STUDIA MATHEMATICA, T. XCII. (1989)

# Nonhomogeneous initial-boundary value problems for linear parabolic systems

by

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**Abstract.** The existence and regularity properties of the strict solution of a linear autonomous parabolic system under general nonhomogeneous boundary conditions are analyzed in detail. Optimal regularity results, generalizing those known by the use of semigroup theory in the homogeneous case, are proved in the Hölder classes of $[L^p(\Omega)]^N$-valued functions.

**0. Introduction.** We are concerned with $C^1([0, T], [L^p(\Omega)]^N)$-solutions of linear parabolic systems of the following kind:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t, x) - A(x, D)u(t, x) = f(t, x), & (t, x) \in [0, T] \times \Omega, \\ u(0, x) = \varphi(x), & x \in \Omega, \\ B(x, D)u(t, x) = g(t, x), & (t, x) \in [0, T] \times \partial\Omega, \end{cases} \tag{0.1}$$

where $f$, $\varphi$, $g$ are prescribed $C^N$-valued functions and $A$, $B$ are matrices of differential operators; suitable hypotheses of regularity and ellipticity are assumed on the data and on the pair $(A, B)$.

Several authors have studied various types of solutions of problem (0.1): in [11], among other things, the case $N = 1$ is considered by means of techniques relying on the abstract theory of sums of linear operators (see also [6]) and on the extensive use of trace spaces. Many results concerning the case $g \equiv 0$ are due to [14], [7], as an application of the theory of analytic semigroups. On the other hand, in [12] weak solutions of an abstract Hilbert space version of (0.1) are studied, and an explicit representation formula for such solutions is exhibited.

Our technique here is in some sense intermediate with respect to those of [11] and [12], since it is based on the extensive use of operator-valued Dunford integrals, as in [11], and on an "a priori" representation formula for the solution of (0.1) which is very similar to that of [12]. Indeed, it can be seen that our formula reduces to that of [12] in the $L^2(\Omega)$-case (see Remark 3.3 below); however, it is to be noted that the former seems to be better from the point of view of applications to linear nonautonomous as well as nonlinear parabolic problems. Such problems will be studied in a forthcoming paper.

Keeping in mind such developments, it is important here to get a solution $u$ of (0.1) possessing an "optimal" regularity with respect to that of data: this is achieved by choosing as evolution space the Hölder space $C^{\alpha}([0, T], [L^p(\Omega)]^N)$ and assuming suitable compatibility conditions on initial and boundary values, which turn out to be necessary and sufficient. Thus we find that $u'$, $A(\cdot, D)u$, $B(\cdot, D)u$ belong to the same space as $f$, $g$, and the related "good" estimates hold: this is precisely what is needed for the above applications.

Let us describe the subjects of the next sections. Section 1 is devoted to preliminaries such as notation, assumptions and basic results to be used later. In Section 2 we introduce the main tools of our technique: "resolvent"-type operators and suitable Poincaré inequalities. In Section 3 we study existence of the solution and derive its representation formula. Section 4 contains some further technicalities which are needed in Section 5, where maximal regularity of the solution is (tediously) proved. Finally, there is an appendix containing the proof of a few results stated in Section 4.

## 1. Preliminaries

**1.A. Some function spaces.** Our basic function classes are the Lebesgue spaces $L^p(\Omega)$ and the Sobolev spaces $W^{k,p}(\Omega)$, $W_0^{k,p}(\Omega)$ ($k \in N$, $p \in [1, \infty]$), endowed with their usual norms; we denote by $[L^p(\Omega)]^N$ and $[W^{k,p}(\Omega)]^N$, $[W_0^{k,p}(\Omega)]^N$ the corresponding spaces for functions $f\colon \Omega \to C^N$, $N \geqslant 1$. In both cases, the $L^p$-norm (resp. the $W^{k,p}$-norm) of $f$ will be denoted by $\|f\|_p$ or $\|f\|_{p,\Omega}$ (resp. $\|f\|_{k,p}$ or $\|f\|_{k,p,\Omega}$).

If $Y$ is a Banach space, we will usually consider spaces of continuous functions $[0, T] \to Y$: namely, the whole space $C^0(Y)$, the Hölder classes $C^{\beta}(Y)$ ($\beta \in ]0, 1[$), the spaces $C^k(Y)$ and $C^{k+\beta}(Y)$, $k \in N^+$, whose definitions and norms are obvious; in particular, the Hölder seminorm of a function $f$ will be denoted by $[f]_{C^{\beta}(Y)}$ ($\beta \in ]0, 1[$). We will also use the space

$$B(Y) := \{u\colon [0, T] \to Y\colon \|u\|_{B(Y)} := \sup_{t \in [0,T]} \|u(t)\|_Y < \infty\}.$$

**1.B. Setting of the problem, assumptions.** Let $\Omega \subset R^n$ be a bounded open set with $C^2$ boundary. We consider the following linear differential operators with complex-valued coefficients, defined for $x \in \bar{\Omega}$:

$$(1.1)\quad A^{h,k}(x, D) := \sum_{s,j=1}^{n} a_{sj}^{hk}(x) D_s D_j + \sum_{j=1}^{n} a_j^{hk}(x) D_j + a^{hk}(x) I, \qquad h,k = 1, \ldots, N,$$

$$(1.2)\quad B^{r,h}(x, D) := \sum_{j=1}^{n} b_j^{rh}(x) D_j + b^{rh}(x) I, \qquad r,h = 1, \ldots, N,$$

where $N \geqslant 1$ is a fixed integer. Under suitable ellipticity assumptions on the coefficients of the operators $\{A^{h,k}\}$ and $\{B^{r,h}\}$, we are going to study the following evolution problem:



$$(1.3)\quad \begin{cases} \dfrac{\partial u^h}{\partial t}(t, x) - \sum_{k=1}^{N} [A^{h,k}(x, D) + \omega \delta^{hk}] u^k(t, x) = f^h(t, x), \\ \qquad\qquad h = 1, \ldots, N,\ (t, x) \in [0, T] \times \Omega, \\ u^h(0, x) = \varphi^h(x), \quad h = 1, \ldots, N,\ x \in \Omega, \\ \sum_{h=1}^{N} B^{r,h}(x, D) u^h(t, x) = g^r(t, x), \quad r = 1, \ldots, N,\ (t, x) \in [0, T] \times \partial\Omega, \end{cases}$$

where $T \in ]0, \infty[$, $\omega \in C$, $\delta^{hk}$ is the Kronecker matrix and $\{f^h\}$, $\{\varphi^h\}$, $\{g^r\}$ are prescribed data.

The assumptions listed below will guarantee the solvability of system (1.3) in suitable function spaces; such assumptions are essentially those required in order to study the spectral properties in $[L^p(\Omega)]^N$, $p \in ]1, \infty[$, of the elliptic boundary value problem

$$(1.4)\quad \begin{cases} \lambda U^h(x) - \sum_{k=1}^{N} A^{h,k}(x, D) U^k(x) = F^h(x), \quad h = 1, \ldots, N, \quad x \in \Omega, \\ B^r(x, D) U(x) := \sum_{h=1}^{N} B^{r,h}(x, D) U^h(x) = G^r(x), \quad r = 1, \ldots, N,\ x \in \partial\Omega; \end{cases}$$

they are formulated in [8], [9], [5]. It is worth while to recall them and quote the related results in the most useful way for the study of problem (1.3).

For each $(\theta, x, \xi, t) \in R \times \bar{\Omega} \times R^n \times R$ set

$$\bar{A}_{\theta}^{h,k}(x, \xi, t) = \sum_{s,j=1}^{n} a_{sj}^{hk}(x) \xi_s \xi_j + e^{i\theta} t^2 \delta^{hk}.$$

Hypothesis 1.1 ($\theta_0$-root condition). *There exist $\theta_0 \in ]\pi/2, \pi[$ and $c_0 > 0$ such that*

$$|\det\{\bar{A}_{\theta}^{h,k}(x, \xi, t)\}| \geqslant c_0 [|\xi|^2 + t^2]^N \qquad \forall (\theta, x, \xi, t) \in [-\theta_0, \theta_0] \times \bar{\Omega} \times R^n \times R;$$

*in addition, for each $(\theta, x, \xi, t) \in [-\theta_0, \theta_0] \times \partial\Omega \times R^n \times R$ with $|\xi|^2 + t^2 > 0$ and $\xi \cdot \nu(x) = 0$, the polynomial*

$$\tau \to \det\{\bar{A}_{\theta}^{h,k}(x, \xi + \tau\nu(x), t)\}$$

*has precisely $N$ roots $\tau_j^+(\theta, x, \xi, t)$, $j = 1, \ldots, N$, with positive imaginary part (here $\nu(x)$ is the unit outward normal vector at $x \in \partial\Omega$).*

In other words, Hypothesis 1.1 says that for each $\theta \in [-\theta_0, \theta_0]$ the system of differential operators

$$A_{\theta}^{h,k}(x, D, \partial/\partial t) := A^{h,k}(x, D) + \delta^{hk} e^{i\theta} \partial^2/\partial t^2$$

is uniformly elliptic in $\bar{\Omega} \times R$ in the sense of [4].

For all $x\in\partial\Omega$ and $\xi\in\mathbf{R}^n-\{0\}$, we denote by $\hat{A}_\theta$ the adjoint matrix of $\{\bar{A}_\theta^{h,k}\}$, i.e.

$$\{\hat{A}_\theta^{h,k}\}:=(\det\{\bar{A}_\theta^{h,k}\})\cdot\{\bar{A}_\theta^{h,k}\}^{-1}.$$

Concerning the boundary operators $\{B^{r,h}\}$, we assume that for each $x\in\partial\Omega$ the order of the operator $B^{r,h}(x,D)$ does not exceed $m_{r,h}$, where $m_{r,h}=0$ or $m_{r,h}=1$. This means that if $m_{r,h}=0$ then no derivatives appear in $B^{r,h}(\cdot,D)\equiv B^{r,h}(\cdot)$, so that each function $b_j^{rh}$ vanishes identically, whereas if $m_{r,h}=1$ then at least one among the functions $b_j^{rh}$ does not vanish. We denote by $\bar{B}^{r,h}(x,D)$ the principal part of $B^{r,h}(x,D)$, i.e. that having precisely order $m_{r,h}$. Note that even if $m_{r,h}=1$ it may happen that $\bar{B}^{r,h}(x,D)=0$ at some $x\in\partial\Omega$; however, we require the following

HYPOTHESIS 1.2 ($\theta_0$-complementing condition). *For each $(\theta,x,\xi,t)\in[-\theta_0,\theta_0]\times\partial\Omega\times\mathbf{R}^n\times\mathbf{R}$ with $|\xi|^2+t^2>0$ and $\xi\cdot\nu(x)=0$ the rows of the matrix*

$$\{\sum_{h=1}^N\bar{B}^{r,h}(x,\xi+\tau\nu(x))\,\hat{A}_\theta^{h,k}(x,\xi+\tau\nu(x))\}_{r,k=1}^N$$

*are linearly independent modulo the polynomial*

$$\tau\to\prod_{j=1}^N(\tau-\tau_j^+(\theta,x,\xi,t)).$$

Finally, concerning the regularity of the coefficients of $\{A^{h,k}\}$, $\{B^{r,h}\}$, we assume:

HYPOTHESIS 1.3. *Set $m_r:=\max\{m_{r,h}:\ h=1,\ldots,N\}$; then*

$$a_{sj}^{hk},\ a_j^{hk},\ a^{hk}\in C^0(\bar{\Omega}),\qquad s,j=1,\ldots,n;\ h,k=1,\ldots,N,$$
$$b_j^{rh},\ b^{rh}\in C^{2-m_r}(\bar{\Omega}),\qquad j=1,\ldots,n;\ h,r=1,\ldots,N.$$

THEOREM 1.4. *Under Hypotheses 1.1–1.3, for each $p\in]1,\infty[$ there exist $\lambda_p>0$ and $C_p>0$ such that for each $\lambda$ in the sector*

$$S_{p,\theta_0}:=\{z\in\mathbf{C}:\ |\arg(z-\lambda_p)|<\theta_0\},$$

*and for all $F\in[L^p(\Omega)]^N$, $G\in\prod_{r=1}^N W^{2-m_r,p}(\Omega)$, problem (1.4) has a unique solution $U\in[W^{2,p}(\Omega)]^N$, which satisfies in addition*

$$(1.5)\qquad \sum_{k=0}^2(|\lambda|+1)^{1-k/2}\|D^kU\|_p$$
$$\leqslant C_p\{\|F\|_p+\sum_{r=1}^N\inf[\|D^{2-m_r}V^r\|_p+(|\lambda|+1)^{1-m_r/2}\|V^r\|_p]\}$$

*where for $r=1,\ldots,N$ the infimum is taken among all $V^r\in W^{2-m_r,p}(\Omega)$ such that $V^r=G^r$ on $\partial\Omega$ (in the sense of $W^{2-m_r-1/p,p}(\partial\Omega)$, i.e. in the sense of traces on $\partial\Omega$).*



Proof. The existence of $U$ is proved in [9, Teorema 5.3]; the dependence on $\lambda$ is proved in [5, Theorems 12.2–13.1]. ■

Remark 1.5. It is not restrictive to suppose that the numbers $m_r$ satisfy $m_r\leqslant m_{r+1}$, $r=1,\ldots,N-1$; as $m_r$ is either 0 or 1, we can define the integer $r_0$, $0\leqslant r_0\leqslant N$, as

$$(1.6)\qquad r_0:=\begin{cases}0 & \text{if } m_r\neq 0 \text{ for } r=1,\ldots,N,\\ \max\{r:\ m_r=0\} & \text{otherwise.}\end{cases}$$

Hence we can divide the $N$ boundary operators $\{B^r(x,D)\}$ (see (1.4)) into two classes, one of them possibly empty:

$$(1.7)\qquad B_0(x,D)\equiv B_0(x):=\{B^r(x)\}_{1\leqslant r\leqslant r_0},\qquad B_1(x,D):=\{B^r(x,D)\}_{r_0<r\leqslant N}.$$

Note that $B_0(\cdot)$ maps $[W^{2,p}(\Omega)]^N$ into $[W^{2,p}(\Omega)]^{r_0}$, whereas $B_1(\cdot,D)$ maps $[W^{2,p}(\Omega)]^N$ into $[W^{1,p}(\Omega)]^{N-r_0}$. Correspondingly, the data $\{G^r\}_{1\leqslant r\leqslant N}$ can also be divided into two classes:

$$(1.8)\qquad G_0:=\{G^r\}_{1\leqslant r\leqslant r_0}\in[W^{2,p}(\Omega)]^{r_0},\qquad G_1:=\{G^r\}_{r_0<r\leqslant N}\in[W^{1,p}(\Omega)]^{N-r_0}.$$

Remark 1.6. If we are able to solve (1.3) for a certain $\bar{\omega}\in\mathbf{C}$, then we can solve it for each $\omega\in\mathbf{C}$. Thus instead of (1.3) we can study

$$(1.9)\qquad \begin{cases}\dfrac{\partial u^h}{\partial t}(t,x)-\displaystyle\sum_{k=1}^N[A^{h,k}(x,D)-\lambda_p\delta^{hk}]u^k(t,x)=f^h(t,x),\\ \qquad\qquad h=1,\ldots,N,\ (t,x)\in[0,T]\times\Omega,\\ u^h(0,x)=\varphi^h(x),\qquad h=1,\ldots,N,\ x\in\Omega,\\ B^r(x,D)u(t,x)=\displaystyle\sum_{h=1}^N B^{r,h}(x,D)u^h(t,x)=g^r(t,x),\\ \qquad\qquad r=1,\ldots,N,\qquad (t,x)\in[0,T]\times\partial\Omega,\end{cases}$$

where $\lambda_p$ is the constant appearing in Theorem 1.4. The elliptic boundary value problem which corresponds to (1.9) is now

$$(1.10)\qquad \begin{cases}\lambda U(x)-A(x,D)U(x)=F(x), & x\in\Omega,\\ B_0(x)U(x)=G_0(x), & x\in\partial\Omega,\\ B_1(x,D)U(x)=G_1(x), & x\in\partial\Omega,\end{cases}$$

where we have set

$$(1.11)\qquad A(x,D)U(x):=\{\sum_{k=1}^N[A^{h,k}(x,D)-\lambda_p\delta^{hk}]U^k(x)\}_{h=1}^N;$$

problem (1.10) is uniquely solvable for each $\lambda$ belonging to the sector $S_{\theta_0} := \{z \in C: |\arg z| < \theta_0\}$ and for all $F \in [L^p(\Omega)]^N$, $G_0 \in [W^{2,p}(\Omega)]^{r_0}$, $G_1 \in [W^{1,p}(\Omega)]^{N-r_0}$. In addition, the estimate (1.5) holds for each $\lambda \in S_{\theta_0}$.

Let us define what we mean by a solution of problem (1.9).

Definition 1.7. A *strict solution* of (1.9) is a function

$$u \in C^1([L^p(\Omega)]^N) \cap C^0([W^{2,p}(\Omega)]^N)$$

such that:

(i) $\partial u/\partial t - A(\cdot, D)u = f$ in the sense of $C^0([L^p(\Omega)]^N)$.

(ii) $B^r(\cdot, D)u = g^r$ in the sense of $C^0(W^{2-m_r-1/p,p}(\partial\Omega))$, $r = 1, \ldots, N$, i.e. in the sense of traces on $\partial\Omega$ for each fixed $t$.

(iii) $u(0, \cdot) = \varphi$ in the sense of $[W^{2,p}(\Omega)]^N$.

**1.C. A little bit of interpolation.** If $X, Y$ are Banach spaces with $X \hookrightarrow Y$ (continuous imbedding) we can construct the interpolation spaces $(X, Y)_{1-\theta,\infty}$ ($\theta \in ]0,1[$) in the following way ([13]):

Definition 1.8. $(X, Y)_{1-\theta,\infty}$ is the set of $x \in X$ such that there exists a function $u$: $]0,1] \to X$ differentiable as a function $]0,1] \to Y$, satisfying:

(i) $\sup\limits_{t \in ]0,1]} t^{1-\theta}\|u(t)\|_X < \infty, \qquad \sup\limits_{t \in ]0,1]} t^{1-\theta}\|u'(t)\|_Y < \infty.$

(ii) $u(0) = x$.

We set

$$\|x\|_{(X,Y)_{1-\theta,\infty}} := \inf_{u(0)=x} \{\sup_{t \in ]0,1]} t^{1-\theta}\|u(t)\|_X + \sup_{t \in ]0,1]} t^{1-\theta}\|u'(t)\|_Y\}.$$

Lemma 1.9. *Let $f \in C^\alpha(X) \cap C^{1+\beta}(Y)$ with $\alpha, \beta \in ]0,1[$. We have:*

(i) *$f' \in B((X, Y)_{1-\theta,\infty})$ where $\theta = \beta/(1+\beta-\alpha)$ and*

$$\|f'\|_{B((X,Y)_{1-\theta,\infty})} \leqslant c(\alpha, \beta)\{[f]_{C^\alpha(X)} + [f']_{C^\beta(Y)}\}.$$

(ii) *If $\sigma \in ]0,\beta[$, then $f' \in C^\sigma((X, Y)_{1-\lambda,\infty})$ where $\lambda = (\beta-\sigma)/(1+\beta-\alpha)$ and*

$$\|f'\|_{C^\sigma((X,Y)_{1-\lambda,\infty})} \leqslant c(\alpha, \beta, \sigma)\{[f]_{C^\alpha(X)} + [f']_{C^\beta(Y)}\}.$$

Proof. (i) It is easy to see that if $t_0 \in [0, T]$, then $f'(t_0)$ satisfies Definition 1.8 with

$$u(t) := v(t^{1/(1+\beta-\alpha)}), \qquad v(r) := r^{-1}\int_{R} f'(t_0+s)\varphi((r-s)/r)\,ds,$$

$\varphi$ being a $C^\infty$ real-valued function with support contained in $]-1, 1[$ and such that $\int_R \varphi(x)\,dx = 1$.

Part (ii) follows by interpolation via the Reiteration Theorem. ∎



If we choose in particular $X = [W^{2,p}(\Omega)]^N$, $Y = [L^p(\Omega)]^N$, $p \in ]1, \infty[$, we can characterize the interpolation spaces $(X, Y)_{1-\theta,\infty}$.

Definition 1.10. (i) Let $N \in N^+$, $s > 0$. The *Besov–Nikol'skii spaces* $[B^{s,p}_\infty(R^n)]^N \equiv [B^{s,p}(R^n)]^N$ are defined in the following way:

(a) If $s \in ]0,1[$,

$$[B^{s,p}(R^n)]^N := \{f \in [L^p(R^n)]^N: \; [f]_{s,p} := \sup_{h \in R^n} |h|^{-s}\Big[\int_{R^n} |f(x+h)-f(x)|^p\,dx\Big]^{1/p} < \infty\}.$$

(b) If $s = 1$,

$$[B^{1,p}(R^n)]^N := \{f \in [L^p(R^n)]^N: \; [f]_{*,1,p} = \sup_{h \in R^n} |h|^{-1}\Big[\int_{R^n} |f(x+h)+f(x-h)-2f(x)|^p\,dx\Big]^{1/p} < \infty\}.$$

(c) If $s = k+\sigma$ where $k \in N$, $\sigma \in ]0,1]$,

$$[B^{s,p}(R^n)]^N := \{f \in [W^{k,p}(R^n)]^N: D^\gamma f \in [B^{\sigma,p}(R^n)]^N \;\; \forall |\gamma| = k\}.$$

A norm in $[B^{s,p}(R^n)]^N$, $s = k+\sigma$, $k \in N$, $\sigma \in ]0,1[$, is given by

$$\|f\|_{s,p,R^n} \equiv \|f\|_{s,p} := \|f\|_{k,p} + \sum_{|\gamma|=k} [D^\gamma f]_{\sigma,p};$$

an obvious modification is required when $s = 1$, in which case we write $\|f\|_{*,1,p}$ to denote the $B^{1,p}$-norm (in order to avoid confusion with the $W^{1,p}$-norm).

(ii) The space $[B^{s,p}(\Omega)]^N$ consists of the restrictions $f|_\Omega$ such that $f \in [B^{s,p}(R^n)]^N$; a norm in $[B^{s,p}(\Omega)]^N$ is

$$\|g\|_{s,p,\Omega} \equiv \|g\|_{s,p} := \inf_{f|_\Omega = g} \|f\|_{s,p,R^n}$$

(writing $\|g\|_{*,s,p}$ instead of $\|g\|_{s,p}$ when $s$ is an integer).

The characterization of the spaces $(X, Y)_{1-\theta,\infty}$ is the following:

Proposition 1.11. *If $\theta \in ]0,1[$ we have:*

(i) $([W^{k,p}(\Omega)]^N, [L^p(\Omega)]^N)_{1-\theta,\infty} = [B^{k\theta,p}(\Omega)]^N \quad (k \in N^+)$;

(ii) $([W^{2,p}(\Omega) \cap W^{1,p}_0(\Omega)]^N, [L^p(\Omega)]^N)_{1-\theta,\infty} = [B^{2\theta,p}_0(\Omega)]^N$, *provided* $\theta \neq 1/(2p)$;

(iii) $([W^{2,p}_\#(\Omega)]^N, [L^p(\Omega)]^N)_{1-\theta,\infty} = [B^{2\theta,p}_\#(\Omega)]^N$, *provided* $\theta \neq 1/(2p)$, $\theta \neq 1/2+1/(2p)$,

*where we have set*

(1.12) $\quad [B^{2\theta,p}_0(\Omega)]^N := \{f \in [B^{2\theta,p}(\Omega)]^N: f = 0$ *on* $\partial\Omega$ *if* $2\theta > 1/p\}$, (1.13) $$[W^{2,p}_{\#}(\Omega)]^N := \{f \in [W^{2,p}(\Omega)]^N\colon\ B^r(\cdot, D)f = 0 \text{ on } \partial\Omega,\ r = 1, \ldots, N\}$$

(1.14) $$[B^{2\theta,p}_{\#}(\Omega)]^N := \{f \in [B^{2\theta,p}(\Omega)]^N\colon\ B^r(\cdot, D)f = 0 \text{ on } \partial\Omega \text{ if } 2\theta > m_r + 1/p,\ r = 1, \ldots, N\}.$$

Proof. For the case $N = 1$ see [15, Theorems 4.3.1/1–4.3.3(a)] and [11, Théorème 7.5]. In the general case, $N \geq 1$, one can proceed by adapting the argument of [1] with the aid of Definition 1.8 and of Lemma 4.2 below. We omit the details. ■

## 2. Tools

**2.A. The operators $R(\lambda)$, $N_0(\lambda)$, $N_1(\lambda)$.** Theorem 1.4 allows us to construct three "resolvent" operators which play a fundamental role in our theory.

For fixed $p \in ]1,\infty[$ and for each $\lambda \in S_{\theta_0}$ we denote by $R(\lambda)\colon [L^p(\Omega)]^N \to [W^{2,p}(\Omega)]^N$ the operator defined by

(2.1) $$U = R(\lambda)F \Leftrightarrow \begin{cases} \lambda U - A(\cdot, D)U = F & \text{in } \Omega, \\ B_0(\cdot)U = 0,\ B_1(\cdot, D)U = 0 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 1.4, $R(\lambda)$ is well defined and we have

(2.2) $$\sum_{k=0}^{2} (1+|\lambda|)^{1-k/2} \|D^k R(\lambda)F\|_p \leq C_p \|F\|_p \quad \forall \lambda \in S_{\theta_0}.$$

If we set $Y := [L^p(\Omega)]^N$, $D_A := [W^{2,p}_{\#}(\Omega)]^N$ (see (1.13)), and $Au := A(\cdot, D)u$, then (2.1) and (2.2) say that $R(\lambda) = (\lambda - A)^{-1} \in \mathscr{L}(Y)$ and $\|R(\lambda)\|_{\mathscr{L}(Y)} \leq C/|\lambda|$; thus $A$ is the infinitesimal generator of an analytic semigroup $\{E(\xi)\}_{\xi \geq 0}$ which is strongly continuous at $\xi = 0$ since $\bar{D}_A = Y$. The semigroup can be represented as

(2.3) $$E(s)\psi = \fint_\gamma e^{s\lambda} R(\lambda)\psi \, d\lambda, \quad s > 0, \quad \psi \in [L^p(\Omega)]^N,$$

where $\fint_\gamma$ means $(2\pi i)^{-1} \int_\gamma$ and $\gamma$ is a curve contained in $S_{\theta_0}$ and joining $\infty e^{-i\theta}$ to $\infty e^{i\theta}$ ($\theta \in ]\pi/2, \theta_0[$); for instance we can take

(2.4) $$\gamma := \{z \in \mathbf{C}\colon z = re^{\pm i\theta},\ r \geq 1\} \cup \{z \in \mathbf{C}\colon z = e^{i\sigma},\ |\sigma| \leq \theta\}.$$

The main properties of $E(\cdot)$ are summarized in the following

Proposition 2.1. *We have*:

(i) $E(\cdot)\psi \in C^0(Y)$, $\lim_{\xi \downarrow 0} \|E(\xi)\psi - \psi\|_Y = 0 \quad \forall \psi \in Y$.

(ii) $E(\cdot)\psi \in C^\beta(Y)$, $\beta \in ]0, 1[ \Leftrightarrow \psi \in [B^{2\beta,p}_{\#}(\Omega)]^N$, $\beta \notin \{1/(2p), 1/2 + 1/(2p)\}$ (see (1.14)).

(iii) $E(\cdot)\psi \in C^1(Y) \Leftrightarrow \psi \in D_A$; *in this case* $\dfrac{d}{d\xi} E(\xi)\psi = E(\xi)A\psi$.

(iv) $E(\cdot)\psi \in C^{1+\beta}(Y)$, $\beta \in ]0,1[ \Leftrightarrow \psi \in D_A$ *and* $A\psi \in [B^{2\beta,p}_{\#}(\Omega)]^N$, $\beta \notin \{1/(2p), 1/2 + 1/(2p)\}$.

*Moreover, in each case $E(\cdot)\psi$ depends continuously on $\psi$ in the corresponding norms.*

Proof. This follows by the results of [14] and by Proposition 1.11(iii). ■

Next, by using again Theorem 1.4 we construct for fixed $p \in ]1,\infty[$ and for each $\lambda \in S_{\theta_0}$ the operators $N_0(\lambda)\colon [W^{2,p}(\Omega)]^{r_0} \to [W^{2,p}(\Omega)]^N$ and $N_1(\lambda)\colon [W^{1,p}(\Omega)]^{N-r_0} \to [W^{2,p}(\Omega)]^N$ ($r_0$ is defined in (1.6)) in the following manner:

(2.5) $$V = N_0(\lambda)G_0 \Leftrightarrow \begin{cases} \lambda V - A(\cdot, D)V = 0 & \text{in } \Omega, \\ B_0(\cdot)V = G_0,\ B_1(\cdot, D)V = 0 & \text{on } \partial\Omega, \end{cases}$$

(2.6) $$W = N_1(\lambda)G_1 \Leftrightarrow \begin{cases} \lambda W - A(\cdot, D)W = 0 & \text{in } \Omega, \\ B_0(\cdot)W = 0,\ B_1(\cdot, D)W = G_1 & \text{on } \partial\Omega. \end{cases}$$

By Theorem 1.4, $N_0(\lambda)$ and $N_1(\lambda)$ are well defined and satisfy

(2.7) $$\sum_{k=0}^{2} (1+|\lambda|)^{1-k/2} \|D^k N_0(\lambda)G_0\|_p \leq C_p \inf\{\|D^2\psi\|_p + (1+|\lambda|)\|\psi\|_p\colon \psi \in [W^{2,p}(\Omega)]^{r_0},\ \psi = G_0 \text{ on } \partial\Omega\},$$

(2.8) $$\sum_{k=0}^{2} (1+|\lambda|)^{1-k/2} \|D^k N_1(\lambda)G_1\|_p \leq C_p \inf\{\|D\psi\|_p + (1+|\lambda|)^{1/2}\|\psi\|_p\colon \psi \in [W^{1,p}(\Omega)]^{N-r_0},\ \psi = G_1 \text{ on } \partial\Omega\}.$$

The reader can amuse himself in proving the following simple

Lemma 2.2. *Fix $p \in ]1, \infty[$ and let $\lambda, \mu \in S_{\theta_0}$. We have*:

(i) $A(\cdot, D)R(\lambda)\psi = \lambda R(\lambda)\psi - \psi \quad \forall \psi \in [L^p(\Omega)]^N$.

(ii) $$\lambda R(\lambda)\psi = \psi + R(\lambda)A(\cdot, D)\psi - N_0(\lambda)B_0(\cdot)\psi - N_1(\lambda)B_1(\cdot, D)\psi \quad \forall \psi \in [W^{2,p}(\Omega)]^N.$$

(iii) $$[A(\cdot, D)R(\lambda) - R(\lambda)A(\cdot, D)]\psi = -N_0(\lambda)B_0(\cdot)\psi - N_1(\lambda)B_1(\cdot, D)\psi \quad \forall \psi \in [W^{2,p}(\Omega)]^N.$$

(iv) $[R(\lambda) - R(\mu)]\psi = (\mu - \lambda)R(\lambda)R(\mu)\psi \quad \forall \psi \in [L^p(\Omega)]^N$.

(v) $[N_0(\lambda)-N_0(\mu)]\psi_0 = (\mu-\lambda)R(\lambda)N_0(\mu)\psi_0$

$$\forall\psi_0 \in [W^{2,p}(\Omega)]^{r_0} \quad \text{(see (1.6)).}$$

(vi) $[N_1(\lambda)-N_1(\mu)]\psi_1 = (\mu-\lambda)R(\lambda)N_1(\mu)\psi_1$

$$\forall\psi_1 \in [W^{1,p}(\Omega)]^{N-r_0} \quad \text{(see (1.6)).} \ \blacksquare$$

Remark 2.3. By Lemma 2.2(v)–(vi) we see that the operators $N_0(\lambda)$, $N_1(\lambda)$, as well as $R(\lambda)$, are holomorphic functions of $\lambda$, defined in $S_{\theta_0}$.

**2.B. The open set $\Omega$.** We have assumed that $\Omega$ is a bounded open set of $\boldsymbol{R}^n$ with $C^2$ boundary. For each $x \in \bar{\Omega}$ set $d(x) := \operatorname{dist}(x, \partial\Omega)$ and for $\varrho > 0$ set $\Omega(\varrho) := \{x \in \Omega: d(x) < \varrho\}$.

LEMMA 2.4. *There exists $\varrho_0 > 0$ such that $d(\cdot) \in C^2(\overline{\Omega(\varrho_0)})$. Moreover, for each $x \in \Omega(\varrho_0)$ there exists a unique $y(x) \in \partial\Omega$ such that $d(x) = \operatorname{dist}(x, y(x))$ and such that $x$, $y(x)$ and $y(x)+v(y(x))$ lie on the same line ($v(y(x))$ is the unit outward normal vector at $y(x)$). Moreover, $x \to y(x)$ and $x \to v(y(x)) \equiv v(x)$ belong to $C^1(\overline{\Omega(\varrho_0)})$ and may be extended as functions in $C^1(\bar{\Omega})$.*

Proof. See [10, Appendix]. $\blacksquare$

LEMMA 2.5. *Fix $p \in ]1, \infty[$. There exists $K_p > 0$ such that for each $\varrho \in ]0, \varrho_0]$ ($\varrho_0$ is given in Lemma 2.4) and for each $u \in [W^{2,p}(\Omega)]^N$ we have*

$$\|Du\|_{p,\Omega(\varrho)} \leqslant K\{\varrho\|D^2u\|_{p,\Omega(\varrho)} + \varrho^{-1}\|u\|_{p,\Omega(\varrho)}\}. \tag{2.9}$$

Proof. Estimate (2.9) is classical, except for the dependence on $\varrho$. The latter can be achieved by adapting e.g. the proof of [2, Theorem 4.14]. $\blacksquare$

LEMMA 2.6 (Poincaré). *Fix $p \in ]1,\infty[$. We have:*

(i) *There exists $K_p > 0$ such that for all $\varrho \in ]0, \varrho_0]$* (see Lemma 2.4) *and $f \in [W_0^{1,p}(\Omega)]^N$*

$$\|f\|_{p,\Omega(\varrho)} \leqslant K_p\varrho\|Df\|_{p,\Omega(\varrho)}. \tag{2.10}$$

(ii) *If $\theta \in ]0,1/(2p)[$ there exists $K_{p,\theta} > 0$ such that for all $\varrho \in ]0,\varrho_0]$ and $f \in [B^{2\theta,p}(\Omega)]^N$*

$$\|f\|_{p,\Omega(\varrho)} \leqslant K_{p,\theta}\varrho^{2\theta}\|f\|_{2\theta,p,\Omega(\varrho)}. \tag{2.11}$$

(iii) *If $\theta \in ]1/(2p), 1/2+1/(2p)[$ there exists $K_{p,\theta} > 0$ such that for all $\varrho \in ]0,\varrho_0]$ and $f \in [B_0^{2\theta,p}(\Omega)]^N$* (see (1.12))

$$\|f\|_{p,\Omega(\varrho)} \leqslant K_{p,\theta}\varrho^{2\theta}\|f\|_{2\theta,p,\Omega(\varrho)}. \tag{2.12}$$

Proof. (2.10) follows easily by using the properties of $\Omega$ and the fact that $f = 0$ on $\partial\Omega$; (2.11) is a consequence of (2.10) and of interpolation theory ([15, Theorem 4.3.3]). Finally, (2.12) follows by applying first (2.10) to $f$ and then (2.11) to $Df$. $\blacksquare$



An immediate consequence of Lemma 2.6 is a refinement of the estimate (2.7).

LEMMA 2.7. *If $\theta \in ]0,1/(2p)[$ there exists $C(p, \varrho_0, \theta)$ such that for each $G_0 \in [W^{2,p}(\Omega)]^{r_0}$*

$$\sum_{k=0}^{2}(1+|\lambda|)^{1-k/2}\|D^kN_0(\lambda)G_0\|_p \leqslant C(p, \varrho_0, \theta)\inf\{\|D^2\psi\|_p + (1+|\lambda|)^{1-\theta}\|\psi\|_{2\theta,p}\colon \psi \in [W^{2,p}(\Omega)]^{r_0},\ \psi = G_0 \text{ on } \partial\Omega\}. \tag{2.13}$$

Proof. It is enough to apply (2.7) with $\psi$ replaced by $\psi\eta_\lambda$, where $\eta_\lambda \in C^\infty(\bar{\Omega})$ is such that:

$$\begin{cases} \eta_\lambda = \begin{cases} 1 & \text{on } \Omega_\lambda := \Omega\left(\frac{1}{2}(\varrho_0 \wedge (1+|\lambda|)^{-1/2})\right), \\ 0 & \text{on } \Omega - \Omega(\varrho_0 \wedge (1+|\lambda|)^{-1/2}), \end{cases} \\ \|D\eta_\lambda\|_\infty \leqslant c[\varrho_0^{-1} v (1+|\lambda|)^{1/2}], \quad \|D^2\eta_\lambda\|_\infty \leqslant c[\varrho_0^{-2} v (1+|\lambda|)]. \end{cases} \tag{2.14}$$

Indeed, by (2.14) and (2.7) we have for each $\psi \in [W^{2,p}(\Omega)]^{r_0}$ with $\psi = G_0$ on $\partial\Omega$

$$\sum_{k=0}^{2}(1+|\lambda|)^{1-k/2}\|D^kN_0(\lambda)G_0\|_p \leqslant c\{\|D^2\psi\|_p + (1+|\lambda|)\|\psi\|_{p,\Omega_\lambda}\};$$

noting that there exists $c(\varrho_0) > 0$ such that

$$c(\varrho_0)^{-1}(1+|\lambda|)^{-1/2} \leqslant \varrho_0 \wedge (1+|\lambda|)^{-1/2} \leqslant c(\varrho_0)(1+|\lambda|)^{-1/2} \qquad \forall\lambda \in S_{\theta_0},$$

we use (2.11) and the result follows. $\blacksquare$

**3. Existence and representation of the solution.** We suppose that Hypotheses 1.1–1.3 hold and we fix $p \in ]1, \infty[$, $\alpha \in ]0,1[$ with $\alpha \notin \{1/(2p), 1/2, 1/2+1/(2p)\}$ (such values of $\alpha$ are critical in some sense; see Remark 3.2 below). For notational simplicity we set

$$\begin{aligned} &Y := [L^p(\Omega)]^N, \quad &&Y_0 := [L^p(\Omega)]^{r_0}, \quad &&Y_1 := [L^p(\Omega)]^{N-r_0}, \\ &X := [W^{2,p}(\Omega)]^N, \quad &&X_0 := [W^{2,p}(\Omega)]^{r_0}, \quad &&X_1 := [W^{1,p}(\Omega)]^{N-r_0}. \end{aligned} \tag{3.1}$$

We choose the data of problem (1.9) in the following way:

$$f = \{f^h\}_{h=1}^N \in C^\alpha(Y), \tag{3.2}$$

$$g_0 = \{g^r\}_{r=1}^{r_0} \in C^\alpha(X_0) \cap C^{\alpha+1}(Y_0), \tag{3.3}$$

$$g_1 = \{g^r\}_{r=r_0+1}^N \in C^\alpha(X_1) \cap C^{\alpha+1/2}(Y_1), \tag{3.4}$$

$$\varphi = \{\varphi^h\}_{h=1}^N \in X. \tag{3.5}$$

In addition, we assume the following compatibility conditions:

$$B_0(x)\,\varphi(x) = g_0(0,\,x), \qquad x\in\partial\Omega, \tag{3.6}$$

$$B_1(x,\,D)\,\varphi(x) = g_1(0,\,x), \qquad x\in\partial\Omega. \tag{3.7}$$

THEOREM 3.1. *Under the above assumptions, problem* (1.9) *has a unique strict solution* $u$, *which is given by the following representation formula*:

$$u(t)\equiv u(t,\,\cdot) = \fint_\gamma e^{t\lambda}R(\lambda)\,\varphi\,d\lambda + \int_0^t \fint_\gamma e^{(t-s)\lambda}R(\lambda)\,f(s)\,d\lambda\,ds + \int_0^t \fint_\gamma e^{(t-s)\lambda}N_0(\lambda)\,g_0(s)\,d\lambda\,ds + \int_0^t \fint_\gamma e^{(t-s)\lambda}N_1(\lambda)\,g_1(s)\,d\lambda\,ds, \tag{3.8}$$

*where* $\gamma$ *is the curve* (2.4), $\fint_\lambda$ *means* $(2\pi i)^{-1}\int_\gamma$ *and* $f(s)$, $g_0(s)$, $g_1(s)$ *stand for* $f(s,\cdot)$, $g_0(s,\cdot)$, $g_1(s,\cdot)$. *Moreover, for each* $\theta\in\,]0,\,1/(2p)[\,\cap\,]0,\,\alpha]$ *the following estimate holds*:

$$\|u'\|_{C^0(Y)}+\|u\|_{C^0(X)} \leqslant C_\theta\,\{\|\varphi\|_X+\|f(0)\|_Y+T^\theta\,[[f]_{C^\theta(Y)}+[g_0]_{C^\theta(X_0)} + [g_0']_{C^\theta(Y_0)}+[g_1]_{C^\theta(X_1)}+[g_1]_{C^{\theta+1/2}(Y_1)}]\}. \tag{3.9}$$

Proof. Uniqueness of the strict solution follows by the theory of analytic semigroups (see e.g. [14]). The proof of existence consists of four steps:

*Step* 1: *Formula* (3.8) *is meaningful.*
*Step* 2: *The function* $u$ *given by* (3.8) *is differentiable in* $Y$ *and* (3.9) *holds.*
*Step* 3: $u$ *solves the equation of* (1.9) *in* $[0,\,T]\times\Omega$.
*Step* 4: $u$ *satisfies the boundary conditions of* (1.9).

Proof of Step 1. This is the simplest part of the proof, but it is worth while performing it carefully since we use a complex variable change which will appear systematically in the sequel; thus the details of this calculation will not be repeated any more.

Proposition 2.1(i) immediately gives $E(\cdot)\varphi\in C^0(Y)$ and $E(0)\varphi=\varphi$ ($E(\cdot)$ is defined in (2.3)).

Fix $0\leqslant s<t\leqslant T$ and set $\mu := (t-s)\lambda$ in the complex integral

$$I(t,\,s) := \fint_\gamma e^{(t-s)\lambda}[R(\lambda)\,f(s)+N_0(\lambda)\,g_0(s)+N_1(\lambda)\,g_1(s)]\,d\lambda.$$

We obtain

$$I(t,\,s) = \fint_{(t-s)\cdot\gamma} e^{\mu}\,[R(\mu(t-s)^{-1})\,f(s)+N_0(\mu(t-s)^{-1})\,g_0(s) + N_1(\mu(t-s)^{-1})\,g_1(s)]\,(t-s)^{-1}\,d\mu.$$

Now we remark that the curve $(t-s)\cdot\gamma$ is homotopic to $\gamma$ in $S_{\theta_0}$ and that the



integrand is a holomorphic function of $\mu$ in $S_{\theta_0}$ (Remark 2.3). Hence we can replace $\fint_{(t-s)\cdot\gamma}$ by $\fint_\gamma$, so that by (2.2), (2.13) and (2.8) we get for a fixed $\theta\in\,]0,1/(2p)[$

$$\|I(t,\,s)\|_Y \leqslant c\int_\gamma e^{\mathrm{Re}\,\mu}\Big[\frac{t-s}{t-s+|\mu|}\|f(s)\|_Y+\Big(\frac{t-s}{t-s+|\mu|}+\frac{(t-s)^\theta}{(t-s+|\mu|)^\theta}\Big)\|g_0(s)\|_{X_0} + \Big(\frac{t-s}{t-s+|\mu|}+\frac{(t-s)^{1/2}}{(t-s+|\mu|)^{1/2}}\Big)\|g_1(s)\|_{X_1}\Big]\frac{|d\mu|}{t-s}$$

$$\leqslant c\,[1+(t-s)^{\theta-1}+(t-s)^{-1/2}]\,\{\|f\|_{C^0(Y)}+\|g_0\|_{C^0(X_0)}+\|g_1\|_{C^0(X_1)}\}.$$

This estimate immediately yields that formula (3.8) makes sense and for each $\theta\in\,]0,1/(2p)[$

$$\|u(t)-\varphi\|_Y \leqslant \|E(t)\,\varphi-\varphi\|_Y+O(t^\theta) \qquad \text{as } t\downarrow 0.$$

A similar calculation shows that $u\in C^0(Y)$.

Proof of Step 2. Actually we will prove that $u'\in C^0(Y)$ and

$$u'(t) = \fint_\gamma \lambda e^{t\lambda}R(\lambda)\,\varphi\,d\lambda+\int_0^t\fint_\gamma \lambda e^{(t-s)\lambda}\,\{R(\lambda)\,[f(s)-f(t)] + \sum_{j=0}^{1} N_j(\lambda)\,[g_j(s)-g_j(t)]\}\,d\lambda\,ds + \fint_\gamma e^{t\lambda}\,[R(\lambda) f(t)+\sum_{j=0}^{1} N_j(\lambda)\,g_j(t)]\,d\lambda. \tag{3.10}$$

The hardest task consists in verifying that formula (3.10) is meaningful, i.e. that all integrals involved are in fact convergent; next, we will show, by a usual approximation procedure on intervals $[\delta,\,T]$, $\delta>0$, that formula (3.10) indeed coincides with $u'(t)$, and finally we will prove that $u'(t)$, constructed in $]0,\,T]$, is in fact a continuous function in $[0,\,T]$.

First of all, fix $\theta\in\,]0,1/(2p)[\,\cap\,]0,\alpha]$. Using Lemma 1.9(i), Proposition 1.11(i) and (3.3), we certainly have

$$\begin{cases} f\in C^\theta(Y), \quad g_1\in C^\theta(X_1)\cap C^{\theta+1/2}(Y_1), \quad g_0\in C^\theta(X_0)\cap C^{\theta+1}(Y_0),\\ g_0'\in B([B^{2\theta,p}(\Omega)]^N),\\ \|g_0'(t)\|_{2\theta,p}\leqslant c\,\{[g_0]_{C^\theta(X_0)}+[g_0']_{C^\theta(Y_0)}\} \qquad \forall t\in[0,\,T]. \end{cases} \tag{3.11}$$

By (2.2), for each $t>0$ we easily get (via the change $\mu := (t-s)\lambda$; we will mention this fact no more!)

$$\Big\|\int_0^t\fint_\gamma \lambda e^{(t-s)\lambda}R(\lambda)\,[f(s)-f(t)]\,d\lambda\,ds\Big\|_Y \leqslant ct^\theta\,[f]_{C^\theta(Y)}; \tag{3.12}$$

on the other hand, (2.8) yields

$$\Big\|\int_0^t\int_\gamma \lambda e^{(t-s)\lambda} N_1(\lambda)[g_1(s)-g_1(t)]\,d\lambda\,ds\Big\|_Y \leqslant ct^\theta\{[g_1]_{C^\theta(X_1)}+[g_1]_{C^{\theta+1/2}(Y_1)}\}, \tag{3.13}$$

and finally by (2.11) and (3.11)

$$\Big\|\int_0^t\int_\gamma \lambda e^{(t-s)\lambda} N_0(\lambda)[g_0(s)-g_0(t)]\,d\lambda\,ds\Big\|_Y \leqslant ct^\theta\{[g_0]_{C^\theta(X_0)}+[g_0']_{C^\theta(Y_0)}\}. \tag{3.14}$$

The remaining integrals appearing in (3.10) are obviously convergent, so that (3.10) is meaningful.

Next, we consider the following functions defined in $[\delta, T]$, $\delta>0$:

$$u^\varepsilon(t)\equiv u^\varepsilon(t,\cdot):=\int_\gamma e^{t\lambda}R(\lambda)\varphi\,d\lambda+\int_0^{t-\varepsilon}\int_\gamma e^{(t-s)\lambda}\Big[R(\lambda)f(s)+\sum_{j=0}^1 N_j(\lambda)g_j(s)\Big]\,d\lambda\,ds,$$

where $\varepsilon\in]0,\delta[$. By using again (2.2), (2.8), (2.13) and (3.11) it is not difficult to show that if $\varepsilon\downarrow 0$ then $u^\varepsilon(t)\to u(t)$ in $Y$, uniformly in $[\delta, T]$, whereas $(u^\varepsilon)'(t)$ tends in $Y$ to the right-hand side of (3.10), uniformly in $[\delta, T]$. Thus $u'$ is given by (3.10) and is continuous in $]0, T]$.

In order to get continuity at $t=0$, we rewrite $u'(t)$ as

$$\begin{aligned} u'(t) &= \int_\gamma e^{t\lambda}\Big[\lambda R(\lambda)\varphi+R(\lambda)f(0)+\sum_{j=0}^1 N_j(\lambda)g_j(0)\Big]\,d\lambda\\ &\quad+\int_0^t\int_\gamma e^{(t-s)\lambda}\Big\{\lambda R(\lambda)[f(s)-f(t)]+\sum_{j=0}^1 \lambda N_j(\lambda)[g_j(s)-g_j(t)]\Big\}\,d\lambda\\ &\quad+\int_\gamma e^{t\lambda}\Big\{R(\lambda)[f(t)-f(0)]+\sum_{j=0}^1 N_j(\lambda)[g_j(t)-g_j(0)]\Big\}\,d\lambda\\ &=: I_1+I_2+I_3. \end{aligned}$$

The same arguments as before now lead to

$$\|I_2\|_Y+\|I_3\|_Y\leqslant ct^\theta\{[f]_{C^\theta(Y)}+[g_1]_{C^\theta(X_1)}+[g_1]_{C^{\theta+1/2}(Y_1)} +[g_0]_{C^\theta(X_0)}+[g_0']_{C^\theta(Y_0)}\}; \tag{3.15}$$

concerning $I_1$, we invoke Lemma 2.2(ii) and the compatibility conditions (3.6), (3.7) and rewrite it in the following way:

$$\begin{aligned} I_1 &= \int_\gamma e^{t\lambda}[\lambda R(\lambda)\varphi+N_0(\lambda)B_0\varphi+N_1(\lambda)B_1(D)\varphi+R(\lambda)f(0)]\,d\lambda\\ &= \int_\gamma e^{t\lambda}R(\lambda)[A(D)\varphi+f(0)]\,d\lambda \end{aligned}$$



(here $A(D)$ stands for $A(\cdot, D)$, see (1.11), and $B_0$, $B_1(D)$ stand for $B_0(\cdot)$, $B_1(\cdot, D)$. see (1.7)). Now by Proposition 2.1(i) and (2.2) we conclude that

$$I_1 = A(D)\varphi+f(0)+o(1)\quad \text{as } t\downarrow 0,\qquad \|I_1\|_Y\leqslant c\|A(D)\varphi+f(0)\|_Y;$$

recalling (3.15) we easily find that $u'(0)=A(D)\varphi+f(0)$ exists, so that $u'\in C^0(Y)$. Moreover, by (3.15) and the equation $u'-A(D)u=f$, (3.9) also follows.

Proof of Step 3. For each $t\in]0, T]$ rewrite (3.8) in the following manner:

$$\begin{aligned} u(t) &= \int_\gamma e^{t\lambda}R(\lambda)\varphi\,d\lambda+\int_0^t\int_\gamma e^{(t-s)\lambda}\Big\{R(\lambda)[f(s)-f(t)]\\ &\quad+\sum_{j=0}^1 N_j(\lambda)[g_j(s)-g_j(t)]\Big\}\,d\lambda\,ds\\ &\quad+\int_\gamma \lambda^{-1}e^{t\lambda}\Big[R(\lambda)f(t)+\sum_{j=0}^1 N_j(\lambda)g_j(t)\Big]\,d\lambda\\ &\quad-\int_\gamma \lambda^{-1}\Big[R(\lambda)f(t)+\sum_{j=0}^1 N_j(\lambda)g_j(t)\Big]\,d\lambda. \end{aligned} \tag{3.16}$$

By using once again the estimates (2.2), (2.8), (2.13) as well as (3.11), it is easy to see that the first three integrals in (3.16) are convergent in $X \equiv [W^{2,p}(\Omega)]^N$, whereas the last one vanishes, since the integrand is holomorphic in $S_{\theta_0}$ and decays as $|\lambda|^{-1-\theta}$ for large $|\lambda|$. Hence we can operate with $A(D)$, obtaining by Lemma 2.2(i)

$$\begin{aligned} A(D)u(t) &= \int_\gamma \lambda e^{t\lambda}R(\lambda)\varphi\,d\lambda+\int_0^t\int_\gamma e^{(t-s)\lambda}\Big\{\lambda R(\lambda)[f(s)-f(t)]\\ &\quad+\sum_{j=0}^1 \lambda N_j(\lambda)[g_j(s)-g_j(t)]\Big\}\,d\lambda\,ds\\ &\quad+\int_\gamma e^{t\lambda}\Big[R(\lambda)f(t)+\sum_{j=0}^1 N_j(\lambda)g_j(t)\Big]\,d\lambda-\int_\gamma \lambda^{-1}e^{t\lambda}f(t)\,d\lambda. \end{aligned} \tag{3.17}$$

By the Residue Theorem the last integral equals $f(t)$ and consequently a comparison between (3.17) and (3.10) shows that

$$A(D)u(t)=u'(t)-f(t),\qquad t\in]0, T].$$

Recalling that $f, u'\in C^0(Y)$ and that $u'(0)=A(D)\varphi+f(0)$, we conclude that $A(D)u\in C^0(Y)$, and the equation of (1.9) holds.

Proof of Step 4. As we saw before, $u(t)\in X$ so that we can operate with $B_0$ and $B_1(D)$ on $u(t)$. By (3.16) and the Residue Theorem we have

$$B_1(D)u(t) = \int_0^t \oint_\gamma e^{(t-s)\lambda} B_1(D) N_1(\lambda)[g_1(s)-g_1(t)]\,d\lambda\,ds$$
$$+\oint_\gamma \lambda^{-1} e^{t\lambda} B_1(D) N_1(\lambda) g_1(t)\,d\lambda = g_1(t),$$

and similarly we get $B_0 u(t) = g_0(t)$.

The proof of Theorem 3.1 is complete. ■

Remark 3.2. The values $\alpha = 1/(2p)$, $\alpha = 1/2$ and $\alpha = 1/2+1/(2p)$ are critical for different reasons. When $\alpha = 1/2$ it is not clear which is the right space for $g_1$ (see (3.4)): it should perhaps be the space $C^{1/2}(X_1)\cap C^{*,1}(Y_1)$, where $C^{*,1}(Y_1)$ is the Zygmund class of functions $h\in C^0(Y_1)$ such that

$$\sup_{0\leqslant s<t\leqslant T} (t-s)^{-1}\,\|h(t)+h(s)-2h((t+s)/2)\|_{Y_1} < \infty .$$

When $\alpha = 1/(2p)$ or $\alpha = 1/2+1/(2p)$ the problem is that the interpolation spaces $([W^{2,p}(\Omega)]^N, [L^p(\Omega)]^N)_{\alpha,\infty}$ have not been concretely characterized, so that we are not able to state the explicit compatibility conditions in our maximal regularity theorem (see Section 5 below).

Remark 3.3. By Lemma 2.2(v),(vi) we have (setting $r_1 := N-r_0$)

$$[N_j(\lambda)-N_j(0)]\,h_j = -\lambda R(\lambda) N_j(0)\,h_j \quad \forall h_j\in[W^{2-j,p}(\Omega)]^{r_j},\ j=0,1;$$

hence formula (3.8) can be rewritten as

$$u(t) = \oint_\gamma e^{t\lambda} R(\lambda)\,\varphi\,d\lambda - \int_0^t \oint_\gamma \lambda e^{(t-s)\lambda} R(\lambda) \sum_{j=0}^{1} N_j(0)\,g_j(s)\,d\lambda\,ds$$
$$+\int_0^t \oint_\gamma e^{(t-s)\lambda} R(\lambda) f(s)\,d\lambda\,ds,$$

or, recalling (2.3) and denoting by $A$ the infinitesimal generator of $E(t)$,

$$u(t) = E(t)\varphi + A\int_0^t E(t-s)\sum_{j=0}^{1} N_j(0)\,g_j(s)\,ds + \int_0^t E(t-s) f(s)\,ds. \tag{3.18}$$

Formula (3.18) looks very close to the representation formula (1.1) of [12]. However, (3.8) and [12, (1.1)] act in different situations: in the latter, the functions $g_j$ are less regular than ours, and consequently there is a need for extending the operators $N_j(0)$ to larger trace spaces: this is possible, due to the Hilbert space framework of [12].

## 4. Technicalities

**4.A. Approximation of $B^{2\sigma,p}$-functions.** Let $f\in[B^{2\sigma,p}(\Omega)]^k$, $\sigma\in]0,1[$, $k\in N^+$. By definition, there exists a function $F\in[B^{2\sigma,p}(\boldsymbol{R}^n)]^k$ such that

$$F|_\Omega \equiv f, \qquad \|F\|_{2\sigma,p,\boldsymbol{R}^n} \leqslant 2\,\|f\|_{2\sigma,p,\Omega}.$$



For $\varepsilon > 0$ set

$$F_{0,\varepsilon}(x) := \varepsilon^{-n}\int_{\boldsymbol{R}^n} F(y)\,\varphi(\varepsilon^{-1}(x-y))\,dy, \qquad x\in\boldsymbol{R}^n,$$

$\varphi$ being a scalar even function in $C^\infty(\boldsymbol{R}^n)$ with support contained in the unit ball and such that $\int_{\boldsymbol{R}^n}\varphi(x)\,dx = 1$; next, define

$$F_\varepsilon(x) := F_{0,\varepsilon}(x) - \varepsilon(\partial F_{0,\varepsilon}/\partial\varepsilon)(x), \qquad x\in\boldsymbol{R}^n, \qquad f_\varepsilon := F_\varepsilon|_\Omega. \tag{4.1}$$

Remark 4.1. The approximating functions of type (4.1) play a fundamental role in this paper; hence the above notation (with subscript $\varepsilon$) will always refer to this, and only this, kind of approximation.

The following properties are proved in the Appendix.

LEMMA 4.2. *Fix $k\in N^+$, $\sigma\in]0,1[-\{1/2\}$ and let $f\in[B^{2\sigma,p}(\Omega)]^k$. Then for each $\varepsilon>0$ we have:*

(i) $\|f_\varepsilon - f\|_{p,\Omega} \leqslant c\varepsilon^{2\sigma}\|f\|_{2\sigma,p,\Omega}$.

(ii) $\|f_\varepsilon - f\|_{2\theta,p,\Omega} \leqslant c\varepsilon^{2(\sigma-\theta)}\|f\|_{2\sigma,p,\Omega} \quad \forall\theta\in]0,\sigma]$.

(iii) $\|D^h f_\varepsilon\|_{p,\Omega} \leqslant c\varepsilon^{2\sigma-h}\|f\|_{2\sigma,p,\Omega} \quad \forall h\in N,\ h>2\sigma$.

(iv) $\|Df_\varepsilon - Df\|_{p,\Omega} \leqslant c\varepsilon^{2\sigma-1}\|f\|_{2\sigma,p,\Omega} \quad$ *if* $\sigma\in]1/2,1[$.

*Finally, if $f\in[W^{1,p}(\Omega)]^k$ then*

(v) $\|f_\varepsilon - f\|_{p,\Omega} \leqslant c\varepsilon\,\|Df\|_{p,\Omega}$.

Proof. See the Appendix below. ■

LEMMA 4.3. *Fix $k\in N^+$, $\sigma\in]0,1[-\{1/2\}$ and let $f\in[B^{2\sigma,p}(\Omega)]^k$. Then for each $\varepsilon>0$ we have*

(i) $\|D^h(\partial f_\varepsilon/\partial\varepsilon)\|_{p,\Omega} \leqslant c\varepsilon^{2\sigma-1-h}\|f\|_{2\sigma,p,\Omega} \quad \forall h\in N$.

(ii) $\|\partial f_\varepsilon/\partial\varepsilon\|_{2\theta,p,\Omega} \leqslant c\varepsilon^{2\sigma-2\theta-1}\|f\|_{2\sigma,p,\Omega} \quad \forall\theta\in]0,1[$.

Proof. See the Appendix below. ■

The above lemmata imply the following simple but useful corollary:

COROLLARY 4.4. *Fix $k\in N^+$, $\sigma\in]0,1[-\{1/2\}$, and let $f\in[B^{2\sigma,p}(\Omega)]^k$. Set $\chi_t := [f_\varepsilon]_{\varepsilon=t^{1/2}}$, $t>0$; then for $t>r>0$ we have:*

(i) $\|D^h\chi_t\|_{p,\Omega} \leqslant ct^{\sigma-h/2}\|f\|_{2\sigma,p,\Omega} \quad \forall h\in N,\ h>2\sigma$.

(ii) $\|D^h(\chi_t-\chi_r)\|_{p,\Omega} \leqslant c\,[t^{\sigma-h/2}-r^{\sigma-h/2}]\,\|f\|_{2\sigma,p,\Omega} \quad \forall h\in N$.

(iii) $\|\chi_t-\chi_r\|_{2\theta,p,\Omega} \leqslant c\,(t-r)^{\sigma-\theta}\|f\|_{2\sigma,p,\Omega} \quad \forall\theta\in]0,\sigma]$.

(iv) $\|D^h(\partial\chi_t/\partial t)\|_{p,\Omega} \leqslant ct^{\sigma-1-h/2}\|f\|_{2\sigma,p,\Omega} \quad \forall h\in N$.

(v) $\|\partial\chi_t/\partial t\|_{2\theta,p,\Omega} \leqslant ct^{\sigma-\theta-1}\|f\|_{2\sigma,p,\infty} \quad \forall\theta\in]0,\sigma]$.

Proof. This is a straightforward consequence of Lemma 4.2 and of the identity

$$\partial\chi_t/\partial t = \tfrac{1}{2} t^{-1/2} [\partial f_\varepsilon/\partial\varepsilon]_{\varepsilon=t^{1/2}}. \quad \blacksquare$$

**4.B. Further properties of $N_0(\lambda)$, $N_1(\lambda)$.** Recall that $r_0$ is the number defined in (1.6), $B_0 \equiv B_0(\cdot)$ and $B_1(D) \equiv B_1(\cdot, D)$ are the boundary operators defined in (1.7), and the spaces $Y$, $X$, $Y_0$, $Y_1$, $X_0$, $X_1$ were introduced in (3.1).

LEMMA 4.5. *Fix $\sigma \in ]0,1[-\{1/2, 1/(2p)\}$ and let $h \in [B^{2\sigma,p}(\Omega)]^N$, $g \in [B^{2\sigma,p}(\Omega)]^{r_0}$ be such that $B_0 h = g$ on $\partial\Omega$ if $\sigma > 1/(2p)$. Then for all $\varepsilon > 0$ and $\theta \in ]0,1/(2p)[\cap]0,\sigma]$ we have:*

$$\text{(4.2)} \quad \sum_{k=0}^{2} (1+|\lambda|)^{1-k/2} \|D^k N_0(\lambda)(B_0 h_\varepsilon - g_\varepsilon)\|_Y$$
$$\leqslant c[\varepsilon^{2\sigma-3}(1+|\lambda|)^{-1/2} + \varepsilon^{2\sigma-2} + \varepsilon^{2\sigma-2\theta}(1+|\lambda|)^{1-\theta} + (1+|\lambda|)^{1-\sigma}]$$
$$\times \{\|h\|_{2\sigma,p,\Omega} + \|g\|_{2\sigma,p,\Omega}\}.$$

Proof. Set

$$I_0 := \sum_{k=0}^{2} (1+|\lambda|)^{1-k/2} \|D^k N_0(\lambda)(B_0 h_\varepsilon - g_\varepsilon)\|_Y.$$

*Case* 1: $\sigma \in ]0,1/2+1/(2p)[-\{1/(2p),1/2\}$. We apply estimate (2.7) with $\psi := (B_0 h_\varepsilon - g_\varepsilon)\eta_\lambda$, where $\eta_\lambda$ is defined by (2.14) (recall also Remark 4.1). We get

$$I_0 \leqslant c\{\|D^2(B_0 h_\varepsilon - g_\varepsilon)\|_{p,\Omega_\lambda} + (1+|\lambda|)\|B_0 h_\varepsilon - g_\varepsilon\|_{p,\Omega_\lambda}\}$$
$$\leqslant c\{\|h_\varepsilon\|_{X_0} + \|g_\varepsilon\|_{X_0} + (1+|\lambda|)[\|B_0(h_\varepsilon - h)\|_{p,\Omega_\lambda}$$
$$+ \|B_0 h - g\|_{p,\Omega_\lambda} + \|g - g_\varepsilon\|_{p,\Omega_\lambda}]\}.$$

Now by Lemma 4.2(iii) and (2.11) we obtain for a fixed $\theta \in ]0,1/(2p)[\cap]0,\sigma]$:

$$I_0 \leqslant c\varepsilon^{2\sigma-2}\{\|h\|_{2\sigma,p,\Omega} + \|g\|_{2\sigma,p,\Omega}\}$$
$$+ c(1+|\lambda|)^{1-\theta}\{\|h_\varepsilon - h\|_{2\theta,p,\Omega} + \|g_\varepsilon - g\|_{2\theta,p,\Omega}\} + c\|B_0 h - g\|_{p,\Omega_\lambda}.$$

Finally, we use Lemma 4.2(i) and (2.11) (or (2.12) if $\sigma > 1/(2p)$), deducing

$$I_0 \leqslant c[\varepsilon^{2\sigma-2} + \varepsilon^{2(\sigma-\theta)}(1+|\lambda|)^{1-\theta} + (1+|\lambda|)^{1-\sigma}]\{\|h\|_{2\sigma,p,\Omega} + \|g\|_{2\sigma,p,\Omega}\},$$

which implies the result.

*Case* 2: $\sigma \in [1/2+1/(2p),1[$. We apply again (2.7) with

$$\psi(x) := \left[B_0(x) h_\varepsilon(x) - g_\varepsilon(x) - d(x)\left(\frac{\partial}{\partial\nu(\cdot)}(B_0 h_\varepsilon - g_\varepsilon)\right)_\varepsilon(x)\right]\eta_\lambda(x)$$

where $\eta_\lambda$ is the function (2.14) and $d(x)$, $\nu(x)$ are defined in Lemma 2.4



(again, recall also Remark 4.1). Set for simplicity $J^\varepsilon := B_0 h_\varepsilon - g_\varepsilon$, $J := B_0 h - g$; then we get

$$\text{(4.3)} \quad I_0 \leqslant c\left\{\left\|D^2\left[J^\varepsilon - d\left(\frac{\partial J^\varepsilon}{\partial\nu}\right)_\varepsilon\right]\right\|_{p,\Omega_\lambda}\right.$$
$$\left. + (1+|\lambda|)\left[\left\|J^\varepsilon - d\frac{\partial J^\varepsilon}{\partial\nu}\right\|_{p,\Omega_\lambda} + \left\|d\left[\frac{\partial J^\varepsilon}{\partial\nu} - \left(\frac{\partial J^\varepsilon}{\partial\nu}\right)_\varepsilon\right]\right\|_{p,\Omega_\lambda}\right]\right\}$$
$$:= I_1 + I_2 + I_3.$$

On the other hand, by Lemma 4.2(iii) and recalling that $d \in C^2(\bar\Omega_\lambda)$ and $\partial J^\varepsilon/\partial\nu \in [C^1(\bar\Omega_\lambda)]^{r_0}$ by Lemma 2.4, we have

$$\|D^2 J^\varepsilon\|_{p,\Omega} \leqslant c\{\|h_\varepsilon\|_{X_0} + \|g_\varepsilon\|_{X_0}\} \leqslant c\varepsilon^{2\sigma-2}\{\|h\|_{2\sigma,p,\Omega} + \|g\|_{2\sigma,p,\Omega}\},$$

$$\left\|D^2\left[d\left(\frac{\partial J^\varepsilon}{\partial\nu}\right)_\varepsilon\right]\right\|_{p,\Omega_\lambda} \leqslant c\left\{\left\|dD^2\left(\frac{\partial J^\varepsilon}{\partial\nu}\right)_\varepsilon\right\|_{p,\Omega_\lambda} + \left\|D\left(\frac{\partial J^\varepsilon}{\partial\nu}\right)_\varepsilon\right\|_{p,\Omega_\lambda} + \left\|\left(\frac{\partial J^\varepsilon}{\partial\nu}\right)_\varepsilon\right\|_{p,\Omega_\lambda}\right\}$$
$$\leqslant c\left\{(1+|\lambda|)^{-1/2}\varepsilon^{2\sigma-3}\left\|\frac{\partial J^\varepsilon}{\partial\nu}\right\|_{2\sigma-1,p,\Omega}\right.$$
$$\left. + \varepsilon^{2\sigma-2}\left\|\frac{\partial J^\varepsilon}{\partial\nu}\right\|_{2\sigma-1,p,\Omega} + \left\|\frac{\partial J^\varepsilon}{\partial\nu}\right\|_{p,\Omega}\right\}$$
$$\leqslant c\{\varepsilon^{2\sigma-3}(1+|\lambda|)^{-1/2} + \varepsilon^{2\sigma-2} + 1\}\|J^\varepsilon\|_{2\sigma,p,\Omega}$$
$$\leqslant c\{\varepsilon^{2\sigma-3}(1+|\lambda|)^{1/2} + \varepsilon^{2\sigma-2} + 1\}\|J\|_{2\sigma,p,\Omega}$$

and therefore

$$\text{(4.4)} \quad I_1 \leqslant c[\varepsilon^{2\sigma-3}(1+|\lambda|)^{-1/2} + \varepsilon^{2\sigma-2} + 1]\{\|h\|_{2\sigma,p,\Omega} + \|g\|_{2\sigma,p,\Omega}\}.$$

Next, concerning $I_3$, as $\partial J^\varepsilon/\partial\nu \in [C^1(\bar\Omega)]^{r_0}$ we deduce by Lemma 4.2(v),(iii)

$$\text{(4.5)} \quad I_3 \leqslant c(1+|\lambda|)^{1/2}\varepsilon\|D^2 J^\varepsilon\|_{p,\Omega} \leqslant c(1+|\lambda|)^{1/2}\varepsilon^{2\sigma-1}\|J^\varepsilon\|_{2\sigma,p,\Omega}$$
$$\leqslant c\varepsilon^{2\sigma-1}(1+|\lambda|)^{1/2}\{\|h\|_{2\sigma,p,\Omega} + \|g\|_{2\sigma,p,\Omega}\}.$$

Finally, we have to estimate $I_2$. Noting that $J - d\cdot\partial J^\varepsilon/\partial\nu = 0$ on $\partial\Omega$ and using (2.11), (2.10) and Lemma 4.2(ii),(iv),(iii), a direct calculation yields for a fixed $\theta \in ]0,1/(2p)[\cap]0,\sigma]$

$$I_2 \leqslant c(1+|\lambda|)\left[\|J^\varepsilon - J\|_{p,\Omega_\lambda} + \left\|J - d\frac{\partial J^\varepsilon}{\partial\nu}\right\|_{p,\Omega_\lambda}\right]$$
$$\leqslant c(1+|\lambda|)^{1-\theta}\|J^\varepsilon - J\|_{2\theta,p,\Omega} + c(1+|\lambda|)^{1/2}\left\|D\left(J - d\frac{\partial J^\varepsilon}{\partial\nu}\right)\right\|_{p,\Omega_\lambda}$$

$$\leqslant c\varepsilon^{2\sigma-2\theta}(1+|\lambda|)^{1-\theta}\|J\|_{2\sigma,p,\Omega}$$
$$+c(1+|\lambda|)^{1/2}\left[\left\|DJ-Dd\frac{\partial J}{\partial v}\right\|_{p,\Omega_\lambda}\right.$$
$$\left.+\left\|Dd\left(\frac{\partial J}{\partial v}-\frac{\partial J^\varepsilon}{\partial v}\right)\right\|_{p,\Omega_\lambda}+\|dD^2J^\varepsilon\|_{p,\Omega_\lambda}\right]$$
$$\leqslant c[\varepsilon^{2\sigma-2\theta}(1+|\lambda|)^{1-\theta}+\varepsilon^{2\sigma-1}(1+|\lambda|)^{1/2}+\varepsilon^{2\sigma-2}]\|J\|_{2\sigma,p,\Omega}$$
$$+c(1+|\lambda|)^{1/2}\left\|DJ-Dd\frac{\partial J}{\partial v}\right\|_{p,\Omega_\lambda}.$$

Now we remark that $Dd\equiv v$ and $DJ=(\partial J/\partial v)v$ on $\partial\Omega$; hence $DJ-Dd(\partial J/\partial v)=0$ on $\partial\Omega$, so that by (2.12) we get

$$\left\|DJ-Dd\frac{\partial J}{\partial v}\right\|_{p,\Omega_\lambda}\leqslant c(1+|\lambda|)^{1/2-\sigma}\|DJ\|_{2\sigma-1,p,\Omega}\leqslant c(1+|\lambda|)^{1/2-\sigma}\|J\|_{2\sigma,p,\Omega}$$

and consequently

$$I_1\leqslant c[\varepsilon^{2(\sigma-\theta)}(1+|\lambda|)^{1-\theta}+\varepsilon^{2\sigma-1}(1+|\lambda|)^{1/2}+\varepsilon^{2\sigma-2}+(1+|\lambda|)^{1-\sigma}]\times\{\|h\|_{2\sigma,p,\Omega}+\|g\|_{2\sigma,p,\Omega}\}. \tag{4.6}$$

By (4.3)–(4.6) we obtain the result. ∎

Lemma 4.6. *Fix $\sigma\in]1/2,1[-\{1/2+1/(2p)\}$ and let $h\in[B^{2\sigma,p}(\Omega)]^N$, $g\in[B^{2\sigma-1,p}(\Omega)]^{N-r_0}$. For each $\varepsilon>0$ we have:*

(i) *If $\sigma\in]1/2,1/2+1/(2p)[$, then*

$$\sum_{k=0}^{2}(1+|\lambda|)^{1-k/2}\|D^kN_1(\lambda)B_1(D)h_\varepsilon\|_Y\leqslant c[\varepsilon^{2\sigma-2}+(1+|\lambda|)^{1-\sigma}]\|h\|_{2\sigma,p,\Omega}. \tag{4.7}$$

(ii) *If $\sigma\in]1/2+1/(2p),1[$ and $B_1(D)h=g$ on $\partial\Omega$, then*

$$\sum_{k=0}^{2}(1+|\lambda|)^{1-k/2}\|D^kN_1(\lambda)[B_1(D)h_\varepsilon-g_\varepsilon]\|_Y \leqslant c[\varepsilon^{2\sigma-2}+\varepsilon^{2\sigma-1}(1+|\lambda|)^{1/2}+(1+|\lambda|)^{1-\sigma}]\{\|h\|_{2\sigma,p,\Omega}+\|g\|_{2\sigma-1,p,\Omega}\}. \tag{4.8}$$

Proof. Set

$$I_0:=\sum_{k=0}^{2}(1+|\lambda|)^{1-k/2}\|D^kN_1(\lambda)(B_1(D)h_\varepsilon)\|_Y,$$
$$J_0:=\sum_{k=0}^{2}(1+|\lambda|)^{1-k/2}\|D^kN_1(\lambda)(B_1(D)h_\varepsilon-g_\varepsilon)\|_Y.$$

(i) We apply (2.8) with $\psi:=(B_1(D)h_\varepsilon)\eta_\lambda$, $\eta_\lambda$ being the function (2.14).



Using (2.8) and Lemma 4.2(iii),(ii) we have

$$I_0\leqslant c\{\|D(B_1(D)h_\varepsilon)\|_{p,\Omega_\lambda}+(1+|\lambda|)^{1/2}\|B_1(D)h_\varepsilon\|_{p,\Omega_\lambda}\}$$
$$\leqslant c\{\|D^2h_\varepsilon\|_{p,\Omega}+\|Dh_\varepsilon\|_{p,\Omega}+(1+|\lambda|)^{1-\sigma}\|B_1(D)h_\varepsilon\|_{2\sigma-1,p,\Omega_\lambda}\}$$
$$\leqslant c\{\varepsilon^{2\sigma-2}+(1+|\lambda|)^{1-\sigma}\}\|h\|_{2\sigma,p,\Omega}.$$

(ii) Choose now in (2.8) $\psi:=(B_1(D)h_\varepsilon-g_\varepsilon)\eta_\lambda$. Then

$$J_0\leqslant c\{\|D(B_1(D)h_\varepsilon-g_\varepsilon)\|_{p,\Omega_\lambda}+(1+|\lambda|)^{1/2}[\|B_1(D)(h_\varepsilon-h)\|_{p,\Omega_\lambda}+\|B_1(D)h-g\|_{p,\Omega_\lambda}+\|g-g_\varepsilon\|_{p,\Omega_\lambda}]\}=:I_1+I_2+I_3+I_4. \tag{4.9}$$

Now, arguing as before, we get

$$I_1\leqslant c\varepsilon^{2\sigma-2}\{\|h\|_{2\sigma,p,\Omega}+\|g\|_{2\sigma-1,p,\Omega}\}, \tag{4.10}$$

and similarly, using Lemma 4.2(iv),

$$I_2+I_4\leqslant c(1+|\lambda|)^{1/2}\varepsilon^{2\sigma-1}\{\|h\|_{2\sigma,p,\Omega}+\|g\|_{2\sigma-1,p,\Omega}\}. \tag{4.11}$$

On the other hand, as $B_1(D)h-g=0$ on $\partial\Omega$, (2.12) yields

$$I_3\leqslant c(1+|\lambda|)^{1-\sigma}\|B_1(D)h-g\|_{2\sigma-1,p,\Omega_\lambda}\leqslant c(1+|\lambda|)^{1-\sigma}\{\|h\|_{2\sigma,p,\Omega}+\|g\|_{2\sigma-1,p,\Omega}\}, \tag{4.12}$$

and by (4.9)–(4.12) we get the result. ∎

**4.C. An approximation result.** Let $g_0$, $g_1$ be as in (3.3), (3.4), i.e. $g_0\in C^\alpha(X_0)\cap C^{\alpha+1}(Y_0)$, $g_1\in C^\alpha(X_1)\cap C^{\alpha+1/2}(Y_1)$. We prove a sort of "Taylor's formula" for such functions.

Lemma 4.7. *For all $\theta\in]0,\alpha[$ and $s,t\in[0,T]$ we have:*

(i) $$\|g_0(s)-g_0(t)-(s-t)(\partial g_0(t)/\partial t)_\varepsilon\|_{X_0}\leqslant c[|t-s|^\alpha+|t-s|\varepsilon^{2\alpha-2}]\{[g_0]_{C^\alpha(X_0)}+[g_0']_{C^\alpha(Y_0)}\}.$$

(ii) $$\|g_0(s)-g_0(t)-(s-t)(\partial g_0(t)/\partial t)_\varepsilon\|_{2\theta,p,\Omega}\leqslant c[|t-s|^{1+\alpha-\theta}+|t-s|\varepsilon^{2\alpha-2\theta}]\{\|g_0\|_{C^\alpha(X_0)}+\|g_0\|_{C^{1+\alpha}(Y_0)}\}.$$

Proof. (i) Due to Lemma 1.9(i) and Proposition 1.11(i)

$$\sup_{t\in[0,T]}\|\partial g_0(t)/\partial t\|_{2\alpha,p,\Omega}\leqslant c\{[g_0]_{C^\alpha(X_0)}+[g_0']_{C^\alpha(Y_0)}\}, \tag{4.13}$$

so that by Lemma 4.2(iii) we obtain

$$\|g_0(s)-g_0(t)-(s-t)(\partial g_0(t)/\partial t)_\varepsilon\|_{X_0}$$
$$\leqslant \|g_0(s)-g_0(t)\|_{X_0}+|s-t|\,\|(\partial g_0(t)/\partial t)_\varepsilon\|_{X_0}$$
$$\leqslant c\,[|t-s|^\alpha+|t-s|\,\varepsilon^{2\alpha-2}]\,\{[g_0]_{C^\alpha(X_0)}+[g_0']_{C^\alpha(Y_0)}\}.$$

(ii) Due to Lemma 1.9(ii) and Proposition 1.11(i)

$$\|\partial g_0/\partial t\|_{C^{\alpha-\theta}([B^{2\theta,p}(\Omega)]^{r_0})}\leqslant c\,\{[g_0]_{C^\alpha(X_0)}+[g_0']_{C^\alpha(Y_0)}\};\tag{4.14}$$

hence by Lemma 4.2(ii), (4.14) and (4.13)

$$\|g_0(s)-g_0(t)-(s-t)(\partial g_0(t)/\partial t)_\varepsilon\|_{2\theta,p,\Omega}$$
$$\leqslant \Big\|\int_t^s[\partial g_0(r)/\partial r-\partial g_0(t)/\partial t]\,dr\Big\|_{2\theta,p,\Omega}+|t-s|\,\|\partial g_0(t)/\partial t-(\partial g_0(t)/\partial t)_\varepsilon\|_{2\theta,p,\Omega}$$
$$\leqslant c\,[|t-s|^{1+\alpha-\theta}+|t-s|\,\varepsilon^{2\alpha-2\theta}]\,\{[g_0]_{C^\alpha(X_0)}+[g_0']_{C^\alpha(Y_0)}\}.$$

Lemma 4.8. *Let $\alpha\in]1/2,1[$. For all $s,t\in[0,T]$ we have:*

(i) $$\|g_1(s)-g_1(t)-(s-t)(\partial g_1(t)/\partial t)_\varepsilon\|_{X_1}\leqslant [|t-s|^\alpha+|t-s|\,\varepsilon^{2\alpha-2}]\,\{[g_1]_{C^\alpha(X_1)}+\|g_1'\|_{C^{\alpha-1/2}(Y_1)}\}.$$

(ii) $$\|g_1(s)-g_1(t)-(s-t)(\partial g_1(t)/\partial t)_\varepsilon\|_{Y_1}\leqslant c\,[|t-s|^{\alpha+1/2}+|t-s|\,\varepsilon^{2\alpha-1}]\,\{[g_1]_{C^\alpha(X_1)}+\|g_1'\|_{C^{\alpha-1/2}(Y_1)}\}.$$

Proof. Similar to the proof of Lemma 4.7. ∎

**5. Maximal regularity.** Theorem 3.1 allows us to solve problem (1.9) in $C^1(Y)\cap C^0(X)$ (see (3.1)), starting from data whose regularity is described in (3.2)–(3.5). This situation is not satisfactory from the point of view of the regularity of the solution, and is also useless for applications to quasilinear parabolic problems. Actually, (3.2)–(3.5) imply more smoothness of the solution: of course, in order to obtain as much smoothness as possible (i.e. $\partial u/\partial t$, $A(D)u$, $B_0(u)$ and $B_1(D)u$ as regular as the data) we shall need to impose, together with (3.6) and (3.7), further compatibility conditions.

The result we are going to prove is the following:

Theorem 5.1. *Assume Hypotheses 1.1–1.3. Let $f$, $g_0$, $g_1$ and $\varphi$ be such that (3.2)–(3.5) hold with $\alpha\in]0,1[-\{1/(2p),1/2,1/2+1/(2p)\}$, and suppose that conditions (3.6) and (3.7) are true. The strict solution $u$ of problem (1.9), which is given by formula (3.8), belongs to $C^{1+\alpha}(Y)\cap C^\alpha(X)$ if and only if the following conditions are fulfilled:*

$$h:=A(D)\varphi+f(0)\in[B^{2\alpha,p}(\Omega)]^N\quad \textit{if } \alpha\in]0,1/(2p)[,\tag{5.1}$$

$$h\in[B^{2\alpha,p}(\Omega)]^N \textit{ and } B_0h=[\partial g_0(t,\cdot)/\partial t]_{t=0} \textit{ on } \partial\Omega \quad \textit{if } \alpha\in]1/(2p),1/2+1/(2p)[-\{1/2\},\tag{5.2}$$

$$h\in[B^{2\alpha,p}(\Omega)]^N \textit{ and } B_0h=[\partial g_0(t,\cdot)/\partial t]_{t=0},\quad B_1(D)h=[\partial g_1(t,\cdot)/\partial t]_{t=0} \textit{ on } \partial\Omega \textit{ if } \alpha\in]1/2+1/(2p),1[.\tag{5.3}$$

*Moreover, if this is the case, then the following estimate holds:*

$$[u']_{C^\alpha(Y)}+[u]_{C^\alpha(X)}\leqslant c\,\{\|A(D)\varphi+f(0)\|_{2\alpha,p}+F_\alpha\},\tag{5.4}$$

*where*

$$\begin{cases} F_\alpha:=[f]_{C^\alpha(Y)}+[g_0]_{C^\alpha(X_0)}+\|g_0'\|_{C^\alpha(Y_0)}+[g_1]_{C^\alpha(X_1)}+G_\alpha,\\ G_\alpha:=\begin{cases}[g_1]_{C^{\alpha+1/2}(Y_1)} & \textit{if } \alpha\in]0,1/2[,\\ \|g_1'\|_{C^{\alpha-1/2}(Y_1)} & \textit{if } \alpha\in]1/2,1[.\end{cases}\end{cases}\tag{5.5}$$

Proof. It consists (as usual!) of four steps:

Step 1: *Conditions (5.1)–(5.3) are necessary.*

Step 2: *If $\alpha\in]0,1/(2p)[$, condition (5.1) is sufficient, and (5.4) holds.*

Step 3: *If $\alpha\in]1/(2p),1/2+1/(2p)[-\{1/2\}$, condition (5.2) is sufficient, and (5.4) holds.*

Step 4: *If $\alpha\in]1/2+1/(2p),1[$, condition (5.3) is sufficient, and (5.4) holds.*

Proof of Step 1. It is just a formality. First of all we remark that the property "$u\in C^\alpha(X)$" is a consequence of "$u\in C^{1+\alpha}(Y)$". Indeed, if $u\in C^{1+\alpha}(Y)$ then $u(t)-u(s)$, $t,s\in[0,T]$, solves the elliptic problem

$$\begin{cases} A(D)[u(t)-u(s)]=f(s)-f(t)+u'(t)-u'(s) & \text{in } \Omega,\\ B_0[u(t)-u(s)]=g_0(t)-g_0(s) & \text{on } \partial\Omega,\\ B_1(D)[u(t)-u(s)]=g_1(t)-g_1(s) & \text{on } \partial\Omega;\end{cases}$$

hence by Theorem 1.4 we have (see (5.5)):

$$\|u(t)-u(s)\|_X\leqslant c\,|t-s|^\alpha\,\{[f]_{C^\alpha(Y)}+[u']_{C^\alpha(Y)}+[g_0]_{C^\alpha(X_0)}+[g_1]_{C^\alpha(X_1)}\}$$
$$+c\,|t-s|^{(\alpha+1/2)\wedge 1}\,G_\alpha+c\,|t-s|\,\|g_0'\|_{C^0(Y_0)},$$

i.e. $u\in C^\alpha(X)$ and

$$[u]_{C^\alpha(X)}\leqslant c\,\{[u']_{C^\alpha(Y)}+[f]_{C^\alpha(Y)}+[g_0]_{C^\alpha(X_0)}+[g_1]_{C^\alpha(X_1)}\}+cT^{(1-\alpha)\wedge 1/2}\,\{G_\alpha+\|g_0'\|_{C^0(Y_0)}\}.\tag{5.6}$$

Now by Lemma 1.9(i) and Proposition 1.11(i) we get $u'(0)\equiv h\in[B^{2\alpha,p}(\Omega)]^N$, $g_0'(0)\in[B^{2\alpha,p}(\Omega)]^{r_0}$ and, provided $\alpha\in]1/2,1[$, $g_1'(0)\in[B^{2\alpha-1,p}(\Omega)]^{N-r_0}$. Thus $B_0h|_{\partial\Omega}$ and $g_0'(0)|_{\partial\Omega}$, as well as $B_1(D)h|_{\partial\Omega}$ and $g_1'(0)|_{\partial\Omega}$, are simultaneously meaningful, and in this case they must coincide, so that conditions (5.1)–(5.3) must hold.

Proof of Step 2. This is an "easy" step: the proof relies on a good decomposition of the difference $u'(t)-u'(r)$ for $0\leqslant r\leqslant t$. Using (3.10) we can write after easy manipulations:

$$(5.7)\qquad u'(t)-u'(r)$$
$$=\int_r^t\int_\gamma \lambda e^{(t-s)\lambda}\Big\{R(\lambda)[f(s)-f(t)]+\sum_{j=0}^{1}N_j(\lambda)[g_j(s)-g_j(t)]\Big\}\,d\lambda\,ds$$
$$+\int_0^r\int_{r-s}^{t-s}\int_\gamma \lambda^2 e^{\lambda q}\Big\{R(\lambda)[f(s)-f(r)]+\sum_{j=0}^{1}N_j(\lambda)[g_j(s)-g_j(r)]\Big\}\,d\lambda\,dq\,ds$$
$$+\int_\gamma e^{(t-r)\lambda}\Big\{R(\lambda)[f(t)-f(r)]+\sum_{j=0}^{1}N_j(\lambda)[g_j(t)-g_j(r)]\Big\}\,d\lambda$$
$$+\int_r^t\int_\gamma \lambda e^{\lambda q}\Big\{R(\lambda)[f(r)-f(0)]+\sum_{j=0}^{1}N_j(\lambda)[g_j(r)-g_j(0)]\Big\}\,d\lambda\,dq$$
$$+\int_\gamma [e^{t\lambda}-e^{r\lambda}]\Big\{\lambda R(\lambda)\varphi+R(\lambda)f(0)+\sum_{j=0}^{1}N_j(\lambda)g_j(0)\Big\}\,d\lambda=:\sum_{i=1}^{5}I_i.$$

We will now use systematically the estimates (2.2), (2.8), (2.13) and the regularity on the data described by (3.11); in addition we recall that, by Lemma 1.9(i),

$$|u-v|^{-1}\|g_0(u)-g_0(v)\|_{2\alpha,p}\leqslant\sup_{t\in[0,T]}\|\partial g_0(t)/\partial t\|_{2\alpha,p}$$
$$\leqslant c\{[g_0]_{C^\alpha(X_0)}+[g_0']_{C^\alpha(Y_0)}\}\qquad\forall u,v\in[0,T],\ u\neq v.$$

Concerning $I_1$ we have (see (5.5))

$$\|I_1\|_Y\leqslant c\int_r^t\{(t-s)^{-1}[\|f(t)-f(s)\|_Y+\|g_0(t)-g_0(s)\|_{X_0}]$$
$$+(t-s)^{\alpha-2}\Big\|\int_s^t(\partial g_0(u)/\partial u)\,du\Big\|_{2\alpha,p,\Omega}$$
$$+(t-s)^{-1}\|g_1(t)-g_1(s)\|_{X_1}+(t-s)^{-3/2}\|g_1(t)-g_1(s)\|_{Y_1}\}\,ds$$
$$\leqslant c(t-r)^\alpha F_\alpha,$$

whereas $I_2$ is estimated by

$$\|I_2\|_Y\leqslant c\int_0^r\int_{r-s}^{t-s}\{q^{-2}[\|f(s)-f(r)\|_Y+\|g_0(s)-g_0(r)\|_{X_0}]$$
$$+q^{\alpha-3}\|g_0(s)-g_0(r)\|_{2\alpha,p,\Omega}+q^{-2}\|g_1(s)-g_1(r)\|_{X_1}$$
$$+q^{-5/2}\|g_1(s)-g_1(r)\|_{Y_1}\}\,dq\,ds\leqslant c(t-r)^\alpha F_\alpha.$$



Next, for $I_3$ and $I_4$ we get

$$\|I_3\|_Y\leqslant c\{\|f(t)-f(r)\|_Y+\|g_0(t)-g_0(r)\|_{X_0}$$
$$+(t-r)^{\alpha-1}\|g_0(t)-g_0(r)\|_{2\alpha,p,\Omega}+\|g_1(t)-g_1(r)\|_{X_1}$$
$$+(t-r)^{-1/2}\|g_1(t)-g_1(r)\|_{Y_1}\}\leqslant c(t-r)^\alpha F_\alpha;$$

$$\|I_4\|_Y\leqslant c\int_r^t\{q^{-1}[\|f(r)-f(0)\|_Y+\|g_0(r)-g_0(0)\|_Y]$$
$$+q^{\alpha-2}\|g_0(r)-g_0(0)\|_{2\alpha,p,\Omega}+q^{-1}\|g_1(r)-g_1(0)\|_{X_1}$$
$$+q^{-3/2}\|g_1(r)-g_1(0)\|_{Y_1}\}\,dq\leqslant c(t-r)^\alpha F_\alpha.$$

Finally, remembering Lemma 2.2(i), the compatibility conditions (3.6), (3.7), and the definition (2.3) of $E(\cdot)$,

$$(5.8)\qquad I_5=\int_\gamma[e^{t\lambda}-e^{r\lambda}]R(\lambda)[A(D)\varphi+f(0)]\,ds=[E(t)-E(r)]h;$$

thus by Proposition 2.1(ii)

$$I_5=O((t-r)^\alpha)\ \text{as}\ t-r\downarrow 0\ \Leftrightarrow\ h\in[B^{2\alpha,p}(\Omega)]^N,$$

and in this case we have also

$$\|I_5\|_Y\leqslant c(t-r)^\alpha\|h\|_{2\alpha,p}.$$

Collecting the above estimates for $I_1,\ldots,I_5$ we get the result, and recalling (5.6) estimate (5.4) also holds.

Remark 5.2. Before starting with Step 3, which is much harder, let us observe that many of the integrals appearing in (5.7) are $O((t-r)^\alpha)$ as $t-r\downarrow 0$ even when $\alpha\notin\,]0,1/(2p)[$: for instance, all integrals in $I_1$, $I_2$, $I_3$, $I_4$ involving $N_1(\lambda)$ are of this kind when $\alpha\in\,]0,1/2+1/(2p)[-\{1/2\}$; similarly, all integrals in $I_1$, $I_2$, $I_3$, $I_4$ involving $R(\lambda)$ are $O((t-r))^\alpha$ as $t-r\downarrow 0$ when $\alpha\in\,]0,1[$. This will relieve our toil.

Proof of Step 3. If we just repeat the estimates of Step 2, we can only obtain $u'\in C^\theta(Y)$ for each $\theta\in\,]0,1/(2p)[$, due to the fact that in estimating the terms which contain $N_0(\lambda)$ we cannot fully use the fact that $g_0\in C^\alpha(X_0)\cap C^{1+\alpha}(Y_0)$. In order to overcome this obstacle, we introduce suitable approximations of $\partial g_0(t)/\partial t$ and of $h:=A(D)\varphi+f(0)$. Namely, recalling (4.1) and Remark 4.1, we set

$$(5.9)\qquad \begin{aligned}\chi_\tau^0(t,\cdot)&\equiv\chi_\tau^0(t):=[(\partial g_0(t)/\partial t)_\varepsilon]_{\varepsilon=\tau^{1/2}},\\ \psi_\tau(\cdot)&\equiv\psi_\tau:=[h_\varepsilon]_{\varepsilon=\tau^{1/2}},\qquad 0<\tau\leqslant T.\end{aligned}$$

Taking into account Remark 5.2 we rewrite (5.7) by separating the terms which are $O((t-r)^\alpha)$ as $t-r\downarrow 0$ from the others; in addition, we add and subtract other integral terms involving the function (5.9). We can evaluate exactly such new integrals (with the help of (2.13) and Corollary 4.4(i)) and after boring but elementary calculations we get as $t-r\downarrow 0$

$$(5.10)\qquad u'(t)-u'(r) = O\big((t-r)^\alpha\big)$$
$$+\int_r^t \oint_\gamma \lambda e^{(t-s)\lambda} N_0(\lambda)[g_0(s)-g_0(t)-(s-t)\chi^0_{t-r}(t)]\,d\lambda\,ds$$
$$+\oint_\gamma e^{(t-r)\lambda}(r-t)N_0(\lambda)\chi^0_{t-r}(t)\,d\lambda+\oint_\gamma \lambda^{-1}e^{(t-r)\lambda}N_0(\lambda)\chi^0_{t-r}(t)\,d\lambda$$
$$+\int_0^r\oint_\gamma\int_{r-s}^{t-s}\lambda^2 e^{\lambda q}N_0(\lambda)[g_0(s)-g_0(r)-(s-r)\chi^0_{t-s}(r)]\,dq\,d\lambda\,ds$$
$$+\int_0^r\oint_\gamma \lambda[e^{(t-s)\lambda}-e^{(r-s)\lambda}]N_0(\lambda)(s-r)\chi^0_{t-s}(r)\,d\lambda\,ds$$
$$+\oint_\gamma e^{(t-r)\lambda}N_0(\lambda)[g_0(t)-g_0(r)-(t-r)\chi^0_{t-r}(r)]\,d\lambda$$
$$+\oint_\gamma e^{(t-r)\lambda}(t-r)N_0(\lambda)\chi^0_{t-r}(r)\,d\lambda$$
$$+\int_r^t\oint_\gamma \lambda e^{\lambda q}N_0(\lambda)[g_0(r)-g_0(0)-r\chi^0_r(r)]\,d\lambda\,dq$$
$$+\oint_\gamma [e^{\lambda t}-e^{\lambda r}]rN_0(\lambda)\chi^0_r(r)\,d\lambda$$
$$+\oint_\gamma [e^{t\lambda}-e^{r\lambda}]R(\lambda)h\,d\lambda =: O\big((t-r)^\alpha\big)+\sum_{j=1}^{10}J_j$$

where $J_{10}$ is just the term $I_5$ of (5.7), written in the form (5.8).

Now we use Lemma 4.7, (2.13) and Corollary 4.4(i),(iii), getting for a fixed $\theta\in]0,1/(2p)[$

$$(5.11)\qquad \|J_1\|_Y \leqslant c\int_r^t\{(t-s)^{-1}\|g_0(s)-g_0(t)-(s-t)\chi^0_{t-r}(t)\|_{X_0}$$
$$+(t-s)^{\theta-2}\|g_0(s)-g_0(t)-(s-t)\chi^0_{t-r}(t)\|_{2\theta,p}\}\,ds \leqslant c(t-r)^\alpha F_\alpha;$$

$$(5.12)\qquad \|J_2+J_7\|_Y = \Big\|\oint_\gamma e^{(t-r)\lambda}(t-r)N_0(\lambda)[\chi^0_{t-r}(r)-\chi^0_{t-r}(t)]\,d\lambda\Big\|_Y$$
$$\leqslant c\{(t-r)[\|\chi^0_{t-r}(r)\|_{X_0}+\|\chi^0_{t-r}(t)\|_{X_0}]+(t-r)^\theta\|\chi^0_{t-r}(r)-\chi^0_{t-r}(t)\|_{2\theta,p}\}$$
$$\leqslant c(t-r)^\alpha F_\alpha;$$



$$(5.13)\qquad \|J_4\|_Y \leqslant c\int_0^r\int_{r-s}^{t-s}\{q^{-2}\|g_0(s)-g_0(r)-(s-r)\chi^0_{t-s}(r)\|_{X_0}$$
$$+q^{\theta-3}\|g_0(s)-g_0(r)-(s-r)\chi^0_{t-s}(r)\|_{2\theta,p}\}\,dq\,ds \leqslant c(t-r)^\alpha F_\alpha;$$

$$(5.14)\qquad \|J_6\|_Y \leqslant c\{\|g_0(t)-g_0(r)-(t-r)\chi^0_{t-r}(r)\|_{X_0}$$
$$+(t-r)^{\theta-1}\|g_0(t)-g_0(r)-(t-r)\chi^0_{t-r}(r)\|_{2\theta,p}\} \leqslant c(t-r)^\alpha F_\alpha;$$

$$(5.15)\qquad \|J_8\|_Y \leqslant c\int_r^t\{q^{-1}\|g_0(r)-g_0(0)-r\chi^0_r(r)\|_{X_0}$$
$$+q^{\theta-2}\|g_0(r)-g_0(0)-r\chi^0_r(r)\|_{2\theta,p}\}\,dq \leqslant c(t-r)^\alpha F_\alpha.$$

Concerning $J_3$, $J_5$, $J_9$, $J_{10}$, here are the worst troubles. We start with $J_5$. Integration by part yields

$$(5.16)\qquad J_5 = -\oint_\gamma [e^{t\lambda}-e^{r\lambda}]rN_0(\lambda)\chi^0_t(r)\,d\lambda$$
$$-\oint_\gamma \lambda^{-1}e^{(t-r)\lambda}N_0(\lambda)\chi^0_{t-r}(r)\,d\lambda$$
$$+\oint_\gamma \lambda^{-1}[e^{t\lambda}-e^{r\lambda}]N_0(\lambda)\chi^0_t(r)\,d\lambda$$
$$+\int_0^r\oint_\gamma \lambda^{-1}[e^{(t-s)\lambda}-e^{(r-s)\lambda}]N_0(\lambda)\frac{\partial}{\partial s}\chi^0_{t-s}(r)\,d\lambda\,ds$$
$$+\int_0^r\oint_\gamma [e^{(t-s)\lambda}-e^{(r-s)\lambda}](s-r)\frac{\partial}{\partial s}\chi^0_{t-s}(r)\,d\lambda\,ds$$
$$=: \sum_{j=1}^{5}J_{5,j}.$$

We couple $J_{5,1}$ with $J_9$ and $J_{5,2}$ with $J_3$. By Corollary 4.4(i),(iii)

$$(5.17)\qquad \|J_{5,1}+J_9\|_Y = \Big\|\oint_\gamma [e^{t\lambda}-e^{r\lambda}]rN_0(\lambda)[\chi^0_r(r)-\chi^0_t(r)]\,d\lambda\Big\|_Y$$
$$\leqslant c\int_r^t\{rq^{-1}[\|\chi^0_r(r)\|_{X_0}+\|\chi^0_t(r)\|_{X_0}]+rq^{\theta-2}\|\chi^0_r(r)-\chi^0_t(r)\|_{2\theta,p}\}\,dq$$
$$\leqslant c(t-r)^\alpha F_\alpha,$$

whereas

$$\|J_{5,2}+J_3\|_Y = \Big\|\oint_\gamma \lambda^{-1}e^{(t-r)\lambda}N_0(\lambda)[\chi^0_{t-r}(t)-\chi^0_{t-r}(r)]\,d\lambda\Big\|_Y$$
$$\leqslant c\{(t-r)[\|\chi^0_{t-r}(t)\|_{X_0}+\|\chi^0_{t-r}(r)\|_{X_0}]+(t-r)^\theta\|\chi^0_{t-r}(t)-\chi^0_{t-r}(r)\|_{2\theta,p}\};$$

now, since $g_0'\in C^{\alpha-\theta}([B^{2\theta,p}(\Omega)]^{r_0})\cap B([B^{2\alpha,p}(\Omega)]^{r_0})$ (Lemma 1.9), we can write by Lemma 4.2(ii)

$$\|\chi^0_{t-r}(t)-\chi^0_{t-r}(r)\|_{2\theta,p}$$
$$\leqslant \|\chi^0_{t-r}(t)-g_0'(t)\|_{2\theta,p}+\|g_0'(t)-g_0'(r)\|_{2\theta,p}+\|g_0'(r)-\chi^0_{t-r}(r)\|_{2\theta,p}$$
$$\leqslant c(t-r)^{\alpha-\theta}\{\|g_0'(t)\|_{2\alpha,p}+\|g_0'(r)\|_{2\alpha,p}+[g_0']_{C^{\alpha-\theta}([B^{2\theta,p}(\Omega)]^{r_0})}\},$$

so that by Corollary 4.4(i) we obtain

$$\|J_{5,2}+J_3\|_Y \leqslant c(t-r)^\alpha F_\alpha. \tag{5.18}$$

Next, we estimate $J_{5,4}$ and $J_{5,5}$: by Corollary 4.4(iv),(v) we have

$$\|J_{5,4}\|_Y \leqslant c\int_0^r\int_{r-s}^{t-s}\left\{\left\|\frac{\partial}{\partial s}\chi^0_{t-s}(r)\right\|_{X_0}+q^{\theta-1}\left\|\frac{\partial}{\partial s}\chi^0_{t-s}(r)\right\|_{2\theta,p}\right\}dq \tag{5.19}$$
$$\leqslant c(t-r)^\alpha F_\alpha,$$

$$\|J_{5,5}\|_Y \leqslant c\int_0^r\int_{r-s}^{t-s}\left\{(r-s)q^{-1}\left\|\frac{\partial}{\partial s}\chi^0_{t-s}(r)\right\|_{X_0}\right. \tag{5.20}$$
$$\left.+(r-s)q^{\theta-2}\left\|\frac{\partial}{\partial s}\chi^0_{t-s}(r)\right\|_{2\theta,p}\right\}dq \leqslant c(t-r)^\alpha F_\alpha.$$

Finally, we couple $J_{5,3}$ with $J_{10}$:

$$J_{5,3}+J_{10} = \oint_\gamma [e^{t\lambda}-e^{r\lambda}]\{\lambda^{-1}N_0(\lambda)\chi^0_t(r)+R(\lambda)h\}\,d\lambda \tag{5.21}$$
$$=\oint_\gamma\lambda^{-1}[e^{t\lambda}-e^{r\lambda}]N_0(\lambda)[\chi^0_t(r)-\chi^0_t(0)]\,d\lambda+\oint_\gamma[e^{t\lambda}-e^{r\lambda}]R(\lambda)[h-\psi_t]\,d\lambda$$
$$+\oint_\gamma\lambda^{-1}[e^{t\lambda}-e^{r\lambda}]\{N_0(\lambda)\chi^0_t(0)+\lambda R(\lambda)\psi_t\}\,d\lambda =: J_1'+J_2'+J_3'.$$

Now, arguing as before, it is easily seen that

$$\|J_1'\|_Y+\|J_2'\|_Y \leqslant c(t-r)^\alpha F_\alpha; \tag{5.22}$$

thus we have only to treat $J_3'$. By Lemma 2.2(ii)

$$N_0(\lambda)\chi^0_t(0)+\lambda R(\lambda)\psi_t = \psi_t+R(\lambda)A(D)\psi_t+N_0(\lambda)[\chi^0_t(0)-B_0\psi_t]-N_1(\lambda)B_1\psi_t$$

so that we can perform the last splitting:

$$J_3' = \int_r^t\oint_\gamma e^{\lambda q}R(\lambda)A(D)\psi_t\,d\lambda\,dq \tag{5.23}$$
$$+\int_r^t\oint_\gamma e^{\lambda q}N_0(\lambda)[\chi^0_t(0)-B_0\psi_t]\,d\lambda\,dq$$
$$-\int_r^t\oint_\gamma e^{\lambda q}N_1(\lambda)B_1(D)\psi_t\,d\lambda\,dq =: J_{3,1}'+J_{3,2}'+J_{3,3}'.$$



By (2.2) and Corollary 4.4(i)

$$\|J_{3,1}'\|_Y \leqslant c\int_r^t t^{\alpha-1}\,dq\,\|h\|_{2\alpha,p} \leqslant c(t-r)^\alpha\|h\|_{2\alpha,p}; \tag{5.24}$$

on the other hand, if $\alpha\in]1/(2p),1/2[$ by (2.8) and Corollary 4.4(i) we have

$$\|J_{3,3}'\|_Y \leqslant c\int_r^t[t^{\alpha-1}+t^{\alpha-1/2}q^{-1/2}]\,dq\,\|h\|_{2\alpha,p} \leqslant c(t-r)^\alpha\|h\|_{2\alpha,p},$$

whereas if $\alpha\in]1/2,1/2+1/(2p)[$ we obtain by (4.7)

$$\|J_{3,3}'\|_Y \leqslant c\int_r^t[t^{\alpha-1}+q^{\alpha-1}]\,dq\,\|h\|_{2\alpha,p};$$

hence in any case we have

$$\|J_{3,3}'\|_Y \leqslant c(t-r)^\alpha\|h\|_{2\alpha,p}. \tag{5.25}$$

Finally (and this is a real end!), we estimate $J_{3,2}'$: as, by (5.2), $B_0h = g_0'(0)$ on $\partial\Omega$, by (4.2) we obtain

$$\|J_{3,2}'\|_Y \leqslant c\int_r^t[q^{1/2}t^{\alpha-3/2}+t^{\alpha-1}+q^{\theta-1}t^{\alpha-\theta}+q^{\alpha-1}]\,dq \tag{5.26}$$
$$\times\{\|h\|_{2\alpha,p}+\|g_0'(0)\|_{2\alpha,p} \leqslant c(t-r)^\alpha\{F_\alpha+\|h\|_{2\alpha,p}\}.$$

Collecting (5.10)–(5.26) and (5.6), we conclude the proof of Step 3.

Remark 5.3. We regret that Step 4 will be as troublesome as Step 3. However, as done in Remark 5.2, some simplifications can be made. In the basic splitting (5.7) we need only consider the terms involving $N_1(\lambda)$. Indeed, those containing $R(\lambda)$ are $O((t-r)^\alpha)$ when $\alpha\in]0,1[$; on the other hand, we can treat those containing $N_0(\lambda)$ as in Step 3, and all terms generated in (5.10), (5.16), (5.21) and (5.23) are $O((t-r)^\alpha)$ even when $\alpha\in]1/2+1/(2p),1[$, with the only exception of $J_{3,3}'$ in (5.23).

Proof of Step 4. Again, if we repeat the estimates of Step 3, we cannot take full advantage of the regularity of $\partial g_1(t)/\partial t$, so that the estimates for $N_1(\lambda)g_1(t)$ are not optimal. We skip this obstacle by adding and subtracting wisely some terms containing the functions $\psi_\tau = [h_\varepsilon]_{\varepsilon=\tau^{1/2}}$ (see (5.9)) and

$$\chi^1_\tau(t,\cdot)\equiv\chi^1_\tau(t) := [(\partial g_1(t)/\partial t)_\varepsilon]_{\varepsilon=\tau^{1/2}},\qquad 0<\tau\leqslant T. \tag{5.27}$$

We recall that, since $g_1\in C^\alpha(X_1)\cap C^{\alpha+1/2}(Y_1)$, Lemma 1.9(i) yields

$$|u-v|^{-1}\|g_1(u)-g_1(v)\|_{2\alpha-1,p} \leqslant \sup_{t\in[0,T]}\|\partial g_1(t)/\partial t\|_{2\alpha-1,p}$$
$$\leqslant c\{[g_1]_{C^\alpha(X_1)}+\|g_1'\|_{C^{\alpha-1/2}(Y_1)}\}\qquad \forall u,v\in[0,T],\ u\neq v.$$

If $t > r \geqslant 0$ we split $u'(t)-u'(r)$ by assembling all terms which are $O\big((t-r)^\alpha\big)$ according to Remark 5.3. We add and subtract suitable integral terms which are exactly evaluable (with the aid of (2.8) and Corollary 4.4(i)). The result is, as $t-r\downarrow 0$:

$$(5.28)\quad u'(t)-u'(r) = O\big((t-r)^\alpha\big)$$
$$+\int_r^t \fint_\gamma \lambda e^{(t-s)\lambda} N_1(\lambda)\,[g_1(s)-g_1(t)-(s-t)\,\chi^1_{t-r}(t)]\,d\lambda\,ds$$
$$+\fint_\gamma e^{(t-r)\lambda}(r-t)\,N_1(\lambda)\,\chi^1_{t-r}(t)\,d\lambda+\fint_\gamma \lambda^{-1}e^{(t-r)\lambda}N_1(\lambda)\,\chi^1_{t-r}(t)\,d\lambda$$
$$+\int_0^r \int_{r-s}^{t-s} \fint_\gamma \lambda^2 e^{\lambda q} N_1(\lambda)\,[g_1(s)-g_1(r)-(s-r)\,\chi^1_{t-s}(r)]\,d\lambda\,dq\,ds$$
$$+\int_0^r \fint_\gamma \lambda\,[e^{(t-s)\lambda}-e^{(r-s)\lambda}](s-r)\,N_1(\lambda)\,\chi^1_{t-s}(r)\,d\lambda\,ds$$
$$+\fint_\gamma e^{(t-r)\lambda}N_1(\lambda)\,[g_1(t)-g_1(r)-(t-r)\,\chi^1_{t-r}(r)]\,d\lambda$$
$$+\fint_\gamma e^{(t-r)\lambda}(t-r)\,N_1(\lambda)\,\chi^1_{t-r}(r)\,d\lambda$$
$$+\int_r^t \fint_\gamma \lambda e^{\lambda q} N_1(\lambda)\,[g_1(r)-g_1(0)-r\chi^1_r(r)]\,d\lambda\,dq$$
$$+\fint_\gamma [e^{\lambda t}-e^{\lambda r}]\,rN_1(\lambda)\,\chi^1_r(r)\,d\lambda$$
$$-\fint_\gamma \lambda^{-1}[e^{t\lambda}-e^{r\lambda}]\,N_1(\lambda)\,B_1(D)\,\psi_t\,d\lambda =: O\big((t-r)^\alpha\big)+\sum_{j=1}^{10} I_j,$$

where $I_{10}$ is just $J'_{3,3}$ of (5.23) and the other terms come from $I_1, I_2, I_3, I_4$ of (5.7). By using Lemma 4.8(i),(ii), (2.8) and Corollary 4.4 (i),(ii) we easily arrive, as in Step 3, at

$$(5.29)\quad \|I_1\|_Y+\|I_2+I_7\|_Y+\|I_4\|_Y+\|I_6\|_Y+\|I_8\|_Y \leqslant c\,(t-r)^\alpha F_\alpha.$$

Concerning $I_5$, we integrate by parts:

$$(5.30)\quad I_5 = -\fint_\gamma [e^{t\lambda}-e^{r\lambda}]\,rN_1(\lambda)\,\chi^1_t(r)\,d\lambda$$
$$-\fint_\gamma \lambda^{-1}e^{(t-r)\lambda}N_1(\lambda)\,\chi^1_{t-r}(r)\,d\lambda$$
$$+\fint_\gamma \lambda^{-1}[e^{t\lambda}-e^{r\lambda}]\,N_1(\lambda)\,\chi^1_t(r)\,d\lambda$$
$$+\int_0^r \fint_\gamma \lambda^{-1}[e^{(t-s)\lambda}-e^{(r-s)\lambda}]\,N_1(\lambda)\,\frac{\partial}{\partial s}\chi^1_{t-s}(r)\,d\lambda\,ds$$
$$+\int_0^r \fint_\gamma [e^{(t-s)\lambda}-e^{(r-s)\lambda}](s-r)\,N_1(\lambda)\,\frac{\partial}{\partial s}\chi^1_{t-s}(r)\,d\lambda\,ds$$
$$:= \sum_{j=1}^{5} I_{5,j}.$$



We couple $I_{5,1}$ with $I_9$ and $I_{5,2}$ with $I_3$. By (2.8) and Corollary 4.4(ii)

$$(5.31)\quad \|I_{5,1}+I_9\|_Y = \Big\|\fint_\gamma [e^{t\lambda}-e^{r\lambda}]\,rN_1(\lambda)\,[\chi^1_t(r)-\chi^1_r(r)]\,ds\Big\|_Y$$
$$\leqslant c\int_r^t \{rq^{-1}\|\chi^1_t(r)-\chi^1_r(r)\|_{X_1}+rq^{-3/2}\|\chi^1_t(r)-\chi^1_r(r)\|_{Y_1}\}\,dq$$
$$\leqslant c\,(t-r)^\alpha F_\alpha,$$

whereas

$$\|I_{5,2}+I_3\|_Y = \Big\|\fint_\gamma \lambda^{-1}e^{(t-r)\lambda}N_1(\lambda)\,[\chi^1_{t-r}(t)-\chi^1_{t-r}(r)]\,d\lambda\Big\|_Y$$
$$\leqslant c\,\{(t-r)\,[\|\chi^1_{t-r}(t)\|_{X_1}+\|\chi^1_{t-r}(r)\|_{X_1}]$$
$$+(t-r)^{1/2}\|\chi^1_{t-r}(t)-\chi^1_{t-r}(r)\|_{Y_1}\};$$

now, since $g'_1\in C^{\alpha-1/2}(Y_1)\cap B\big([B^{2\alpha-1,p}(\Omega)]^{N-r_0}\big)$ (Lemma 1.9), we can write by Lemma 4.2(i)

$$(5.32)\quad \|\chi^1_{t-r}(t)-\chi^1_{t-r}(r)\|_{Y_1} \leqslant \|\chi^1_{t-r}(t)-g'_1(t)\|_{Y_1}$$
$$+\|g'_1(t)-g'_1(r)\|_{Y_1}+\|g'_1(r)-\chi^1_{t-r}(r)\|_{Y_1}$$
$$\leqslant c\,(t-r)^{\alpha-1/2}\{\|g'_1(t)\|_{2\alpha-1,p}+\|g'_1(r)\|_{2\alpha-1,p}+[g'_1]_{C^{\alpha-1/2}(Y_1)}\},$$

so that Corollary 4.4(i) implies

$$(5.33)\quad \|I_{5,2}+I_3\|_Y \leqslant c\,(t-r)^\alpha F_\alpha.$$

Next, we estimate $I_{5,4}$ and $I_{5,5}$ by (2.8) and Corollary 4.4(iv):

$$(5.34)\quad \|I_{5,4}\|_Y \leqslant c\int_0^r\int_{r-s}^{t-s}\Big\{\Big\|\frac{\partial}{\partial s}\chi^1_{t-s}(r)\Big\|_{X_1}+q^{-1/2}\Big\|\frac{\partial}{\partial s}\chi^1_{t-s}(r)\Big\|_{Y_1}\Big\}\,dq\,ds$$
$$\leqslant c\int_0^r\int_{r-s}^{t-s}[(t-s)^{\alpha-2}+q^{-1/2}(t-s)^{\alpha-3/2}]\,dq\,ds\,\|g'_1(r)\|_{2\alpha-1,p}$$
$$\leqslant c\,(t-r)^\alpha F_\alpha,$$

$$(5.35)\quad \|I_{5,5}\|_Y \leqslant c\int_0^r\int_{r-s}^{t-s}\Big\{(r-s)\,q^{-1}\Big\|\frac{\partial}{\partial s}\chi^1_{t-s}(r)\Big\|_{X_1}$$
$$+(r-s)\,q^{-3/2}\Big\|\frac{\partial}{\partial s}\chi^1_{t-s}(r)\Big\|_{Y_1}\Big\}\,dq\,ds \leqslant c\,(t-r)^\alpha F_\alpha.$$

The end is closer and closer! It remains to estimate $I_{5,3}+I_{10}$, which we rewrite as

$$I_{5,3}+I_{10} = \int_\gamma \lambda^{-1}[e^{t\lambda}-e^{r\lambda}]N_1(\lambda)[\chi_t^1(r)-\chi_t^1(0)]\,d\lambda$$
$$+\int_\gamma \lambda^{-1}[e^{t\lambda}-e^{r\lambda}]N_1(\lambda)[\chi_t^1(0)-B_1(D)\psi_t]\,d\lambda =: I_1'+I_2';$$

but, using (5.32), we find that

$$(5.36)\qquad \|I_1'\|_Y \leqslant c\int_r^t \{[\|\chi_t^1(r)\|_{X_1}+\|\chi_t^1(0)\|_{X_1}]+q^{-1/2}\|\chi_t^1(r)-\chi_t^1(0)\|_{Y_1}\}\,dq$$
$$\leqslant c\int_r^t [t^{\alpha-1}+q^{-1/2}r^{\alpha-1/2}]\,dq\cdot F_\alpha \leqslant c(t-r)^\alpha F_\alpha,$$

and concerning $I_2'$ we have by (5.3) and (4.8)

$$(5.37)\qquad \|I_2'\|_Y \leqslant c\int_r^t [t^{\alpha-1}+t^{\alpha-1/2}q^{-1/2}+q^{1-\alpha}]\,dq\,\{\|h\|_{2\alpha,p}+\|g_1'(0)\|_{2\alpha-1,p}\}$$
$$\leqslant c(t-r)^\alpha\{F_\alpha+\|h\|_{2\alpha,p}\}.$$

By (5.28)–(5.31), (5.33)–(5.37), taking into account (5.6), we conclude the proof of Step 4. Theorem 5.1 is completely proved. ∎

**Appendix**

**A. Proof of Lemma 4.3.** We recall that $f_\varepsilon$ is defined in (4.1), and it is plain that

$$\varepsilon \to f_\varepsilon \in C^\infty(]0,\infty[,\ \bigcap_{m\in N}[W^{m,p}(\Omega)]^k).$$

Let us start by remarking that if $f\in[B^{2\sigma,p}(\Omega)]^k$ and $F$ is any $B^{2\sigma,p}$-extension of $f$ to $\boldsymbol{R}^n$, then we have

$$(A.1)\qquad D^h\frac{\partial}{\partial\varepsilon}F_\varepsilon = -\varepsilon\frac{\partial^2}{\partial\varepsilon^2}D^hF_{0,\varepsilon}.$$

Let us prove (i). Suppose first $\sigma\in]0,1/2[$; then it is easily seen that

$$\left|\frac{\partial^2}{\partial\varepsilon^2}D^hF_{0,\varepsilon}(x)\right| = \varepsilon^{-h-2}\Big|\int_{\boldsymbol{R}^n}F(x-\varepsilon z)\big[(n+h+1)(n+h)D^h\varphi(z)$$
$$+2(n+h+1)\sum_{i=1}^n D_iD^h\varphi(z)z_i+\sum_{i,j=1}^n D_iD_jD^h\varphi(z)z_iz_j\big]\,dz\Big|$$
$$\leqslant c\varepsilon^{-h-2}\Big[\int_{B(0,1)}|F(x-\varepsilon z)-F(x)|^p\,dz\Big]^{1/p},$$

since the integral of the expression in square brackets vanishes; similarly, if $\sigma\in]1/2,1[$, we see that

$$\left|\frac{\partial^2}{\partial\varepsilon^2}D^hF_{0,\varepsilon}(x)\right| = \varepsilon^{-h-1}\Big|\int_{\boldsymbol{R}^n}DF(x-\varepsilon z)\big[(n+h)(n+h-1)D^{h-1}\varphi(z)$$
$$+2(n+h)\sum_{i=1}^n D_iD^{h-1}\varphi(z)z_i+\sum_{i,j=1}^n D_iD_jD^{h-1}\varphi(z)z_iz_j\big]\,dz\Big|$$
$$\leqslant c\varepsilon^{-h-1}\Big[\int_{B(0,1)}|DF(x-\varepsilon z)-DF(x)|^p\,dz\Big]^{1/p};$$

hence in both cases we get

$$(A.2)\qquad \left\|\frac{\partial^2}{\partial\varepsilon^2}D^hF_{0,\varepsilon}\right\|_{p,\boldsymbol{R}^n} \leqslant c\varepsilon^{2\sigma-2-h}\|F\|_{2\sigma,p,\boldsymbol{R}^n}.$$

By (A.1), (A.2) and the arbitrariness of the extension $F$, we obtain the result.

Part (ii) follows (i) via an interpolation argument (recall Proposition 1.11(i)). ∎

Remark A.1. We can prove quite similarly the following generalization of (A.2):

$$(A.3)\qquad \left\|\frac{\partial^m}{\partial\varepsilon^m}D^hF_{0,\varepsilon}\right\|_{p,\boldsymbol{R}^n} \leqslant c(m,h)\,\varepsilon^{2\sigma-m-h}\|F\|_{2\sigma,p,\boldsymbol{R}^n}$$
$$\forall m,h\in N \text{ with } m+h>2\sigma,\ \forall\varepsilon>0.$$

**B. Proof of Lemma 4.2.** (i) For each $\theta\in]0,1/2[\cap]0,\sigma]$ we have

$$(A.4)\qquad \|f_\varepsilon-f\|_{p,\Omega} \leqslant \Big[\int_{\boldsymbol{R}^n}\Big|\int_{B(0,1)}[F(x-\varepsilon z)-F(x)]\varphi(z)\,dz\Big|^p dx\Big]^{1/p}$$
$$+\Big[\int_{\boldsymbol{R}^n}\Big|\int_{B(0,1)}[F(x-\varepsilon z)-F(x)][n\varphi(z)+D\varphi(z)\cdot z]\,dz\Big|^p dx\Big]^{1/p} \leqslant c\varepsilon^{2\theta}\|F\|_{2\theta,p,\boldsymbol{R}^n},$$

so that the result follows provided $\sigma\in]0,1/2[$. On the other hand, if $\sigma\in]1/2,1[$ we write

$$F_\varepsilon-F = \int_0^\varepsilon\frac{\partial}{\partial r}F_r\,dr = -\int_0^\varepsilon r\frac{\partial^2}{\partial r^2}F_{0,r}\,dr,$$

and by (A.2) we get

$$\|f_\varepsilon-f\|_{p,\Omega} \leqslant \|F_\varepsilon-F\|_{p,\boldsymbol{R}^n} \leqslant c\varepsilon^{2\sigma}\|F\|_{2\sigma,p,\boldsymbol{R}^n};$$

this implies (i).

(iii) If $h>2\sigma$ we have

$$D^hF_\varepsilon = D^hF_{0,\varepsilon}-\varepsilon\frac{\partial}{\partial\varepsilon}D^hF_{0,\varepsilon},$$

and by (A.3) the result follows.

(iv) As $\sigma \in ]1/2,1[$ we have $DF \in [B^{2\sigma-1,p}(\Omega)]^{nk}$ and

$$DF_\varepsilon = (DF)_{0,\varepsilon} - \varepsilon \frac{\partial}{\partial \varepsilon}(DF)_{0,\varepsilon};$$

thus the result follows by (A.3) with $f$ replaced by $DF$ and $\theta = \sigma - 1/2$.

(v) follows obviously by a direct calculation.

Finally, we prove (ii). Suppose first $\theta = \sigma$, in which case we have to show that

$$\|f_\varepsilon\|_{2\sigma,p,\Omega} \leqslant c\|f\|_{2\sigma,p,\Omega}. \tag{A.5}$$

If $\sigma \in ]0,1/2[$ we set $g(t) := F_{\varepsilon+t}$, $t > 0$, so that, by (A.1), $g'(t) = -(\varepsilon+t)[\partial^2 F_{0,r}/\partial r^2]_{r=\varepsilon+t}$. Thus (A.3) yields

$$\|g'(t)\|_{p,\mathbf{R}^n} + \|D^2 g(t)\|_{p,\mathbf{R}^n} \leqslant c(\varepsilon+t)^{2\sigma-1}\|F\|_{2\sigma,p,\mathbf{R}^n}.$$

Hence for $u(t) := g(t^{1/2})$ we obtain $u(0) = F_\varepsilon$ and

$$t^{1-\sigma}\|u'(t)\|_{p,\mathbf{R}^n} + t^{1-\sigma}\|D^2 u(t)\|_{p,\mathbf{R}^n} \leqslant c\|F\|_{2\sigma,p,\mathbf{R}^n}.$$

By Definition 1.8 and Proposition 1.11(i) we deduce

$$\|F_\varepsilon\|_{2\sigma,p,\mathbf{R}^n} \leqslant c\|F\|_{2\sigma,p,\mathbf{R}^n},$$

and by the arbitrariness of the extension $F$ we get (A.5).

On the other hand, if $\theta = \sigma \in ]1/2,1[$, we can apply the above argument to $Df \in [B^{2\sigma-1,p}(\Omega)]^{nk}$, and again (A.5) follows.

Finally, if $\theta \in ]0,\sigma[$ we interpolate between $[B^{2\sigma,p}(\Omega)]^k$ and $[L^p(\Omega)]^k$ (via the Reiteration Theorem), obtaining by (i) and (A.5)

$$\|f_\varepsilon - f\|_{2\theta,p,\Omega} \leqslant c\|f_\varepsilon - f\|_{p,\Omega}^{1-\theta/\sigma}\|f_\varepsilon - f\|_{2\sigma,p,\Omega}^{\theta/\sigma} \leqslant c\varepsilon^{2\sigma-2\theta}\|f\|_{2\sigma,p,\Omega}.$$

The proof is complete.

Remark A.2. Lemma 4.1 holds also in the case $\sigma = 1/2$; the proof is analogous, but it is now crucial that the mollifier $\varphi$ is even.

**Acknowledgements.** The author wishes to thank P. Acquistapace and J. Trzeciak whose wise suggestions and kind assistance were necessary conditions for the plan, growth and completion of this paper.

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*Received February 16, 1987*
*Revised version October 22, 1987* (2276)