of the language $L_{\mathrm{PA}}(G)$. Without difficulties we can also relativize the notions of approximation and an approximation formula φ^S.

2.13 (a relativized form of the Main Lemma). *Assume that* $\psi(x, y) \in L_{PA}$ *is c-absolute. Let* $n \geqslant 1$ *and let*

$$\varphi_1(x, \bar{y}), \ldots, \varphi_m(x, \bar{y}) \in \Pi_{n-1}(G), \quad \varphi(x, y) \in \Delta_0(G).$$

If $I^-[\exists\bar{y}\varphi_1, \ldots, \exists\bar{y}\varphi_m] + \forall x, y(G(x) = y \Leftrightarrow \psi(x, y)) \vdash \forall x \exists y \varphi(x, y)$ *then there exists* $k \in N$ *such that for every* $J \subseteq [1, m]$ *the theory*

$$I\Delta_0 + \exp + \forall x, y(G(x) = y \Leftrightarrow \psi(x, y))$$

proves:

for all x_0, S, f, *if* S *is an approximation to* $\varphi_1, \ldots, \varphi_m$ *and* $x_0 \leqslant dm f$, $f\colon \subseteq S \to S$, $\forall a \in dm f\ a < f(a)$ *and* $\forall a \in S \setminus \{\max S\}\ 2^a$, $G(a) \leqslant a^+$, *then the following implication holds*:

if f *is a witness function to*

$$\forall x \exists \bar{y} \varphi_j^S\colon \text{ for } j \in J$$

then f^k *is a witness function to*

$$\bigvee_{j \in [1, m] \setminus J} \operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j^S(x, \bar{y}) \vee \exists y \varphi(x_0, y).$$

Proof. It is enough to show that for every fixed $J \subseteq [1, m]$ our claim is true. Let us fix $J \subseteq [1, m]$. Assume, to the contrary, that a suitable $k \in N$ does not exist. Hence, by compactness, there exists a model

$$(M, G) \models I\Delta_0 + \exp + \forall x, y(G(x) = y \Leftrightarrow \psi(x, y))$$

and elements a_0, $S, f \in M$ such that $a_0 \leqslant \min S$, $f\colon \subseteq S \to S$,

$$\forall a \in S \setminus \{\max S\} 2^a,\ G(a) \leqslant a^+,$$

and $\forall a \in dm f\ a < f(a)$ and $(M, G) \models$ "f is an approximation to $\varphi_1, \ldots, \varphi_m$" and

(1) $(M, G) \models$ "f is a witness function to $\forall x \exists \bar{y} \varphi_j^S$ for $j \in J$" and $\forall k \in \omega\ (M, G) \models$ "f^k is not a witness function to $\bigvee_{j \in [1, m] \setminus J} \operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j^S \vee \exists y \varphi(a_0, y)$".

It also follows by compactness that we can have a model (M, G) satisfying (1) in which, for some $k_0 > \omega$, $(M, G) \models$ "f^{k_0} is not a witness function to

$$\bigvee_{j \in [1, m] \setminus J} \operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j^S \vee \exists y \varphi(a_0, y)\text{"}.$$

Hence by 2.3 there exists $a \in S$ such that $f^{k_0}(a) \downarrow$ and for each $j \in [1, m] \setminus J$

(2) $(M, G) \models$ "$\{(a, f^{k_0}(a))\}$ is not a witness function to $\operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j^S$" $\wedge$ "$\{(a, f^{k_0}(a))\}$ is not a witness function to $\exists y \varphi(a_0, y)$".

Let $I = \{b \in M\colon \exists k \in \omega\ b \leqslant f^k(a)\}$. The set $\{f^k(a)\colon k \in \omega\}$ is unbounded in I and included in S. Hence $S \cap I$ is unbounded in I. Since $f^{k+1}(a) \geqslant (f^k(a))^+ \geqslant G(f^k(a))$, $2^{f^k(a)}$, we see that I is closed under G and exponentiation. It follows by c-absoluteness of ψ that

(3) $(I, G \restriction I) \models I\Delta_0 + \exp + \forall x, y(G(x) = y \Leftrightarrow \psi(x, y))$.

Since S is an approximation to $\varphi_1, \ldots, \varphi_m$ in M, we infer by an obvious relativized version of 2.9 that

(4) $(I, G \restriction I) \models \varphi_i[\bar{b}] \Leftrightarrow (M, G) \models \varphi_i^S[\bar{b}]$ for all $\bar{b} \in I$ and for $i = 1, \ldots, m$.

Now we show that $(I, G) \models \forall x \exists \bar{y} \varphi_j(x, \bar{y})$ for $j \in J$ and that

$$(I, G) \models \neg \operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j \quad \text{for} \quad j \in [1, m] \setminus J \quad \text{and} \quad (I, G) \models \forall y \neg \varphi(a_0, y).$$

This is enough to obtain a contradiction, because then

$$(I, G) \models I^-[\exists \bar{y} \varphi_1, \ldots, \exists \bar{y} \varphi_m] \quad \text{and} \quad (I, G) \models \neg \forall x \exists y \varphi(x, y)$$

and this, in view of (3), contradicts our initial assumption.

Let $j \in J$. Take $c \in I$. Hence there exists $k \in N$ such that $c \leqslant f^k(a)$. By (1) $M \models \exists \bar{y} \leqslant f^{k+1}(a)\ \varphi_j^S(c, \bar{y})$, whence it follows by (4) that $\exists \bar{d} \leqslant f^{k+1}(a) (I, G) \models \varphi_j(c, \bar{d})$. Thus

$$(I, G) \models \forall x \exists \bar{y} \varphi_j(x, \bar{y}).$$

Let $j \in [1, m] \setminus J$. We show that $(I, G) \models \neg \operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j$. By (2) we know that $\{(a, f^{k_0}(a))\}$ is not a witness function for $\operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j^S$ in (M, G). According to Definition 2.4 this means that

$$(M, G) \models \exists x \exists \bar{y} \leqslant a [\varphi_j^S(x, \bar{y}) \wedge \forall \bar{y}_1 \leqslant f^{k_0}(a)\ \neg \varphi_j^S(x+1, \bar{y}_1)].$$

Let us denote suitable x, $\bar{y}$ by c, $\bar{d}$. Then $c, \bar{d} \in I$, since $a \in I$. In particular we have $(M, G) \models \varphi_j^S(c, \bar{d})$ and so $(I, G) \models \varphi_j(c, \bar{d})$. Moreover, $\forall \bar{d}_1 \in I (M, G) \models \neg \varphi_j^S(c+1, \bar{d}_1)$, since $I < f^{k_0}(a)$, and thus $(I, G) \models \forall \bar{y}_1 \neg \varphi_j(c+1, \bar{d}_1)$, i.e. $(I, G) \models \neg \operatorname{Ind} \operatorname{st} \exists \bar{y} \varphi_j$.

Finally, because $\{(a, f^{k_0}(a))\}$ is not a witness function for $\exists y \varphi(a_0, y)$ in (M, G) and $a_0 \in I$, it follows that $(I, G) \models \forall y \neg \varphi(a_0, y)$. This finishes the proof. ■

§ 3. The combinatorial part. We have shown in the previous section that the proof of Theorem 1.3 on functions provably total and Σ_r can be reduced to the proof of Theorem 2.12, which is a relativized version of Theorem 1.2 on functions provably total and Σ_1.

All syntactical information which we need to prove Theorem 2.12 is contained in Lemma 2.13. What we have to do now is of character of purely combinatorial considerations.

At first we derive Theorem 1.2 (which is not just a special case of 2.12) using Theorem 1.3. Theorem 1.3 for $r = 1$ gives us an estimation of the rapidity of growth of functions which are provably total and Σ_1 in finite fragments of the theory $I^-\Sigma_n$ by means of the hierarchy H_α^1 basing on the function $H^1(x) = 2^x$. Obviously it is enough to show that H_α^1 can be bounded by a function from the usual Hardy hierarchy in such a way that we obtain the estimate from Theorem 1.2.

The following lemma says that this is possible:

3.1. Lemma. (a) H^1 *with the index* $\omega^m \cdot k$ *is dominated by* H *with the index* $\omega^{m+2} \cdot k$,

(b) *For $m, n \geqslant 1$ the function H^1 with the index $\omega_n^{\omega^m \cdot k}$ is dominated by H with the index $\omega_n^{\omega^m \cdot (k+1)}$.*

Proof. (a) Let $\alpha \gg \beta$ mean that the smallest exponent in the Cantor representation of α is greater or equal to the biggest exponent for β. We assume that $0 \gg \alpha$. The following property is fundamental for the hierarchy G_α: for every $G\colon N \to N$, if $\alpha \gg \beta$, then $G_{\alpha+\beta} = G_\alpha \circ G_\beta$. The proof is by induction on β and is basing on the following property: if $\alpha \gg \beta$ and $\beta \in \mathrm{Lim}$ then $\{\alpha+\beta\}(x) = \alpha+\{\beta\}(x)$ (cf. Lemma 3 [W]). Hence, for every function G we have $G_{\omega^m \cdot k} = (G_{\omega^m})^k$; consequently, for the proof of assertion (a) it is enough to show that $H^1_{\omega^m}$ is dominated by $H_{\omega^{m+2}}$. We show by induction on $\alpha < \varepsilon_0$ a more general sublemma:

For $x > 0$ and $\alpha < \varepsilon_0$, $H^1_{\omega^\alpha}(x) \leqslant H_{\omega^{\alpha+2}}(x)$.

It is easy to check that $2^x \leqslant H_{\omega^2}(x)$ for $x > 0$. Hence the sublemma is true for $\alpha = 0$. We consider the inductive step $\alpha \to \alpha+1$. Let $x > 0$. Since $\{\omega^{\beta+1}\}(x) = \omega^\beta \cdot x$ for each $\beta < \varepsilon_0$, then

$$H^1_{\omega^{\alpha+1}}(x) = (H^1_{\omega^\alpha})^x(x) \leqslant (H_{\omega^{\alpha+2}})^x(x) = H_{\omega^{\alpha+3}}(x).$$

Assume now that the sublemma is true for ordinals $< \alpha$, where $\alpha \in \mathrm{Lim}$. Let $x > 0$. Since $\{\omega^\alpha\}(x) = \omega^{\{\alpha\}(x)}$ for $\alpha \in \mathrm{Lim}$, then $H^1_{\omega^\alpha}(x) = H^1_{\omega^{\{\alpha\}(x)}}(x) \leqslant H_{\omega^{\{\alpha\}(x)+2}}(x)$. Before going further let us mention a fact implicitly contained in [L–W], which can be found in an explicit form in [K–S], see Th. 2.4.

Let $\beta \Rightarrow_n \gamma$ mean that there exists a finite sequence $\beta_0, \ldots, \beta_k$ such that $\beta_0 = \beta$, $\beta_k = \gamma$ and $\beta_{i+1} = \{\beta_i\}(n)$ or $\beta_{i+1} = \beta_i - 1$ when $\beta_i \notin \mathrm{Lim}$ for $i = 0, \ldots, k-1$. Then fact is the following:

For every $n \geqslant x$ and for every $\beta < \varepsilon_0$, $\beta \Rightarrow_n \{\beta\}(x)$. In view of this fact we easily infer that for $n \geqslant x$, $H_{\omega^{\{\alpha\}(x)}}(n) \leqslant H_{\omega^\alpha}(n)$ (cf. Lemma 2(ii) [W]). Reasoning as in in the proof of the nonlimit step we show that for each $n \geqslant x$,

$$H_{\omega^{\{\alpha\}(x)+2}}(n) \leqslant H_{\omega^{\alpha+2}}(n).$$

Substitution $n = x$ finishes the proof of point (a).

(b) To finish the proof of assertion (b) it is enough to show that for all $m, n \geqslant 1$, $x \geqslant 2$ we have

$$\omega_n^{\omega^m \cdot (k+1)} \Rightarrow_x \omega^{(\omega_{n-1}^{\omega^m \cdot k})+2}.$$

According to the fact quoted above, the conjunction $\beta \Rightarrow_y \gamma$ and $y \leqslant x$ implies $\beta \Rightarrow_x \gamma$; and since $\beta \Rightarrow_x \gamma$ implies $\omega^\beta \Rightarrow_x \omega^\gamma$, it is enough to show that for all $m, n \geqslant 1$

$$\omega_{n-1}^{\omega^m(k+1)} \Rightarrow_2 \omega_{n-1}^{\omega^m \cdot k}+2.$$

We leave to the reader a verification of this fact.

The application of Lemma 2.13 to the proof of Theorem 2.12 requires an answer to the question how large the set S ought to be, in order that for a fixed sequence of formulas $\varphi_1, \ldots, \varphi_m$ it should be possible to find a common α-large approximation $S_1 \subseteq S$. An answer to this question is contained in Lemma 3.3.

3.2. Remark. We conclude immediately from Lemmas 2.6, 2.7 that for every sequence $\varphi_1, \ldots, \varphi_m$ of Σ_1-formulas there exists a natural number $l_{p,m}$, which depends on the number p of quantifiers in $\varphi_1, \ldots, \varphi_m$ and on m, such that ($I\Sigma_1$ proves that) for every $\alpha < \varepsilon_0$ and for every ω^α-large set S with $\min S \geqslant l_{p,m}$ there exists an α-large set $S_1 \subseteq S \setminus \{\max S\}$ which is a common approximation to the formulas $\varphi_1, \ldots, \varphi_m$.

The provability in $I\Sigma_1$ follows from provability of 2.7 in $I\Sigma_1$. A similar conclusion is true for Π_1-formulas in view of the definition of an approximation.

3.3. Lemma. *For every sequence $\varphi_1, \ldots, \varphi_m \in \Sigma_k(\Pi_k)$, where $k \geqslant 1$ there exists an $l \in \omega$ such that (even $I\Sigma_1$ proves that) for every $\alpha < \varepsilon_0$ and for every ω_k^α-large set S with $\min S \geqslant l$ there exists an α-large set $S_1 \subseteq S$ which is a common approximation to $\varphi_1, \ldots, \varphi_m$.*

Proof. We can assume that every formula among $\varphi_1, \ldots, \varphi_m$ has exactly k maximal blocks of uniform and unbounded quantifiers, shortly, k blocks of u.q.

Let $\varphi_{1i}, \ldots, \varphi_{mi}$ denote formulas which we obtain from $\varphi_1, \ldots, \varphi_m$ by deleting the initial $k-i$ blocks of u.q. respectively. Hence $\varphi_{1i}, \ldots, \varphi_{mi} \in \Sigma_i$ or $\varphi_{1i}, \ldots, \varphi_{mi} \in \Pi_i$. Let p_j denote the number of quantifiers appearing in the jth blocks in formulas $\varphi_1, \ldots, \varphi_m$. Take $l \geqslant \max(l_{p1,m}, \ldots, l_{pk,m})$, where $l_{p,m}$ is the number from 3.2.

We show by induction on $i = 1, \ldots, k$ that for every i there exists $S_i \subseteq S$ which is an approximation to $\varphi_{1i}, \ldots, \varphi_{mi}$ such that $S_i \setminus \{l_1 S_i, \ldots, l_{i-1} S_i\}$ is ω^α_{k-i}-large.

The initial step is true by 3.2. Assume that our inductive hypothesis is true for $i < k$. Take S_i as in the inductive assumption. Then for every $j = 1, \ldots, m$ we place the full first block of u.q. from $\varphi_{j,i+1}$ before the formula $\varphi^{S_i}_{j,i}$. Let us denote the resulting formulas by $\varphi'_{1,i+1}, \ldots, \varphi'_{m,i+1}$.

Let $S'_{i+1} \subseteq S_i \setminus \{l_1 S_i, \ldots, l_{i-1} S_i, l_i S_i\}$ be an $\omega^\alpha_{k-(i+1)}$-large set which is a common approximation to $\varphi'_{m,i+1}, \ldots, \varphi'_{m,i+1}$; it exists by 3.2. By Definition 2.8 of an approximation the set $S_{i+1} = S'_{i+1} \cup \{l_1 S_i, \ldots, l_i S_i\}$ is an approximation to formulas $\varphi_{1,i+1}, \ldots, \varphi_{m,i+1}$, which proves the inductive step.

For $i = k$ our inductive thesis assert the existence of a set $S_k \subseteq S$ which is a common approximation to $\varphi_1, \ldots, \varphi_m$ such that $S_k \setminus \{l_1 S_k, \ldots, l_{k-1} S_k\}$ is α-large. Obviously S_k is also α-large, and this finishes the proof.

Now we can pass to the announced proof of Theorem 2.12. there are given formulas $\exists \bar{y}\, \varphi_1(x, \bar{y}), \ldots, \exists \bar{y}\, \varphi_m(x, \bar{y}) \in \Pi_{n-1}(G)$ and a c-absolute formula $\psi(x, y)$ which defines in N a function $\bar{G}$ such that $\forall a\, \bar{G}(a) \geqslant 2^a$ (in the simplest case ψ is a Δ_0-formula defining the exponential function).

Assume that

$$I^-[\exists \bar{y}\, \varphi_1, \ldots, \exists \bar{y}\, \varphi_m] + \forall x, y\, [G(x) = y \Leftrightarrow \psi(x, y)] \vdash \forall x \exists y\, \varphi(x, y).$$

Let $k \in \omega$ be the appropriate number from Lemma 3.3. Denote by l_0 a number greater than the number l for $\varphi_1, \ldots, \varphi_m$ from 3.3, such that the condition $c \geqslant l_0$ implies $2^{m+1} kmc \leqslant 2^c$ — this inequality will be used in the proof of Claim 1.

Let

$$h(x) = \min_y (N, \bar{G}) \models \varphi(x, y).$$

We show that for every $a \geqslant l_0$, $h(a) \leqslant \bar{G}_\beta(a)$, where $\beta = \omega_{n-1}^{\omega m(k+1)}$. We shall work in the model (N, G). Let us emphasize that our reasoning does not change if we replace the model $(N, \bar{G})$ by a model $(M, \bar{G})$ for

$$I\Delta_0 + \exp + \forall x (G(x) = y \Leftrightarrow \psi(x, y)) + \forall x G_\beta(x) \downarrow,$$

in which (for $n > 1$) the hypothesis of Lemma 3.3 is true. We use this observation in a further refinement of Theorem 1.3.

Now take an $a \in N$ such that $a \geqslant l_0$ and let $b = \bar{G}_\beta(a)$. Assume, contrary to the assertion of our theorem, that

$$\forall y \leqslant b \neg (N, \bar{G}) \models \varphi(a, y).$$

Let

$$S_0 = \{\bar{G}^n a\colon n \in N \wedge \bar{G}^n(a) \leqslant b\}.$$

It is easy to check that $H_\beta^{S_0}(a) = \bar{G}_\beta(a)$. Hence the set S_0 is β-large and $\forall c \in S_0 \setminus \{\max S_0\}\ \bar{G}(c) \leqslant c^+$.

By Lemma 3.3 there exists a $\omega^m \cdot (k+1)$-large set $S \subseteq S_0$ which is an approximation to $\varphi_1, \ldots, \varphi_m$. Obviously

$$\forall c \in S \setminus \{\max S\}\ \bar{G}(c) \leqslant c^+.$$

By the choice of k the claim of Lemma 2.13 concerning provability in

$$I\Delta_0 + \exp + \forall x, y (G(x) = y \Leftrightarrow \psi(x, y))$$

is satisfied in $(N, \bar{G})$.

In the sequel we will repeteadly refer to this conclusion with $x = a$ and the set S as above. In these references we recall 2.13.

We will need some counterpart of the Grzegorczyk hierarchy for functions from S to S. Let $F_0^S = H_1^S$. Then we define $F_{j+1}^S(x) \simeq (F_j^S)^x(x)$ for $x \in S$. We have the equality $F_j^S = H_{\omega^j}^S$. Since the set S is $\omega^m \cdot (k+1)$-large, it follows that $(F_m^S)^{k+1}(\min S) \downarrow$.

The fast iteration which appears in the definition of the Grzegorczyk hierarchy is strictly connected with passing from a witness function for $\operatorname{Indst} \exists \bar{y} \varphi_i^S$ to the witness function for $\forall x \exists \bar{y} \varphi_i^S$. Indeed, if g is a witness function for $\operatorname{Indst}(\exists \bar{y} \varphi_i^S)$ and $\varphi_i^S(0, \bar{0})$ then obviously $g^z(z)$ is the witness function for $\forall x \exists \bar{y} \varphi_i^S$. Without loss of generality we may assume that $\varphi_i^S(0, \bar{0})$. Indeed; let $\psi_i(x, \bar{y})$ denote the natural normal form for the formula $x = 0 \vee \varphi_i(x+1, \bar{y})$. Then $I^-[\exists \bar{y} \varphi_i] \equiv I^-[\exists \bar{y} \psi_i]$ and $\psi_i^S(0, \bar{0})$.

Since F_0^S is a witness function for $\forall x \exists \bar{y} \varphi_i^S\colon i \in J = \emptyset$, then by 2.3 $(F_0^S)^k$ is a witness function for

$$\bigvee_{i \in [1,m]} \operatorname{Indst}(\exists \bar{y} \varphi_i^S) \vee \exists y \varphi(a, y).$$

By our assumption $(\neg \exists y \leqslant b \varphi(a, y))$, the last component of the alternative can be omitted.

CLAIM 1. *For every $j = 1, \ldots, m$ the function $F_j^S \circ F_0^S$ is a witness function for $\bigvee_{J \in P_j[1,m]} \bigwedge_{i \in J} \forall x \exists \bar{y} \varphi_i^S$, where $P_j[1, m]$ denotes the set of all j element subsets of* $[1, m]$.

Let us leave a proof of this rather intuitive Claim 1 to a later stage. First we show how the contradiction claimed above follows from Claim 1.

By Claim 1 for $j = m$ we infer that $F_m^S \circ F_0^S$ is a common witness function for formulas $\forall x \exists \bar{y} \varphi_i^S$, $i = 1, \ldots, m$. By 2.13 $(F_m^S \circ F_0^S)^k$ is a witness function for $\exists y \varphi(a, y)$. To obtain a contradiction it is enough to show that $(F_m^S \circ F_0^S)^k(\min S) \downarrow$. Since $F_0^S \circ F_m^S \leqslant F_m^S \circ F_0^S$ and $(F_0^S)^k(c) \leqslant F_m^S(c)$ for $c \geqslant k$ then $(F_m^S \circ F_0^S)^k(\min S) \leqslant (F_m^S)^{k+1}(\min S) \downarrow$ and the contradiction follows.

In the proof of Claim 1 we use the following:

CLAIM 2. *If $f\colon \subseteq S \to S$ is a witness function for $\bigvee \operatorname{Indst}(\exists \bar{y} \varphi_i^S)$ and $\forall c \in dm f$ $c < f(c)$, then $f^{mc}(c)$ is a witness function for $\bigvee_{i \in J} \forall x \exists \bar{y} \varphi_i^S$.*

For the proof of Claim 2 let take a $c \in S$ such that $f^{mc}(c) \downarrow$. By the definition of a witness function for alternative (cf. Def. 2.2), there exists for every $d < mc$ an $i \in J$ such that $\{(f^d(c), f^{d+1}(c))\}$ is a witness function for $\operatorname{Indst}(\exists \bar{y} \varphi_i^S)$. It follows by the pigeonhole principle that there exists an $i_0 \in J$ for which we have c elements as above — denote them by $d_0 < d_1 < \ldots < d_{c-1}$. Hence

$$g = \{(c, f^{d_1}(c)), (f^{d_1}(c), f^{d_2}(c)), \ldots, (f^{d_{c-1}}(c), f^{mc}(c))\}$$

is a witness function for $\operatorname{Indst}(\exists \bar{y} \varphi_{i_0}^S)$. And thus $g^c(c)$ is a witness function for $\forall x \exists \bar{y} \varphi_{i_0}^S$.

Since $g^c(c) = f^{mc}(c)$, Claim 2 is proved.

We now prove Claim 1. Since $(F_0^S)^k$ is a witness function for $\bigvee_{i \in [1,m]} \operatorname{Indst} \exists \bar{y} \varphi_i^S$ then $(F_0^S)^{kmc}(c)$ is a witness function for $\bigvee_{i \in [1,m]} \forall x \exists \bar{y} \varphi_i^S$. By the choice of l_0 we have $2^c \geqslant kmc$ for $c \in S$. Since $F_0^S(c) \geqslant 2^c$, our witness function is $\leqslant (F_0^S)^{F_0^S(c)}(c) \leqslant F_1^S \circ F_0^S(c)$, which proves that $F_1^S \circ F_0^S$ is the function for $j = 1$ with properties as claimed.

Now we perform the inductive step. We prove it under the assumption that $g = F_j^S \circ F_0^S$ is the function for a j as in Claim 1, where $1 \leqslant j \leqslant m$. Take $c \in S$ such that $F_{j+1}^S \circ F_0^S(c) \downarrow$. Let $g_0(c) = \binom{m}{j} kmc + 1$. By the choice of l_0, $2 g_0(c) \leqslant F_0^S(c)$ and hence $g^{g_0(c)}(c) \leqslant (F_j^S)^{F_0^S(c)}(c) \downarrow$. By the inductive assumption, for every $d < g_0(c)$ there exists a $J \in P_j[1, m]$ such that $\{(g^d(c), g^{d+1}(c))\}$ is a witness function for $\bigwedge_{i \in J} \forall x \exists \bar{y} \varphi_i^S$. By the pigeonhole principle we find a $J_0 \in P_j[1, m]$ and we find a function $f_c\colon \subseteq S \to S$ such that $c \in dm f_c$, $f_c^{kmc}(c) \leqslant g^{g_0(c)}(c)$ and f_c is a witness function for $\bigwedge_{i \in J_0} \forall x \exists \bar{y} \varphi_i^S$. Let $J' = [1, m] \setminus J$.

By 2.13 f_c^k is a witness function for $\bigvee_{i \in J'} \operatorname{Indst} \exists \bar{y} \varphi_i^S$. Hence by Claim 2 the function $f_c^{kmc}(c)$ is a witness function for $\bigvee_{i \in J'} \forall x \exists \bar{y} \varphi_i^S$. In particular, there is $j_0 \in J'$ such that for each $x \leqslant c$ there exists a $\bar{y} \leqslant f_c^{kmc}(c)$ with $\varphi_{i_0}^S(x, \bar{y})$. Since $f_c^{kmc}(c) \leqslant (F_j^S)^{F_0^S(c)}(c)$, it follows that the function $\{(c, F_{j+1}^S \circ F_0^S(c))\}$ is a witness function for $\bigwedge\!\!\!\bigwedge_{i \in J_0 \cup \{i_0\}} \forall x \exists \bar{y}\, \varphi_i^S$. This finishes the proof of Claim 1 and the proof of Theorem 2.12. ■

We have remarked in the proof of Theorem 2.12 that the above proof can be carried over in $I\Delta_0 + \exp + \forall x, y(G(x) = y \Leftrightarrow \psi(x, y)) + \forall x\, G_\beta(x)\downarrow$ if the assertion of 3.3 is (in the case $n > 1$) provable in this theory. In fact, this is true because the assertion of Lemma 3.3 is provable in $I\Sigma_1$ and as a Π_1-sentence is also provable in the formal theory of the Grzegorczyk hierarchy: $I\Delta_0 + \exp + \{\forall x\, H_{\omega^p}(x)\downarrow : p \in \omega\}$ which for $n \geqslant 1$, is a subtheory of the theory $I\Delta_0 + \exp + \forall x\, G_\beta(x)$.

The sentence $\forall a \geqslant l_0\, h(a) \leqslant \bar{G}_\beta(a)$, which we have proved, can be reformulated as follows:

$$\forall x \geqslant l_0 \exists y \leqslant G_\beta(x)\, \varphi(x, y)\,.$$

Hence the claim of Theorem 2.12 can be strengthened to the following form: there exist $k, l \in \omega$ such that for $\beta = \omega_{n-1}^{\omega m \cdot k}$

$$I\Delta_0 + \exp + \forall x, y(G(x) = y \Leftrightarrow \psi(x, y)) + \forall x\, G_\beta(x) \vdash$$
$$\forall x \geqslant l\ \exists y < G_\beta(x)\, \varphi(x, y)\,.$$

Hence we obtain the following theorem, in the same way as we have obtained 1.3 from 2.12:

3.4. THEOREM. *Let $1 \leqslant r \leqslant n$ and let $\varphi_1, \ldots, \varphi_m \in \Sigma_n$. Then for every formula $\varphi \in \Pi_{r-1}$ the following implication is true:*

If $I\Sigma_{r-1} + I^-[\varphi_1, \ldots, \varphi_m] \vdash \forall x \exists y\, \varphi(x, y)$ then there exist $k, l \in \omega$ such that for $\beta = \omega_{n-r}^{\omega m \cdot k}$

$$I\Sigma_{r-1} + \forall x\, H_\beta^r(x)\downarrow\ \vdash \forall x \geqslant l \exists y < H_\beta^r(x)\, \varphi(x, y)\,.$$

Note added in proof. If the basic theory of $I^-[\varphi_1, \ldots, \varphi_m]$, i.e. $I\Delta_0 + \exp$ is replaced by $I\Sigma_{n-1}$ then the appropriate reformulation of Theorem 3.4 can also be proved. The proof requires some minor changes of the proof of 1.2. In Lemma 2.13 we change the basic theory and we add the assumption that S is an approximation for a formula universal for $\Sigma_{n-1}(G)$-formulas.

Similar results have been obtained independently by Richard Kaye in his thesis, University of Manchester 1987. He uses an almost purely model-theoretic method. The method of this paper is different, it relies on strict separation of model theory and combinatorics, which provides some special information.

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