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An approximation theorem in higher order Orlicz–Sobolev spaces and applications

by

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Abstract. An approximation result of the Hedberg type is established in the Orlicz–Sobolev space $W^m L_A(\mathbf{R}^n)$ when the N-function A and its conjugate $\bar{A}$ satisfy the Δ_2 condition near infinity. Applications to the description of the action of some distributions in $W^{-m} L_{\bar{A}}(\mathbf{R}^n)$ as well as to some strongly nonlinear elliptic boundary value problems are given.

1. Introduction. This paper is motivated by the study of the so-called "strongly nonlinear" boundary value problems, i.e. boundary value problems for equations of the form

$$A(u)+g(x, u) = f. \tag{1.1}$$

Here A is a quasilinear elliptic differential operator in divergence form

$$A(u) \equiv \sum_{|\alpha| \leq m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla u, \ldots, \nabla^m u) \tag{1.2}$$

whose coefficients A_α satisfy conditions (including growth conditions) which guarantee the solvability of the problem

$$A(u) = f. \tag{1.3}$$

The function g satisfies a sign condition but has otherwise completely unrestricted growth with respect to u. One is interested in the solvability of (1.1).

Such problems were first considered by Browder [5] as an application of the then newly developed theory of non-everywhere defined mappings of monotone type. For $m = 1$, i.e. A of second order, their solvability under fairly general and natural assumptions was proved by Hess [16]. The treatment of the case $m > 1$ is more involved due to the lack of a simple

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truncation operation in higher order Sobolev spaces. Webb [24] observed that a rather delicate approximation procedure introduced in nonlinear potential theory by Hedberg [14] could be used in place of truncation. This yielded the solvability of (1.1) for $m > 1$. Brézis and Browder [4] then used this approximation procedure to solve a question which they had considered earlier [3] about the action of some distributions. They also showed that their result on the action of some distributions could itself be used in place of truncation in the study of (1.1). For a related and recent approach to the study of (1.1), by means of degree theory, see [6].

The functional setting in all the results mentioned above is that of the usual Sobolev spaces $W^{m,p}$. Accordingly the functions A_α in (1.2) are supposed to satisfy polynomial growth conditions with respect to u and its derivatives. When trying to relax this restriction on the A_α's, one is led to replace $W^{m,p}$ by a Sobolev space $W^m L_A$ built from an Orlicz space L_A instead of L^p. Here the N-function A which defines L_A is related to the actual growth of the A_α's. A solvability theory for (1.3) in this setting has been developed in the last fifteen years (see [12] and the references therein). Moreover, the strongly nonlinear problem (1.1) was studied in [11] in the case $m = 1$.

It is our purpose in this paper to start the investigation in this setting of Orlicz–Sobolev spaces of the harder higher order case $m > 1$. We consider problem (1.1) as well as Hedberg's approximation theorem and Brézis–Browder's question on the action of some distributions. Our results are obtained under the assumption that both the N-function A and its conjugate $\bar{A}$ satisfy the Δ_2 condition near infinity. The role played by these conditions is discussed at various places in the paper. We believe that the final result relative to (1.1) (Theorem 5.3) holds true for general N-functions, but it is not clear whether the present approach can be further adapted to get it (see in particular the comments following Theorem 4.1).

A large part of the paper is concerned with the extension to the setting of Orlicz spaces of results about the Riesz potential which are standard in L^p. The representation formula for a function in $W^m L_A$ by means of this potential plays a central role (Theorem 3.9). Its proof uses a theorem by O'Neil [19] on fractional integration in Orlicz spaces, an interpolation theorem of Torchinsky [23] and the Calderón–Zygmund theory of singular integral operators [7]. Some connections between this Riesz potential and the Orlicz–Sobolev imbedding theorem of Donaldson–Trudinger [8] are also of importance for our purposes (Subsection 3.e).

Section 2 contains some preliminaries, Section 3 the study of the Riesz potential, Section 4 the proof of the approximation theorem and Section 5 the result on the action of some distributions and its application to (1.1).

2. Preliminaries. In this section we list briefly some definitions and well-known facts about Orlicz spaces and Orlicz–Sobolev spaces. Standard references are [1], [17], [18]. We also include some complements to be used later (Subsection 2.f).

2.a. Let $A\colon \mathbf{R}^+ \to \mathbf{R}^+$ be an N-function, i.e. A is continuous, convex, with $A(t) > 0$ for $t > 0$, $A(t)/t \to 0$ as $t \to 0$ and $A(t)/t \to +\infty$ as $t \to +\infty$. Equivalently, A admits the representation

$$A(t) = \int_0^t a(\tau)\, d\tau,$$

where $a\colon \mathbf{R}^+ \to \mathbf{R}^+$ is nondecreasing, right-continuous, with $a(0) = 0$, $a(t) > 0$ for $t > 0$ and $a(t) \to +\infty$ as $t \to +\infty$.

The N-function $\bar{A}$ *conjugate to* A is defined by

$$\bar{A}(t) = \int_0^t \bar{a}(\tau)\, d\tau,$$

where $\bar{a}\colon \mathbf{R}^+ \to \mathbf{R}^+$ is given by

$$\bar{a}(t) = \sup\{s;\ a(s) \leqslant t\}.$$

Clearly $\bar{\bar{A}} = A$ and one has Young's inequality: $ts \leqslant A(t) + \bar{A}(s)$ for all $t, s \geqslant 0$.

Replacing A by

$$B(t) = \int_0^t (A(\tau)/\tau)\, d\tau,$$

we get an N-function whose derivative is continuous and strictly increasing. Moreover, A and B are equivalent on $\mathbf{R}^+$ in the sense of [17; p. 15], i.e. they satisfy an inequality like

$$B(t/l) \leqslant A(t) \leqslant B(lt) \qquad \forall t \geqslant 0$$

for some positive constant l. For the problems to be considered in this paper, we can always replace A by B and thus assume from the beginning that a and $\bar{a}$ are continuous and strictly increasing. From now on, this will be understood as a part of the definition of an N-function.

We will also extend these N-functions into even functions on all $\mathbf{R}$.

2.b. The N-function A is said to satisfy the Δ_2 *condition* if, for some k (necessarily > 2),

$$A(2t) \leqslant kA(t) \qquad \forall t \geqslant 0. \tag{2.1}$$

Before stating our first lemma, it is worth observing that for any N-function A, $A(t) < ta(t)$ for all $t > 0$.

LEMMA 2.1. *Let A be an N-function. The following conditions are equivalent:*

(i) *A satisfies the Δ_2 condition.*
(ii) *For some α (necessarily >1), $ta(t) \leqslant \alpha A(t)$ $\forall t \geqslant 0$.*
(iii) *For some $\beta > 1$, $t\bar{a}(t) \geqslant \beta \bar{A}(t)$ $\forall t \geqslant 0$.*
(iv) *For some $d > 0$, $(\bar{A}(t)/t)' \geqslant d\bar{a}(t)/t$ $\forall t > 0$.*

Moreover, β in (iii) *can be taken as the Hölder conjugate of α in* (ii).

Proof. The equivalence of (i) and (ii) is proved in [16; p. 24] and that of (ii) and (iii) in [17; p. 26]. A direct calculation shows that (iii) implies (iv) with $d = 1-1/p$. The converse implication is also obtained by direct calculation. ∎

We will denote by $\alpha(A)$ the smallest number α such that (ii) holds. Observe that (ii) implies

$$A(t) \leqslant A(1)t^{\alpha} \quad \text{for } t \geqslant 1, \qquad A(t) \geqslant A(1)t^{\alpha} \quad \text{for } 0 \leqslant t \leqslant 1. \tag{2.2}$$

Similarly, (iii) implies

$$\bar{A}(t) \geqslant \bar{A}(1)t^{\beta} \quad \text{for } t \geqslant 1, \qquad \bar{A}(t) \leqslant \bar{A}(1)t^{\beta} \quad \text{for } 0 \leqslant t \leqslant 1. \tag{2.3}$$

Remark 2.2. When (2.1) holds only for $t \geqslant$ some $t_0 > 0$ (resp. $t \leqslant$ some $t_0 > 0$), then A is said to satisfy the *Δ_2 condition for t near infinity* (resp. *near zero*). The equivalences of Lemma 2.1 are still valid in these cases provided one requires the various inequalities to hold only for t large (resp. t small).

2.c. Let Ω be an open subset of $\boldsymbol{R}^n$. The *Orlicz space* $L_A(\Omega)$ is defined as the set of (equivalence classes of) real-valued measurable functions u on Ω such that

$$\int_{\Omega} A(u(x)/\lambda)\,dx < \infty$$

for some $\lambda > 0$. It is a Banach space under the norm

$$\|u\|_A = \inf\{\lambda > 0;\ \int_{\Omega} A(u(x)/\lambda)\,dx \leqslant 1\}.$$

The closure in $L_A(\Omega)$ of the bounded measurable functions with compact support in $\bar{\Omega}$ is denoted by $E_A(\Omega)$. The equality $E_A(\Omega) = L_A(\Omega)$ holds if and only if A satisfies the Δ_2 condition, for all t or for t large according to whether Ω has infinite measure or not.

The dual of $E_A(\Omega)$ can be identified with $L_{\bar{A}}(\Omega)$ by means of the pairing $\int_{\Omega} u(x)v(x)\,dx$, and the dual norm on $L_{\bar{A}}(\Omega)$ is equivalent to $\|\ \|_{\bar{A}}$. The space $L_A(\Omega)$ is reflexive if and only if A and $\bar{A}$ satisfy the Δ_2 condition, for all t or for t large, according to whether Ω has infinite measure or not.

2.d. Young's classical theorem on the convolution of an L^p function with an L^q function has been extended by O'Neil [19] to the setting of Orlicz spaces. The following particular case of this result will be sufficient for our purposes.

Lemma 2.3 (cf. [19]). *Let A be an N-function. Let $u \in L_A(\boldsymbol{R}^n)$ and $v \in L^1(\boldsymbol{R}^n)$. Then the integral in the definition of $(u*v)(x)$ is absolutely convergent for a.e. x, $u*v \in L_A(\boldsymbol{R}^n)$ and*

$$\|u*v\|_A \leqslant \|u\|_A \|v\|_{L^1}. \tag{2.4}$$

We indicate a two-line proof of this lemma which, in addition, can be used to show that $u*v \in E_A(\boldsymbol{R}^n)$ if $u \in E_A(\boldsymbol{R}^n)$ and $v \in L^1(\boldsymbol{R}^n)$.

Proof of Lemma 2.3. Take $\lambda > 0$ such that $\int_{\boldsymbol{R}^n} A(u(x)/\lambda)\,dx \leqslant 1$. Then, by Jensen's inequality,

$$A\big((u*v/\|v\|_{L^1})/\lambda\big) \leqslant A(u/\lambda)*|v|/\|v\|_{L^1}$$

a.e. in $\boldsymbol{R}^n$, and the conclusion follows by integration. ∎

2.e. We now turn to the Orlicz–Sobolev spaces. $W^m L_A(\Omega)$ (resp. $W^m E_A(\Omega)$) is the space of all functions u such that u and its distributional derivatives up to order m lie in $L_A(\Omega)$ (resp. $E_A(\Omega)$). It is a Banach space under the norm

$$\|u\|_{m,A} = \sum_{|\alpha| \leqslant m} \|D^{\alpha}u\|_A.$$

Thus $W^m L_A(\Omega)$ and $W^m E_A(\Omega)$ can be identified with subspaces of the product of a suitable number of copies of $L_A(\Omega)$. Denoting this product by $\prod L_A$, we will use the weak topologies $\sigma(\prod L_A, \prod E_{\bar{A}})$ and $\sigma(\prod L_A, \prod L_{\bar{A}})$.

The space $W_0^m E_A(\Omega)$ is defined as the (norm) closure of $\mathcal{D}(\Omega)$ in $W^m E_A(\Omega)$ and the space $W_0^m L_A(\Omega)$ as the $\sigma(\prod L_A, \prod E_{\bar{A}})$ closure of $\mathcal{D}(\Omega)$ in $W^m L_A(\Omega)$.

2.f. We will have to estimate the norm of the intermediate derivatives of a function in $W^m L_A(\boldsymbol{R}^n)$ in terms of the norm of the function and of its derivatives of order m. Such an estimate is implied by the following interpolation inequality, whose L^p version is classical. Let us write

$$|u|_{j,A} = \sum_{|\alpha| = j} \|D^{\alpha}u\|_A.$$

Lemma 2.4. *Let A be an N-function. There is a constant $K = K(n, m)$ such that for any $\varepsilon > 0$, $1 \leqslant j \leqslant m-1$ and $u \in W^m L_A(\boldsymbol{R}^n)$,*

$$|u|_{j,A} \leqslant K\varepsilon |u|_{m,A} + K\varepsilon^{-j/(m-j)} \|u\|_A.$$

Proof. It is carried out in three steps.

First step. We start by calculating as in [1; p. 71] to find that for any $f \in C^2(\boldsymbol{R})$,

$$|f'(t)| \leqslant 9\int_0^1 |f(\tau)|\,d\tau + \int_0^1 |f''(\tau)|\,d\tau \qquad \forall t \in [0, 1].$$

It then follows, by using the convexity of A and Jensen's inequality, that

$$A(f'(t)) \leqslant \tfrac{1}{2}\int_0^1 A(18f(\tau))\,d\tau + \tfrac{1}{2}\int_0^1 A(2f''(\tau))\,d\tau \qquad \forall t\in[0,1].$$

Integrating between 0 and 1 and applying the resulting inequality to the function $t \to f((1-t)a+tb)/(b-a)$, where $a<b$, we obtain, after a change of variables,

$$\int_a^b A(f'(t))\,dt \leqslant \tfrac{1}{2}\int_a^b A(18f(t)/(b-a))\,dt + \tfrac{1}{2}\int_a^b A(2(b-a)f''(t))\,dt. \tag{2.5}$$

Take $\varepsilon>0$ and let p be the smallest integer in $\boldsymbol{Z}$ such that $2^{-p}<\varepsilon$. Thus $2^p \leqslant 2/\varepsilon$. Let q be any positive integer. Using inequality (2.5) on each interval in the sum below, we obtain

$$\int_{-q/2^p}^{q/2^p} A(f'(t))\,dt = \sum_{i=-q}^{q-1}\int_{i/2^p}^{(i+1)/2^p} A(f'(t))\,dt$$
$$\leqslant \tfrac{1}{2}\int_{-q/2^p}^{q/2^p} A(36f(t)/\varepsilon)\,dt + \tfrac{1}{2}\int_{-q/2^p}^{q/2^p} A(2\varepsilon f''(t))\,dt.$$

Letting $q\to+\infty$ yields

$$\int_{\boldsymbol{R}} A(f'(t))\,dt \leqslant \tfrac{1}{2}\int_{\boldsymbol{R}} A(36f(t)/\varepsilon)\,dt + \tfrac{1}{2}\int_{\boldsymbol{R}} A(2\varepsilon f''(t))\,dt. \tag{2.6}$$

In (2.6), as well as in similar inequalities below, $+\infty$ is allowed as a value for the left- or right-hand side.

Second step. Let $u\in W^2L_A(\boldsymbol{R}^n)$ and denote by u_δ its mollification. We can assume, for a subsequence, that $D^\alpha u_\delta \to D^\alpha u$ a.e. for $|\alpha|\leqslant 2$. We apply (2.6) to u_δ, fixing all variables but x_i, and then integrate with respect to the $n-1$ other variables. This gives

$$\int_{\boldsymbol{R}^n} A(\partial u_\delta/\partial x_i) \leqslant \tfrac{1}{2}\int_{\boldsymbol{R}^n} A(36u_\delta/\varepsilon) + \tfrac{1}{2}\int_{\boldsymbol{R}^n} A(2\varepsilon\partial^2 u_\delta/\partial x_i^2)$$
$$\leqslant \tfrac{1}{2}\int_{\boldsymbol{R}^n}[A(36u/\varepsilon)]_\delta + \tfrac{1}{2}\int_{\boldsymbol{R}^n}[A(2\varepsilon\partial^2 u/\partial x_i^2)]_\delta,$$

where we have used Jensen's inequality. Letting $\delta\to 0$ and using Fatou's lemma in the left-hand side, we obtain

$$\int_{\boldsymbol{R}^n} A(\partial u/\partial x_i) \leqslant \tfrac{1}{2}\int_{\boldsymbol{R}^n} A(36u/\varepsilon) + \tfrac{1}{2}\int_{\boldsymbol{R}^n} A(2\varepsilon\partial^2 u/\partial x_i^2). \tag{2.7}$$

Now we choose $\lambda>0$ with $\lambda \geqslant \|u\|_A$, $\mu>0$ with $\mu\geqslant \|\partial^2 u/\partial x_i^2\|_A$, and apply (2.7) to the function $u/(36\lambda/\varepsilon+2\varepsilon\mu)$. This gives

$$\int_{\boldsymbol{R}^n} A\left(\frac{\partial u/\partial x_i}{36\lambda/\varepsilon+2\varepsilon\mu}\right) \leqslant \tfrac{1}{2}\int_{\boldsymbol{R}^n} A\left(\frac{36}{\varepsilon}\frac{u}{36\lambda/\varepsilon}\right) + \tfrac{1}{2}\int_{\boldsymbol{R}^n} A\left(2\varepsilon\frac{\partial^2 u/\partial x_i^2}{2\varepsilon\mu}\right)$$
$$\leqslant \tfrac{1}{2}+\tfrac{1}{2} = 1,$$

which means that $\|\partial u/\partial x_i\|_A \leqslant 36\lambda/\varepsilon + 2\varepsilon\mu$. Consequently,

$$\|\partial u/\partial x_i\|_A \leqslant 36\varepsilon^{-1}\|u\|_A + 2\varepsilon\|\partial^2 u/\partial x_i^2\|_A$$

and we conclude that

$$|u|_{1,A} \leqslant 36n\varepsilon^{-1}\|u\|_A + 2\varepsilon|u|_{2,A}. \tag{2.8}$$

This proves the lemma for $m=2$.

Third step. Once (2.8) is proved, one can derive the inequality of the lemma by a double induction as in [1; p. 73]. ∎

COROLLARY 2.5. *Let Ω be an open subset of $\boldsymbol{R}^n$. Then the conclusion of Lemma 2.4 holds for u in $W_0^m L_A(\Omega)$.*

Proof. Extend u outside Ω by zero and apply Lemma 2.4. ∎

Remark 2.6. Interpolation inequalities of the above type can also be derived for u in $W^m L_A(\Omega)$ under suitable regularity assumptions on Ω (cf. [2]).

COROLLARY 2.7. *Let A be an N-function. Let $u\in L_A(\boldsymbol{R}^n)$ with $D^\alpha u\in L_A(\boldsymbol{R}^n)$ for $|\alpha|=m$. Then $u\in W^m L_A(\boldsymbol{R}^n)$.*

Proof. Let $u_\delta = u*\varrho_\delta$ be the mollification of u. It is well known [9] that $u_\delta\to u$ in $L_A(\boldsymbol{R}^n)$ for $\sigma(L_A, E_{\bar A})$ and that, for $|\alpha|=m$, $D^\alpha u_\delta = D^\alpha u*\varrho_\delta \to D^\alpha u$ in $L_A(\boldsymbol{R}^n)$ for $\sigma(L_A, E_{\bar A})$. For the intermediate derivatives $D^\beta u_\delta$ with $0<|\beta|<m$, we have $D^\beta u_\delta = u*D^\beta\varrho_\delta\in L_A(\boldsymbol{R}^n)$ by Lemma 2.3. Thus $u_\delta\in W^m L_A(\boldsymbol{R}^n)$. Application of Lemma 2.4 then shows that $D^\beta u_\delta$ remains bounded in $L_A(\boldsymbol{R}^n)$. Thus, for a subsequence, $D^\beta u_\delta\to v_\beta\in L_A(\boldsymbol{R}^n)$ for $\sigma(L_A, E_{\bar A})$. Clearly $D^\beta u = v_\beta$. ∎

3. Riesz potential. The main results in this section are the representation formula for a function in $W^m L_A(\boldsymbol{R}^n)$ by means of a Riesz potential (Theorem 3.9) and the differentiability properties of the Riesz potential (Proposition 3.5 and Lemma 3.7). The results in Subsection 3.e are also of importance for our purposes. We essentially show there that the largest integer m for which the representation formula holds corresponds to the separating order in the Orlicz–Sobolev imbedding theorem.

3.a. Let m satisfy $0<m<n$ (not necessarily an integer for the moment). The *Riesz potential* of order m of a function u on $\boldsymbol{R}^n$ is defined by

$$(I_m u)(x) = \frac{1}{\gamma_m}\int_{\boldsymbol{R}^n}\frac{u(y)}{|x-y|^{n-m}}\,dy,$$

where the constant γ_m is given by

$$\gamma_m = \pi^{n/2}2^m\Gamma(m/2)/\Gamma((n-m)/2)$$

(see e.g. [21]). We are interested here in the situation where $u \in L_A(\boldsymbol{R}^n)$. The following lemma is a particular case of O'Neil's theorem on fractional integration in Orlicz spaces [19]. It can also be derived by interpolation from standard L^p results, as was observed by Torchinsky [23]. We denote by A^{-1}: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ the reciprocal function of A.

LEMMA 3.1 (cf. [19]). *Let A be an N-function. Assume:*

(3.1) $$\bar{A} \text{ satisfies the } \Delta_2 \text{ condition},$$

(3.2) $$\int_0^1 \left(A^{-1}(t)/t^{1+m/n}\right) dt < \infty,$$

(3.3) $$\int_1^\infty \left(A^{-1}(t)/t^{1+m/n}\right) dt = +\infty.$$

Define an N-function A_m: $\boldsymbol{R}^+ \to \boldsymbol{R}^+$ by

(3.4) $$A_m^{-1}(t) = \int_0^t \left(A^{-1}(\tau)/\tau^{1+m/n}\right) d\tau.$$

If $u \in L_A(\boldsymbol{R}^n)$, then the integral in the definition of $I_m u$ is absolutely convergent for a.e. x, $I_m u \in L_{A_m}(\boldsymbol{R}^n)$ and

$$\|I_m u\|_{A_m} \leqslant K \|u\|_A$$

for some $K = K(n, m, A)$.

Remark 3.2. The conclusion of this lemma, as well as most of the results in this section, remain true if (3.1) is replaced by the slightly weaker condition:

(3.5) $$\int_0^t \left(a(\tau)/\tau\right) d\tau \leqslant cA(t)/t \quad \forall t > 0$$

for some constant $c > 0$. The implication (3.1) $\Rightarrow$ (3.5) follows by integrating the inequality in part (iv) of Lemma 2.1. Conversely, one can prove that if $(A(t)/t)\, a(t)$ has a limit as $t \to 0$ and as $t \to \infty$, then (3.5) $\Rightarrow$ (3.1).

Remark 3.3. If

(3.6) A satisfies the Δ_2 condition with $1/\alpha(A) > m/n$,

then (3.2) and (3.3) hold. Indeed, put $\alpha = \alpha(A)$. Replacing t by $A^{-1}(t)$ in formula (ii) of Lemma 2.1, we obtain

$$(A^{-1})'(t)/A^{-1}(t) \geqslant 1/(\alpha t).$$

This gives, after integration,

$$A^{-1}(t) \leqslant A^{-1}(1)\, t^{1/\alpha} \quad \forall t \in [0, 1],$$

$$A^{-1}(t) \geqslant A^{-1}(1)\, t^{1/\alpha} \quad \forall t \geqslant 1,$$

which implies (3.2) and (3.3) since $1/\alpha > m/n$.

Remark 3.4. It is easily seen that if (3.6) holds, then the integral in the definition of $I_m u$ is absolutely convergent for a.e. x. Indeed, denoting the kernel $1/|x|^{n-m}$ by $K(x)$ and the characteristic function of the unit ball in $\boldsymbol{R}^n$ by χ, we have $\chi K \in L^1(\boldsymbol{R}^n)$ and $(1-\chi) K \in L_{\bar{A}}(\boldsymbol{R}^n)$ (the latter follows from (2.3) since $(1-\chi) K \in L^{\alpha'}(\boldsymbol{R}^n)$, where α' is the Hölder conjugate of α). Since

$$I_m u = (\chi K) * u + \left((1-\chi) K\right) * u,$$

the conclusion follows by applying Lemma 2.3 to the first term and Young's inequality to the second.

From now on, we will often assume that A and $\bar{A}$ satisfy the Δ_2 condition. This will be used not only in the application of Lemma 3.1 but also in connection with an interpolation lemma to be applied to the Riesz transform and to the maximal function (Lemma 3.6 below).

3.b. We now consider the differentiability properties of $I_m u$.

PROPOSITION 3.5. *Let A be an N-function. Assume* (3.1)–(3.3) *and that A has the Δ_2 property. Let $u \in L_A(\boldsymbol{R}^n)$. If $|\beta| < m$, then $D^\beta(I_m u) \in L_{A_{m-|\beta|}}(\boldsymbol{R}^n)$ and*

(3.7) $$|D^\beta(I_m u)| \leqslant c \left|I_{m-|\beta|} |u|\right| \quad \text{a.e. in } \boldsymbol{R}^n,$$

where $c = c(n, m, A)$. If $|\beta| = m$, then $D^\beta(I_m u) \in L_A(\boldsymbol{R}^n)$ and

(3.8) $$\|D^\beta(I_m u)\|_A \leqslant c \|u\|_A,$$

where $c = c(n, m, A)$.

The following interpolation lemma will be needed in the proof. It can be seen as a particular case of Theorem 2.3 in Torchinsky [23]. See [2] for a direct simple proof.

LEMMA 3.6 (cf. [23]). *Let A be an N-function such that A and $\bar{A}$ satisfy the Δ_2 condition. Take $p > \alpha(A)$. Let T be a sublinear operator in the space of measurable real-valued functions on $\boldsymbol{R}^n$, with domain $D(T)$ containing $L^1(\boldsymbol{R}^n)$ and $L^p(\boldsymbol{R}^n)$, and which is simultaneously of weak type L^1 and L^p. Then $D(T) \supset L_A(\boldsymbol{R}^n)$, $Tu \in L_A(\boldsymbol{R}^n)$ if $u \in L_A(\boldsymbol{R}^n)$, and*

$$\|Tu\|_A \leqslant c \|u\|_A$$

with a constant c independent of u.

Proof of Proposition 3.5. If $|\beta| < m$ then, by differentiating under the integral sign as in Lemma 4.1 of [25] and observing that

$$\left|D^\beta(1/|x|^{n-m})\right| \leqslant c \cdot 1/|x|^{n-(m-|\beta|)},$$

we obtain (3.7) for $u \in \mathcal{D}(\boldsymbol{R}^n)$. Consequently, by Lemma 3.1,

$$\|D^\beta(I_m u)\|_{A_{m-|\beta|}} \leqslant c\|u\|_A$$

for all $u \in \mathcal{D}(\boldsymbol{R}^n)$, and the conclusion (3.7) follows by density.

To deal with the case $|\beta| = m$, we recall that

$$(3.9)\qquad (I_m \varphi)^\wedge = (2\pi|x|)^{-m}\hat{\varphi} \quad \forall \varphi \in \mathcal{S}(\boldsymbol{R}^n)$$

(cf. [21; p. 117]). Consequently, for $\varphi \in \mathcal{S}(\boldsymbol{R}^n)$,

$$\begin{aligned}(D^\beta(I_m\varphi))^\wedge(x) &= (-2\pi i x)^\beta (2\pi|x|)^{-m}\hat{\varphi}(x)\\ &= (-1)^m (ix_1/|x|)^{\beta_1}\ldots(ix_n/|x|)^{\beta_n}\hat{\varphi}(x)\\ &= (-1)^m (R_1^{\beta_1}\circ\ldots\circ R_n^{\beta_n}\varphi)^\wedge(x),\end{aligned}$$

where R_j denotes the *Riesz transform*:

$$(R_j\Psi)^\wedge = (ix_j/|x|)\hat{\Psi}$$

for Ψ, say, in $L^2(\boldsymbol{R}^n)$. Thus we see that the conclusion (3.8) will follow by density from an estimate like

$$(3.10)\qquad \|R_j\Psi\|_A \leqslant c\|\Psi\|_A \quad \forall \Psi \in \mathcal{S}(\boldsymbol{R}^n).$$

To prove (3.10) we use an equivalent definition of the Riesz transform (cf. [21; p. 57]):

$$(R_j\Psi)(x) = \delta_n \lim_{\varepsilon\to 0}\int_{|y|\geqslant\varepsilon}\frac{y_j}{|y|^{n+1}}\Psi(x-y)\,dy$$

where the constant δ_n depends only on n. This shows that R_j is a singular integral operator to which the Calderón–Zygmund theory [7] applies. Thus R_j is of weak type L^1 and bounded in L^p for any $1 < p < \infty$. The estimate (3.10) now follows from Lemma 3.6 by choosing $p > \alpha(A)$. ∎

We will also use the following pointwise estimate for the lower order derivatives of $I_m u$. Its L^p version is due to Hedberg [13].

LEMMA 3.7. *Let A be an N-function. Assume* (3.1)–(3.3). *Let $u \in E_A(\boldsymbol{R}^n)$. Then, for $|\beta| < m$,*

$$|D^\beta(I_m u)| \leqslant c(\mathcal{M}u)^{|\beta|/m}(I_m|u|)^{1-|\beta|/m} \quad \text{a.e.},$$

where $c = c(n, m, A)$.

Here $\mathcal{M}u$ denotes the *maximal function* of u:

$$(\mathcal{M}u)(x) = \sup_{r>0} r^{-n}\int_{|y-x|\leqslant r}|u(y)|\,dy \leqslant +\infty.$$

Proof of Lemma 3.7. Once (3.7) is obtained, the argument is identical to that in [13] (see also [4]). ∎

3.c. The *Bessel potential* G_m is defined for $m \in \boldsymbol{R}$ by

$$(G_m\varphi)^\wedge = \frac{1}{(1+4\pi^2|x|^2)^{m/2}}\hat{\varphi}$$

for $\varphi \in \mathcal{S}(\boldsymbol{R}^n)$. Clearly G_m is one-to-one from $\mathcal{S}(\boldsymbol{R}^n)$ onto itself, and $G_{m_1}\circ G_{m_2} = G_{m_1+m_2}$.

For $0 < m < n$, G_m is related to I_m by the formula

$$(3.11)\qquad G_m = I_m \circ T_m \quad \text{on } \mathcal{S}(\boldsymbol{R}^n),$$

where T_m is defined by

$$(T_m\varphi)^\wedge = \frac{(2\pi|x|)^m}{(1+4\pi^2|x|^2)^{m/2}}\hat{\varphi}$$

for $\varphi \in \mathcal{S}(\boldsymbol{R}^n)$. Indeed, for m an even integer, $(T_m\varphi)^\wedge \in \mathcal{S}(\boldsymbol{R}^n)$ and (3.11) then follows directly from (3.9). For a general m, one has, by Lemma 3.8 below, $(T_m\varphi)^\wedge = \hat{\varphi} + \hat{t}_m\hat{\varphi}$ where $t_m \in L^1(\boldsymbol{R}^n)$; (3.11) can then be deduced from (3.9) by approximating t_m by smooth functions.

The following lemma will allow the extension to $E_A(\boldsymbol{R}^n)$ of the connection (3.11) between the Riesz potential and the Bessel potential.

LEMMA 3.8 (cf. [21]; Ch. V]). *Let $m > 0$. The inverse Fourier transform of $1/(1+4\pi^2|x|^2)^{m/2}$ is a real L^1 function, denoted by g_m. The inverse Fourier transforms of $(2\pi|x|)^m/(1+4\pi^2|x|^2)^{m/2}$ and of $(1+4\pi^2|x|^2)/(1+(2\pi|x|)^m)$ are measures of the form*

Dirac measure at 0 + a real L^1 function,

which will be denoted respectively by $\delta_0 + t_m$ and $\delta_0 + s_m$.

We deduce from the first part of the lemma that for $m > 0$,

$$G_m\varphi = g_m * \varphi \quad \forall \varphi \in \mathcal{S}(\boldsymbol{R}^n).$$

Let now A be an N-function. The above formula combined with Lemma 2.3 allows us to extend G_m by density to a continuous linear mapping in $E_A(\boldsymbol{R}^n)$. Clearly the relation $G_{m_1} * G_{m_2} = G_{m_1+m_2}$ still holds on $E_A(\boldsymbol{R}^n)$. Similarly

$$T_m\varphi = \varphi + t_m * \varphi \quad \forall \varphi \in \mathcal{S}(\boldsymbol{R}^n).$$

This formula combined with Lemma 2.3 allows us to extend T_m by density to a continuous linear mapping in $E_A(\boldsymbol{R}^n)$. It then follows by density from (3.11), by using Lemma 3.1, that if $0 < m < n$ and if (3.1)–(3.3) hold, then the relation $G_m = I_m * T_m$ still holds on $E_A(\boldsymbol{R}^n)$.

3.d. The following representation formula plays a central role in the next sections.

THEOREM 3.9. *Let m be an integer with $0 < m < n$. Consider $W^m L_A(\mathbf{R}^n)$. Assume* (3.1)–(3.3) *and that A has the Δ_2 property. Then, for each $u \in W^m L_A(\mathbf{R}^n)$, there exists $f \in L_A(\mathbf{R}^n)$ such that*

$$u = I_m f.$$

Moreover, the following estimate holds:

$$\|f\|_A \leqslant c\,\|u\|_{m,A},$$

with a constant c independent of u.

Proof. To prove the existence of f and the corresponding estimate, it suffices to prove the existence of $v \in L_A(\mathbf{R}^n)$ such that $u = G_m v$, with an estimate like $\|v\|_A \leqslant c\,\|u\|_{m,A}$. Indeed, the conclusion then follows from the discussion in Subsection 3.c by putting $f = T_m v$. The existence of v together with the desired estimate will be proved by induction on m. The argument here is adapted from [21; Ch. V].

Suppose first $m = 1$. We start by approximating u in $W^1 L_A(\mathbf{R}^n)$ by a sequence $u_k \in \mathcal{D}(\mathbf{R}^n)$. Put

$$v_k = (\delta_0 + s_1) * \Big(u_k + \sum_{j=1}^{n} R_j(\partial u_k/\partial x_j)\Big)$$

where $\delta_0 + s_1$ is the measure involved in Lemma 3.8 and R_j is the Riesz transform. Observe that $u_k + \sum_{j=1}^{n} R_j(\partial u_k/\partial x_j)$ and v_k belong at least to $L^2(\mathbf{R}^n)$. We claim that

$$G_1 v_k = u_k. \tag{3.12}$$

Indeed, using twice the formula which gives the Fourier transform of the convolution of an L^1 by an L^2 function (cf. e.g. [22]), we obtain

$$\begin{aligned}(G_1 v_k)^{\wedge} &= (g_1 * v_k)^{\wedge} = \hat{g}_1 \hat{v}_k \\ &= \hat{g}_1 (1 + 4\pi^2 |x|^2)^{1/2} (1 + 2\pi |x|)^{-1} \Big(u_k + \sum_{j=1}^{n} R_j(\partial u_k/\partial x_j)\Big)^{\wedge} \\ &= (1 + 2\pi |x|)^{-1} (1 + 2\pi |x|)\,\hat{u}_k = \hat{u}_k,\end{aligned}$$

which yields (3.12).

We now deduce from Lemma 2.3 and the continuity of R_j in $L_A(\mathbf{R}^n)$ (cf. the proof of Proposition 3.5) that v_k converges in $L_A(\mathbf{R}^n)$ to

$$v = (\delta_0 + s_1) * \Big(u + \sum_{j=1}^{n} R_j(\partial u/\partial x_j)\Big).$$

Thus, (3.12) implies $G_1 v = u$, and we clearly have the estimate $\|v\|_A \leqslant c\,\|u\|_{1,A}$.

Let us now assume that the desired property holds true for spaces up to order $m-1$. Take $u \in W^m L_A(\mathbf{R}^n)$. Since u and $\partial u/\partial x_j$ for $j = 1, \ldots, n$ belong to $W^{m-1} L_A(\mathbf{R}^n)$, there exist w and w_j in $L_A(\mathbf{R}^n)$ such that $u = G_{m-1} w$ and $\partial u/\partial x_j = G_{m-1} w_j$ with the estimates $\|w\|_A \leqslant c\,\|u\|_{m-1,A}$ and $\|w_j\|_A \leqslant c\,\|\partial u/\partial x_j\|_{m-1,A}$.

We claim that

$$w_j = \partial w/\partial x_j. \tag{3.13}$$

Indeed, we have, for $\varphi \in \mathcal{S}(\mathbf{R}^n)$,

$$\begin{aligned}\int w_j G_{m-1}\varphi &= \int G_{m-1}(w_j)\,\varphi = \int (\partial u/\partial x_j)\,\varphi \\ &= -\int u\,\partial\varphi/\partial x_j = -\int G_{m-1}(w)\,\partial\varphi/\partial x_j \\ &= -\int w G_{m-1}(\partial\varphi/\partial x_j) = -\int w\,(\partial/\partial x_j)(G_{m-1}\varphi),\end{aligned}$$

and consequently, since G_{m-1} transforms $\mathcal{S}(\mathbf{R}^n)$ onto itself,

$$\int w_j \Psi = -\int w\,\partial\Psi/\partial x_j \quad \forall \Psi \in \mathcal{S}(\mathbf{R}^n),$$

which proves (3.13).

Thus $w \in W^1 L_A(\mathbf{R}^n)$. This implies the existence of $v \in L_A(\mathbf{R}^n)$ such that $w = G_1 v$, with the estimate $\|v\|_A \leqslant c\,\|w\|_{1,A}$. It now follows that

$$u = G_{m-1} w = G_{m-1}(G_1 v) = G_m v,$$

with the desired estimate. ■

Remark 3.10. The function f in Theorem 3.9 is unique. Indeed, if A satisfies (3.1)–(3.3) and if $I_m f = 0$ with $f \in E_A(\mathbf{R}^n)$, then $(T_m \circ I_m) f = 0$. But it is easily seen, by decomposing the kernel of I_m into its component near zero and its component near infinity, that $T_m \circ I_m = I_m \circ T_m$ on $\mathcal{S}(\mathbf{R}^n)$ and so, by density, on $E_A(\mathbf{R}^n)$. Thus $G_m f = 0$ and consequently

$$0 = \int (G_m f)\,\varphi = \int f\,(G_m \varphi)$$

for all $\varphi \in \mathcal{S}(\mathbf{R}^n)$, which implies $f = 0$.

Remark 3.11. The arguments in [21; Ch. V] can be adapted further to show that if A and $\bar{A}$ satisfy the Δ_2 condition, then G_m is a one-to-one continuous linear mapping from $L_A(\mathbf{R}^n)$ onto $W^m L_A(\mathbf{R}^n)$ (cf. [2]).

3.e. Theorem 3.9 will be applied in close connection with the Orlicz–Sobolev imbedding theorem of Donaldson–Trudinger [8]. Let us start by recalling this theorem.

Let C_0 be an N-function. Replacing if necessary C_0 by an N-function equivalent to it near infinity, one can always assume $\int_0^1 (C_0^{-1}(\tau)/\tau^{1+1/n})\,d\tau < \infty$. If $\int_1^\infty (C_0^{-1}(\tau)/\tau^{1+1/n})\,d\tau = +\infty$, then one defines a new N-function C_1

by the formula

$$C_1^{-1}(t) = \int_0^t (C_0^{-1}(\tau)/\tau^{1+1/n})\,d\tau.$$

Repeating this process, one obtains a finite sequence of N-functions C_0, $C_1, \ldots, C_q$:

$$C_i^{-1}(t) = \int_0^t (C_{i-1}^{-1}(\tau)/\tau^{1+1/n})\,d\tau,$$

where $q = q(C_0, n)$ is such that $\int_1^\infty (C_{q-1}^{-1}(\tau)/\tau^{1+1/n})\,d\tau = +\infty$ but $\int_1^\infty (C_q^{-1}(\tau)/\tau^{1+1/n})\,d\tau < +\infty$. If $\int_1^\infty (C_0^{-1}(\tau)/\tau^{1+1/n})\,d\tau < \infty$, we put $q(C_0, n) = 0$.

Observe that $q \leqslant n$ since $C_0^{-1}(\tau)$ has at most linear growth at infinity. Observe also that two N-functions which are equivalent near infinity lead to the same value of q and to corresponding N-functions $C_1, \ldots, C_q$ which are equivalent near infinity.

LEMMA 3.12 (cf. [8]). *Let Ω be a bounded open subset of $\boldsymbol{R}^n$ with the cone property. Let C_0 be an N-function. If $m \leqslant q(C_0, n)$, then $W^m L_{C_0}(\Omega) \subset L_{C_m}(\Omega)$ with continuous injection. If $m > q(C_0, n)$, then $W^m L_{C_0}(\Omega) \subset C(\Omega) \cap L^\infty(\Omega)$ with continuous injection.*

Remark 3.13. If $\bar{C}_0$ satisfies the Δ_2 condition near infinity, then $q(C_0, n) < n$. Indeed, we then have, by (2.3),

$$C_0(t) \geqslant C_0(1)\,t^\beta \quad \forall t \geqslant 1$$

for some $\beta > 1$, which implies

$$C_0^{-1}(t) \leqslant (t/C_0(1))^{1/\beta} \quad \forall t \geqslant 1.$$

Our purpose in this subsection is to compare, for a given N-function A, this number $q(A, n)$ with the largest integer m for which Theorem 3.9 can be applied. The latter depends on the behaviour of A near zero while $q(A, n)$ does not. This will lead us to replace A by another N-function equivalent to it near infinity and for which the conditions near zero in Theorem 3.9 are automatically satisfied. The following simple lemma will prove useful for this purpose.

LEMMA 3.14. *Let A be an N-function. Define, for $r > 1$,*

$$\tilde{A}(t) = A(t) \textit{ for } t \geqslant 1,\ A(1)\,t^r \textit{ for } 0 \leqslant t \leqslant 1,$$

$$B(t) = \int_0^t (\tilde{A}(\tau)/\tau)\,d\tau.$$

Then B is an N-function which is equivalent to A near infinity. Moreover, B and $\bar{B}$ satisfy the Δ_2 condition near zero.

Proof. Since $B'(t) = A(t)/t$ for $t \geqslant 1$, the equivalence near infinity of A and B follows easily from a lemma in [17; p. 17]. The Δ_2 condition near zero for B as well as for $\bar{B}$ is obvious. ∎

Consider now an N-function A such that $\bar{A}$ has the Δ_2 property near infinity. Take r with $1 < r < n/(n-1)$ and consider the N-function B as given by Lemma 3.14. Clearly $q(B, n) = q(A, n)$ and $\bar{B}$ has the Δ_2 property on all $\boldsymbol{R}^+$. Moreover, in the construction of the finite sequence $C_0, C_1, \ldots, C_q$ which starts with $C_0 = B$, no modification of the N-functions near zero is needed. This follows from the choice of r since, up to some multiplicative constants,

$$C_i^{-1}(t) = t^{1/r - i/n} \quad \text{for } 0 \leqslant t \leqslant 1,$$

where i varies from 1 to $q(B, n)$, which is $< n$ by Remark 3.13.

Denote by $p(B, n)$ the largest integer m for which

$$\int_0^1 (B^{-1}(t)/t^{1+m/n})\,dt < +\infty, \tag{3.14}$$

$$\int_1^\infty (B^{-1}(t)/t^{1+m/n})\,dt = +\infty. \tag{3.15}$$

The fulfilment of (3.15) implies, by the last inequality of Remark 3.13, that $p(B, n) \leqslant n-1$. Moreover, by the choice of r, condition (3.14) automatically holds for all m with $m \leqslant n-1$. Thus we see that condition (3.14) is not really involved in the definition of $p(B, n)$.

LEMMA 3.15. *$p(B, n) = q(B, n)$. Moreover, if B_m denotes the N-function defined for $m = 1, \ldots, p(B, n)$ by*

$$B_m^{-1}(t) = \int_0^t (B^{-1}(\tau)/\tau^{1+m/n})\,d\tau, \tag{3.16}$$

and if C_m denotes the N-functions of the finite sequence $C_0, \ldots, C_q$ which starts with $C_0 = B$, then B_m and C_m are equivalent on $\boldsymbol{R}^+$ for all m.

The following result will be needed in the proof of Lemma 3.15. It implies that all the N-functions $\bar{C}_1, \ldots, \bar{C}_q, \bar{B}_1, \ldots, \bar{B}_p$ have the Δ_2 property on $\boldsymbol{R}^+$.

LEMMA 3.16. *Let D be an N-function such that*

$$\int_0^1 (D^{-1}(t)/t^{1+m/n})\,dt < \infty \quad \textit{and} \quad \int_1^\infty (D^{-1}(t)/t^{1+m/n})\,dt = \infty$$

for some m with $0 < m < n$ (not necessarily an integer here). If $\bar{D}$ has the Δ_2 property on $\boldsymbol{R}^+$, then the N-function D_m defined by

$$D_m^{-1}(t) = \int_0^t (D^{-1}(\tau)/\tau^{1+m/n})\,d\tau$$

is such that $\bar{D}_m$ also has the Δ_2 property on $\boldsymbol{R}^+$. More precisely, if D satisfies

(3.17) $$D(t) < (1/\beta)\,tD'(t) \quad \forall t \geqslant 0$$

for some $\beta > 1$, then $1/\beta - m/n$ is > 0 and D_m satisfies

(3.18) $$D_m(t) \leqslant (1/\beta - m/n)\,tD_m'(t) \quad \forall t \geqslant 0.$$

Proof. The Δ_2 property on $\boldsymbol{R}^+$ for $\bar{D}$ means, by formula (iii) of Lemma 2.1, that (3.17) holds for some $\beta > 1$. Replacing in (3.17) t by $D^{-1}(\tau)$, we easily deduce

$$(D^{-1}(\tau)/\tau^{m/n})' \leqslant (1/\beta - m/n)\,D^{-1}(\tau)/\tau^{1+m/n} \quad \forall \tau > 0.$$

Integration and use of the fact that

$$\liminf_{\tau \to 0} D^{-1}(\tau)/\tau^{m/n} = 0$$

(a consequence of the integrability of $D^{-1}(\tau)/\tau^{1+m/n}$ near zero) give

(3.19) $$D^{-1}(\tau)/\tau^{m/n} \leqslant (1/\beta - m/n)\,D_m^{-1}(\tau) \quad \forall \tau > 0.$$

This implies $1/\beta - m/n > 0$. Using the relation $(D_m^{-1})'(\tau) = D^{-1}(\tau)/\tau^{1+m/n}$ and replacing τ in (3.19) by $D_m(\tau)$, we obtain (3.18). ∎

Proof of Lemma 3.15. We will show by induction that for $m = 0, 1, \ldots, q(B, n)$, one has

(3.20) $$m \leqslant p(B, n) \quad \text{and} \quad C_m^{-1}(t)/l \leqslant B_m^{-1}(t) \leqslant C_m^{-1}(t) \quad \forall t \geqslant 0$$

for some positive constant l. This clearly holds for $m = 0$ and 1 since $C_0 = B_0$ and $C_1 = B_1$. Assume (3.20) for $m = 0, 1, \ldots, k-1$ with $k \leqslant q(B, n)$. Since (3.20) implies that C_{k-1} is equivalent on $\boldsymbol{R}^+$ to B_{k-1}, C_k is equivalent on $\boldsymbol{R}^+$ to the N-function C defined by

$$C^{-1}(t) = \int_0^t (B_{k-1}^{-1}(\tau)/\tau^{1+1/n})\,d\tau.$$

By using the definition of B_{k-1}^{-1} in this expression and then permuting the resulting integrals, we obtain

(3.21) $$C^{-1}(t) = n\int_0^t (B^{-1}(s)/s^{1+k/n})\,ds - nt^{-1/n} B_{k-1}^{-1}(t) \quad \forall t > 0.$$

Neglecting the last term in (3.21), we first deduce $k \leqslant p(B, n)$ and then

$$C^{-1}(t) \leqslant nB_k^{-1}(t) \quad \forall t \geqslant 0.$$

On the other hand, denoting by β the constant associated with the Δ_2 property for $\bar{B}$ in formula (iii) of Lemma 2.1 and applying (3.19) with $m = k-1$ to the N-function B, we deduce

$$B^{-1}(\tau)/\tau^{(k-1)/n} \leqslant \left(\frac{1}{\beta} - \frac{k-1}{n}\right) B_{k-1}^{-1}(\tau) \quad \forall \tau > 0.$$

Multiplication by $1/t^{1+1/n}$ and integration give

$$B_k^{-1}(t) \leqslant \left(\frac{1}{\beta} - \frac{k-1}{n}\right) C^{-1}(t) \quad \forall t \geqslant 0.$$

This concludes the proof of (3.20). Thus $p(B, n) \geqslant q(B, n)$ and C_m and B_m are equivalent on $\boldsymbol{R}^+$ for $m = 0, 1, \ldots, q(B, n)$.

We claim that $p(B, n) = q(B, n)$. Suppose, by contradiction, $p(B, n) > q(B, n)$. Put $q(B, n) = q$. A computation identical to the preceding one leads to

$$B_{q+1}^{-1}(t) \leqslant (1/\beta - q/n)\int_0^t (B_q(\tau)/\tau^{1+1/n})\,d\tau \quad \forall t \geqslant 0.$$

Since the above integral converges at zero, we deduce that

$$\int_1^\infty (B_q^{-1}(\tau)/\tau^{1+1/n})\,d\tau = \infty.$$

This equality implies a similar equality with B_q replaced by C_q since these two N-functions are already known to be equivalent on $\boldsymbol{R}^+$. This contradicts the definition of $q(B, n)$. ∎

3.f. To conclude this section, we show how in the previous situation, the Riesz potential of order m can be obtained by composition from the Riesz potential of order 1.

Let A be an N-function such that $\bar{A}$ has the Δ_2 property near infinity. Take B as in Lemma 3.14, with $1 < r < n/(n-1)$. Assume $q(B, n) \geqslant 2$ and let m be an integer with $2 \leqslant m \leqslant q(B, n)$.

By Lemma 3.1, I_1 maps continuously $L_B(\boldsymbol{R}^n)$ into L_{C_1}, where we take, as above, $C_0 = B$. Lemma 3.16 implies that Lemma 3.1 can be applied again, which yields that I_1 maps continuously $L_{C_1}(\boldsymbol{R}^n)$ into $L_{C_2}(\boldsymbol{R}^n)$. After m steps we get a continuous mapping

$$\mathscr{I}_m = I_1 \circ \ldots \circ I_1 : L_B(\boldsymbol{R}^n) \to L_{C_m}(\boldsymbol{R}^n).$$

On the other hand, by Lemma 3.15, the Riesz potential of order m, I_m, can also be considered. By Lemma 3.1, I_m maps continuously $L_B(\boldsymbol{R}^n)$ into $L_{B_m}(\boldsymbol{R}^n)$. The mappings $\mathcal{I}_m$ and I_m coincide on $\mathcal{S}(\boldsymbol{R}^n)$ (cf. [21]; p. 118]). Consequently, by density, $\mathcal{I}_m$ and I_m coincide on $E_B(\boldsymbol{R}^n)$.

4. Approximation theorem. Let u belong to $W^m L_A(\boldsymbol{R}^n)$. If $m = 1$, then, combining truncation with the use of a smooth cut-off function, one easily gets an approximation of u by bounded functions with compact supports which, at a.e. point, have the same sign as u and are dominated by u (cf. [11]). It is precisely this type of approximation which is needed in the study of (1.1) as well as in the work of Brézis–Browder on the action of some distributions. We consider here the higher order case $m > 1$.

THEOREM 4.1. *Let A be an N-function such that A and $\bar{A}$ satisfy the Δ_2 condition near infinity. Let $u \in W^m L_A(\boldsymbol{R}^n)$. Then there exists a sequence u_k such that:*

(i) *$u_k \in W^m L_A(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$ and $\operatorname{supp} u_k$ is compact.*
(ii) *$|u_k(x)| \leqslant |u(x)|$ and $u_k(x)u(x) \geqslant 0$ a.e. in $\boldsymbol{R}^n$.*
(iii) *u_k converges to u for the modular convergence in $W^m L_A(\boldsymbol{R}^n)$.*

If, in addition, A satisfies the Δ_2 property near zero, then (iii) *can be strengthened to norm convergence.*

We recall that u_k converges to u for the modular convergence in $W^m L_A(\boldsymbol{R}^n)$ if, for some $\lambda > 0$,

$$\int_{\boldsymbol{R}^n} A\big((D^\alpha u - D^\alpha u_k)/\lambda\big)\,dx \to 0$$

for all $|\alpha| \leqslant m$. This implies convergence for $\sigma(\prod L_A, \prod L_{\bar{A}})$. If A satisfies the Δ_2 condition on $\boldsymbol{R}^+$, then modular convergence coincides with norm convergence.

Proof of Theorem 4.1. Some arguments are similar to those in [4, 24] and we will only sketch them. Take $\zeta \in \mathcal{D}(\boldsymbol{R}^n)$ with $0 \leqslant \zeta \leqslant 1$, $\zeta = 0$ outside the unit ball and $\zeta = 1$ in a neighbourhood of zero. Put $\zeta_l(x) = \zeta(x/l)$ and define $v_l = u\zeta_l$. Fix $\lambda > 0$ such that

$$\int_{\boldsymbol{R}^n} A(4D^\alpha u/\lambda) < \infty \qquad \forall |\alpha| \leqslant m.$$

Then, using the convexity of A, it is easily seen as in [10, p. 22] that

$$\int_{\boldsymbol{R}^n} A\big(2(D^\alpha u - D^\alpha v_l)/\lambda\big) \to 0 \qquad \forall |\alpha| \leqslant m \tag{4.1}$$

as $l \to \infty$. Thus v_l converges to u for the modular convergence in $W^m L_A(\boldsymbol{R}^n)$.

We distinguish two cases: either $m > q(A, n)$ or $m \leqslant q(A, n)$, where

$q(A, n)$ is the number arising in the Orlicz–Sobolev imbedding theorem (see Subsection 3.e). In the first case, this imbedding theorem implies that $v_l \in L^\infty(\boldsymbol{R}^n)$, and consequently the sequence v_l satisfies all the requirements of the theorem. We now consider the second case. Let B be the N-function associated with A as in Subsection 3.e. Since B is equivalent to A near infinity, $v_l \in W^m L_B(\boldsymbol{R}^n)$. We will associate with each v_l a sequence of functions $v_{l,k}$ such that:

(i)′ $v_{l,k} \in W^m L_B(\boldsymbol{R}^n) \cap L^\infty(\boldsymbol{R}^n)$ and $\operatorname{supp} v_{l,k} \subset \operatorname{supp} v_l$.
(ii)′ $|v_{l,k}| \leqslant |v_l|$ and $v_{l,k} v_l \geqslant 0$ a.e. in $\boldsymbol{R}^n$.
(iii)′ $v_{l,k} \to v_l$ in $W^m L_B(\boldsymbol{R}^n)$ as $k \to \infty$.

Let us admit the existence of such a sequence for a moment. The norm convergence in (iii)′ implies that for any $\mu > 0$,

$$\int_{\boldsymbol{R}^n} B\big((D^\alpha v_l - D^\alpha v_{l,k})/\mu\big) \to 0 \qquad \forall |\alpha| \leqslant m \tag{4.2}$$

as $k \to \infty$. Moreover, for a subsequence,

$$D^\alpha v_{l,k} \to D^\alpha v_l \quad \text{a.e. in } \boldsymbol{R}^n,\ \forall |\alpha| \leqslant m. \tag{4.3}$$

By using the convexity of A and the inequality $A(t) \leqslant B(2t)$ for $t \geqslant 1$, we have

$$\begin{aligned}\int_{\boldsymbol{R}^n} A\big((D^\alpha u - D^\alpha v_{l,k})/\lambda\big) \leqslant \tfrac{1}{2} &\int_{\boldsymbol{R}^n} A\big(2(D^\alpha u - D^\alpha v_l)/\lambda\big)\\ &+ \tfrac{1}{2} \int_{|x| \leqslant l} A\big(2(D^\alpha v_l - D^\alpha v_{l,k})/\lambda\big)\chi^\alpha_{l,k}\\ &+ \tfrac{1}{2} \int_{|x| \leqslant l} B\big(4(D^\alpha v_l - D^\alpha v_{l,k})/\lambda\big)(1 - \chi^\alpha_{l,k}),\end{aligned}$$

where $\chi^\alpha_{l,k}$ is the characteristic function of $\{x \in \boldsymbol{R}^n;\ 2|D^\alpha v_l(x) - D^\alpha v_{l,k}(x)|/\lambda \leqslant 1\}$. We deduce from this inequality, using (4.1)–(4.3), that for any given $\varepsilon > 0$, a function $v_{l,k}$ can be chosen such that

$$\int_{\boldsymbol{R}^n} A\big((D^\alpha u - D^\alpha v_{l,k})/\lambda\big) \leqslant \varepsilon \qquad \forall \alpha \leqslant m.$$

The proof of the theorem is then completed.

In order to construct this sequence $v_{l,k}$, we observe that Lemmas 3.14 and 3.15 imply that Theorem 3.9 can be applied to $v_l \in W^m L_B(\boldsymbol{R}^n)$:

$$v_l = I_m f_l$$

for some $f_l \in L_B(\boldsymbol{R}^n)$, with $\|f_l\|_B \leqslant c\,\|v_l\|_{m,B}$. Put

$$w_l = I_m(|f_l|) \in L_{B_m}(\boldsymbol{R}^n),$$

take $H \in C^\infty(\mathbf{R})$ such that $0 \leqslant H \leqslant 1$ and $H = 1$ on $[-1/2, +1/2]$ and 0 outside $[-1, +1]$, and define for $k = 1, 2, \ldots$

$$v_{l,k}(x) = H(k^{-1} w_l(x)) v_l(x).$$

We claim that this sequence satisfies (i)′–(iii)′.

(ii)′ is obvious, as well as $\operatorname{supp} v_{l,k} \subset \operatorname{supp} v_l$. Moreover,

$$|v_{l,k}(x)| \leqslant H(k^{-1} w_l(x)) w_l(x),$$

which implies

$$|v_{l,k}(x)| \leqslant k \quad \text{a.e. in } \mathbf{R}^n. \tag{4.4}$$

Thus it only remains to prove that $v_{l,k} \in W^m L_B(\mathbf{R}^n)$ and that (iii)′ holds. Clearly $v_{l,k} \in L_B(\mathbf{R}^n)$ and $v_{l,k} \to v_l$ in $L_B(\mathbf{R}^n)$ as $k \to \infty$. Consequently, by Corollary 2.7 and Lemma 2.4, it suffices to prove that for $|x| = m$, $D^\alpha v_{l,k} \in L_B(\mathbf{R}^n)$ and $D^\alpha v_{l,k} \to D^\alpha v_l$ in $L_B(\mathbf{R}^n)$. Using Leibniz' rule to compute $D^\alpha v_{l,k}$, we are reduced to proving that for $|\alpha| = m$ and $0 < \beta \leqslant \alpha$,

$$D^\beta [H(k^{-1} w_l)] D^{\alpha-\beta} v_l \in L_B(\mathbf{R}^n) \quad \text{and} \quad \to 0 \text{ in } L_B(\mathbf{R}^n).$$

Some computation based on Lemma 3.7 shows that for $0 < \beta < \alpha$:

$$|D^\beta [H(k^{-1} w_l)]| \leqslant c(\mathcal{M}f_l)^{|\beta|/m} k^{-|\beta|/m} \quad \text{a.e.,} \tag{4.5}$$

$$|D^{\alpha-\beta} v_l| \leqslant c(\mathcal{M}f_l)^{|\alpha-\beta|/m} |w_l|^{1-|\alpha-\beta|/m} \quad \text{a.e.,} \tag{4.6}$$

and for $\beta = \alpha$:

$$|D^\beta [H(k^{-1} w_l)]| \leqslant ck^{-1}(\mathcal{M}f_l + D^\beta w_l) \quad \text{a.e.} \tag{4.7}$$

The constant c above depends only on n, m, B, H. Consider first the case $\beta = \alpha$. By Lemma 4.2 below, $\mathcal{M}f_l \in L_B(\mathbf{R}^n)$, and by Proposition 3.5, $D^\beta w_l \in L_B(\mathbf{R}^n)$. It then follows from (4.7) that $D^\beta H(w_l/k) \to 0$ in $L_B(\mathbf{R}^n)$. Consider now the case $0 < \beta < \alpha$. Inequalities (4.5) and (4.6) imply

$$|D^\beta [H(k^{-1} w_l)] D^{\alpha-\beta} v_l| \leqslant c\mathcal{M}f_l.$$

Moreover, by (4.5), the left-hand side converges to zero a.e. in $\mathbf{R}^n$. It follows from Lemma 4.2 below that it converges to zero in $L_B(\mathbf{R}^n)$. ∎

LEMMA 4.2. *Suppose that A and $\bar{A}$ satisfy the Δ_2 condition on $\mathbf{R}^+$. Then $\mathcal{M}u \in L_A(\mathbf{R}^n)$ if $u \in L_A(\mathbf{R}^n)$. Moreover,*

$$\|\mathcal{M}u\|_A \leqslant c\|u\|_A$$

with a constant c independent of u.

Proof. It is well known that the maximal function operator is of weak type L^1 and bounded in L^p for any p with $1 < p \leqslant \infty$ (cf. [2; p. 5]). The conclusion then follows from Lemma 3.6 by choosing $p > \alpha(A)$. ∎

Remark 4.3. The conclusion of Lemma 4.2 holds under the sole assumption that $\bar{A}$ satisfies the Δ_2 property on $\mathbf{R}^+$. This follows from an interpolation result in [2].

Remark 4.4. The functions $v_{l,k}$ constructed above satisfy

$$\|v_{l,k}\|_{m,B} \leqslant c\|v_l\|_{m,B}$$

with a constant c depending only on n, m, B, H.

We observe that for $m = 1$, the result of Theorem 4.1 holds without any restriction on the N-function A (cf. [11]). It is not clear whether the present approach can be adapted to deal with general N-functions. For instance, it appears that the conclusion of Lemma 4.2 is false in $L\log L$ (cf. [21; p. 23], [20]). We also observe that Theorem 4.1 treats the approximation problem on all $\mathbf{R}^n$, not on an open set Ω as in [11] for $m = 1$. For a survey of the situation relative to $W_0^{m,p}(\Omega)$, see [15].

Remark 4.5. Typical examples of N-functions which do not satisfy the Δ_2 condition near infinity are those which grow at infinity more rapidly than any power. For such an N-function A, the conclusion of Theorem 4.1 trivially holds. Indeed, one then has $m > q(A, n) = 0$, so that the sequence v_l constructed at the beginning of the proof of Theorem 4.1 yields the desired approximation.

Remark 4.5 indicates that the main cases left open by our approach are those of slowly increasing N-functions, for instance $L\log L$. It is worth observing that in the solvability theory for (1.3), those cases also turned out to be more delicate. The difficulty there was the absence of a priori bounds.

5. Applications

5.a. We first deal with the question considered by Brézis–Browder [3, 4] about the action of some distributions.

Let $W^{-m}L_{\bar{A}}(\Omega)$ (resp. $W^{-m}E_{\bar{A}}(\Omega)$) denote the space of distributions on Ω which can be written as sums of derivatives up to order m of functions in $L_{\bar{A}}(\Omega)$ (resp. $E_{\bar{A}}(\Omega)$). It is a Banach space under the usual quotient norm. We will assume that the open set Ω has the segment property. This implies that $\mathcal{D}(\Omega)$ is dense in $\mathring{W}_0^m L_A(\Omega)$ for the modular convergence and thus for $\sigma(\prod L_A, \prod L_{\bar{A}})$ (cf. [9, 10]). Consequently, the action of a distribution $S \in W^{-m}L_{\bar{A}}(\Omega)$ on an element $u \in W_0^m L_A(\Omega)$ is well defined. It will be denoted by $\langle S, u\rangle$.

Consider now the situation where the distribution $S \in W^{-m}L_{\bar{A}}(\Omega)$ is also a function, i.e. $S \in L^1_{\mathrm{loc}}(\Omega)$. It is then tempting to write $\langle S, u\rangle$ as $\int_\Omega S(x)u(x)\,dx$. This is not possible in general (cf. [3]). However, the following is true. We first deal with the case $\Omega = \mathbf{R}^n$.

THEOREM 5.1. *Assume that A and $\bar{A}$ satisfy the Δ_2 condition near infinity. Let $S \in W^{-m} L_{\bar{A}}(\mathbf{R}^n) \cap L^1_{\mathrm{loc}}(\mathbf{R}^n)$ and let $u \in W^m L_A(\mathbf{R}^n)$. Suppose that for some $h \in L^1(\mathbf{R}^n)$, $S(x)u(x) \geqslant h(x)$ a.e. on $\mathbf{R}^n$. Then $Su \in L^1(\mathbf{R}^n)$ and*

$$\int_{\mathbf{R}^n} S(x)u(x)\,dx = \langle S, u\rangle.$$

Proof. The arguments are easily adapted from [4] by using Theorem 4.1 and the fact that the mollification of a function in $L_A(\mathbf{R}^n)$ converges to that function for $\sigma(L_A, L_{\bar{A}})$ (cf. [9]). ∎

We now consider the case of an open set Ω. Here S will be supposed to be locally summable up to the boundary of Ω, i.e. $S \in L^1_{\mathrm{loc}}(\bar{\Omega})$.

THEOREM 5.2. *Assume that A and $\bar{A}$ satisfy the Δ_2 property near infinity and that Ω has the segment property. Let $S \in W^{-m} L_{\bar{A}}(\Omega) \cap L^1_{\mathrm{loc}}(\bar{\Omega})$ and $u \in W_0^m L_A(\Omega)$. Suppose that for some $h \in L^1(\Omega)$, $S(x)u(x) \geqslant h(x)$ a.e. on Ω. Then $Su \in L^1(\Omega)$ and*

$$\int_{\Omega} S(x)u(x)\,dx = \langle S, u\rangle. \tag{5.1}$$

Proof. Some arguments are easily adapted from [4] and we will only sketch them. Let ζ_l be as in the proof of Theorem 4.1 and put $v_l = \zeta_l u$. If we can prove that $Sv_l \in L^1(\Omega)$ and that

$$\int_{\Omega} S(x)v_l(x)\,dx = \langle S, v_l\rangle,$$

then the conclusion follows as in [4] by letting $l \to \infty$. It follows that we can assume from the beginning that u has compact support in $\bar{\Omega}$. By multiplying S by a function in $\mathcal{D}(\mathbf{R}^n)$ which is equal to 1 on a neighbourhood of the support of u, we see that we can also assume without loss of generality that S has compact support in $\bar{\Omega}$.

Consider first the case $m > q(A, n)$. Then $u \in L^\infty(\Omega)$. The proof of Theorem 4 in [10] yields the existence of a sequence $\varphi_i \in \mathcal{D}(\Omega)$ such that $\varphi_i \to u$ for $\sigma(\prod L_A, \prod L_{\bar{A}})$ as well as a.e. in Ω, $|\varphi_i(x)| \leqslant$ some constant a.e. in Ω and $\operatorname{supp}\varphi_i \subset$ a fixed compact set $\subset \bar{\Omega}$. Using this sequence, one easily obtains (5.1).

Consider now the case $m \leqslant q(A, n)$. We extend u by zero outside Ω and denote by $\bar{u}$ the resulting function. As in the proof of Theorem 4.1, we consider the associated N-function B and carry out the construction

$$\bar{u} = I_m f \quad \text{with } f \in L_B(\mathbf{R}^n), \qquad w = I_m(|f|),$$

$$u_k(x) = H\big(k^{-1} w(x)\big)\bar{u}(x).$$

Since u has compact support in $\bar{\Omega}$, $u \in W_0^m L_B(\Omega)$, and there exists a sequence $\varphi_j \in \mathcal{D}(\Omega)$ such that $\varphi_j \to u$ in $W_0^m L_B(\Omega)$ and a.e. in Ω. For each j we perform

the same construction:

$$\bar{\varphi}_j = I_m f_j \quad \text{with } f_j \in L_B(\mathbf{R}^n), \qquad w_j = I_m(|f_j|),$$

$$\varphi_{j,k}(x) = H\big(k^{-1} w_j(x)\big)\bar{\varphi}_j(x).$$

Since $\varphi_{j,k} \in W_0^m L_B(\Omega) \cap L^\infty(\Omega)$ has compact support in Ω and since $S \in W^{-m} L_{\bar{B}}(\Omega)$ (because S has compact support in $\bar{\Omega}$), a simple mollification yields

$$\int_{\Omega} S\varphi_{j,k} = \langle S, \varphi_{j,k}\rangle.$$

Keeping k fixed and letting $j \to \infty$, we deduce by dominated convergence that, for a subsequence,

$$\int_{\Omega} S\varphi_{j,k} \to \int_{\Omega} Su_k$$

since $|\varphi_{j,k}| \leqslant k$ by (4.4) and $\varphi_{j,k} \to u_k$ a.e. in Ω (the latter follows from: $f_j \to f$ in $L_B(\mathbf{R}^n)$, cf. Theorem 3.9, and $w_j \to w$ in $L_{B_m}(\mathbf{R}^n)$, cf. Lemma 3.1). On the other hand, Remark 4.4 implies that $\varphi_{j,k}$ remains bounded in $W_0^m L_B(\Omega)$ independently of j and k. Consequently, for a subsequence, $\varphi_{j,k} \to u_k$ for the $\sigma(\prod L_B, \prod E_{\bar{B}})$ topology of $W_0^m L_B(\Omega)$. Thus $\langle S, \varphi_{j,k}\rangle \to \langle S, u_k\rangle$ since $S \in W^{-m} E_{\bar{B}}(\Omega)$, and we obtain

$$\int_{\Omega} Su_k = \langle S, u_k\rangle.$$

Finally, letting $k \to \infty$, we conclude as in [4] that $Su \in L^1(\Omega)$ and that (5.1) holds. ∎

5.b. We now turn to the strongly nonlinear boundary value problems.

Let Ω be an open subset of $\mathbf{R}^n$ with the segment property. The quadruple $\big(W_0^m L_A(\Omega), W_0^m E_A(\Omega);\ W^{-m} L_{\bar{A}}(\Omega), W^{-m} E_{\bar{A}}(\Omega)\big)$ will be denoted by $(Y, Y_0; Z, Z_0)$. It is a complementary system in the sense of [9].

We consider a mapping T from $D(T) \subset Y$ into Z which satisfies the following conditions:

(i) (finite continuity) $D(T) \supset Y_0$ and T is continuous from each finite-dimensional subspace of Y_0 into Z for $\sigma(Z, Y_0)$.

(ii) (pseudo-monotonicity) For any sequence $u_n \in D(T)$ such that $u_n \to u$ for $\sigma(Y, Z_0)$, $Tu_n \to \chi \in Z$ for $\sigma(Z, Y_0)$ and $\limsup \langle u_n, Tu_n\rangle \leqslant \langle u, \chi\rangle$, it follows that $u \in D(T)$, $Tu = \chi$ and $\langle u_n, Tu_n\rangle \to \langle u, \chi\rangle$.

(iii) Tu remains bounded in Z whenever $u \in D(T)$ remains bounded in Y and $\langle u, Tu\rangle$ remains bounded from above.

(iv) For any given $f \in Z_0$, $\langle u, Tu - f\rangle > 0$ when $u \in D(T)$ has sufficiently large norm in Y.

It is known that these properties imply that the range of T contains Z_0. Moreover, given an equation of the form

$$\sum_{|\alpha| \leqslant m} (-1)^{|\alpha|} D^\alpha A_\alpha(x, u, \nabla u, \ldots, \nabla^m u) = f, \tag{5.2}$$

concrete analytical conditions on the coefficients A_α are known which guarantee that the formula

$$\int_\Omega \sum_{|\alpha| \leqslant m} A_\alpha(x, u, \nabla u, \ldots, \nabla^m u) D^\alpha v\,dx = \langle Tu, v\rangle$$

gives rise to a (non-everywhere defined) mapping T from $W_0^m L_A(\Omega)$ which satisfies the above four properties. Existence theorems for the Dirichlet problem associated to equation (5.2) can thus be derived in this way. See [12] and the references therein.

Let g: $\Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function such that for each $r > 0$, there exists $h_r \in L^1(\Omega)$ with

$$|g(x, u)| \leqslant h_r(x) \tag{5.3}$$

for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$ with $|u| \leqslant r$. Assume the sign condition

$$g(x, u)u \geqslant 0 \tag{5.4}$$

for a.e. $x \in \Omega$ and all $u \in \mathbf{R}$.

THEOREM 5.3. *Assume that A and $\bar{A}$ have the Δ_2 property near infinity and that Ω has the segment property. Let T: $D(T) \subset W_0^m L_A(\Omega) \to W^{-m} L_{\bar{A}}(\Omega)$ satisfy conditions* (i)–(iv). *Let g: $\Omega \times \mathbf{R} \to \mathbf{R}$ be a Carathéodory function with* (5.3) *and* (5.4). *Then, given $f \in W^{-m} E_{\bar{A}}(\Omega)$, there exists $u \in W_0^m L_A(\Omega)$ such that $g(x, u(x)) \in L^1(\Omega)$, $g(x, u(x))u(x) \in L^1(\Omega)$ and*

$$\langle Tu, v\rangle + \int_\Omega g(x, u(x))v(x)\,dx = \langle f, v\rangle$$

for all $v \in W_0^m L_A(\Omega) \cap L^\infty(\Omega)$ and for $v = u$.

Proof. It is easily adapted from that given in [11] in the case $m = 1$, by using Theorem 5.2. ∎

Remark 5.4. As in [4, 11], one can show that if g is nondecreasing in u and if u_1 and u_2 are two solutions corresponding to f_1 and f_2 respectively, then

$$\langle Tu_1 - Tu_2, u_1 - u_2\rangle + \int_\Omega \big(g(x, u_1(x)) - g(x, u_2(x))\big)(u_1(x) - u_2(x))\,dx$$
$$= \langle f_1 - f_2, u_1 - u_2\rangle.$$

Remark 5.5. Just as for Theorem 4.1, the conclusions of Theorems 5.1–5.3 hold for any N-function A if $m > q(A, n)$.

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