converges in $L_{\rm loc}(-\infty, \infty)$ if $\sum_n \lambda_n \alpha_n^{-1} < \infty$. Consequently, by (9), the series

$$\sum_{n=1}^{\infty} E_{\alpha_n}^{+} * \varphi_k$$

converges in $L_{\rm loc}(-\infty, \infty)$ if $\sum_n \lambda_n \alpha_n^{-1} < \infty$, which proves the theorem.

In [7] it is shown that $E_{\alpha_n}^{+} \to 0$ with respect to type I convergence.

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On vector-valued Fourier multiplier theorems *

by

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Abstract. The classical Fourier multiplier theorems of Littlewood–Paley, Marcinkiewicz, and Mikhlin are generalized to the vector-valued setting in d dimensions. A direct and a tensor product approach yield slightly different results. While the direct approach works in general UMD-spaces, the tensor product technique requires some unconditional structure and it is shown that the latter results fail for the Schatten classes S_p with $p \neq 2$.

0. Introduction and results. Let X be a complex Banach space. We first consider the d-dimensional periodic case. For $1 \leqslant p < \infty$ let $\tilde{L}_p$ resp. $\tilde{L}_p(X)$ be the usual Lebesgue space on $[0, 1]^d$ with the normalized Lebesgue measure. A sequence $a = (a_\varkappa)_{\varkappa \in \boldsymbol{Z}^d} \subseteq \boldsymbol{C}$ is said to be an $\tilde{L}_p(X)$-*Fourier multiplier* if there is a constant $C < \infty$ such that

(0.1) For all finitely nonzero sequences $(x_\varkappa)_{\varkappa \in \boldsymbol{Z}^d} \subseteq X$

$$\Big\| \sum_{\varkappa \in \boldsymbol{Z}^d} a_\varkappa x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle} \Big\|_{\tilde{L}_p(X)} \leqslant C \Big\| \sum_{\varkappa \in \boldsymbol{Z}^d} x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle} \Big\|_{\tilde{L}_p(X)}.$$

The set of all $\tilde{L}_p(X)$-Fourier multipliers will be denoted by $\tilde{M}_p(X)$ and the smallest constant C such that (0.1) holds by $\|a\|_{\tilde{M}_p(X)}$. For an $\tilde{L}_p(X)$-Fourier multiplier a the operator

$$\sum_{\varkappa \in \boldsymbol{Z}^d} x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle} \mapsto \sum_{\varkappa \in \boldsymbol{Z}^d} a_\varkappa x_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle}$$

extends uniquely to an operator on $\tilde{L}_p(X)$ which will be denoted by T_a.

To state the Littlewood–Paley theorem we need a decomposition of $\boldsymbol{Z}^d$. Actually, we will work with two different ones:

1) A coarse decomposition arising as differences of dyadic cuboids: $D_0 = \{0\}$ and for $n = dr + j$, $r \in \boldsymbol{N}_0$, $j \in \{1, \ldots, d\}$ let

$$D_n = \{\varkappa = (\varkappa_1, \ldots, \varkappa_d) \in \boldsymbol{Z}^d \mid |\varkappa_i| < 2^{r+1} \text{ for } i \in \{1, \ldots, j-1\},$$
$$2^r \leqslant |\varkappa_j| < 2^{r+1},\ |\varkappa_i| < 2^r \text{ for } i \in \{j+1, \ldots, d\}\}.$$

* This is a part of the author's Ph.D. thesis written under the supervision of Prof. H. König.

2) A fine decomposition arising as products of the one-dimensional dyadic decomposition: let $I_0 = \{0\} \subseteq \boldsymbol{Z}$ and for $n \in \boldsymbol{N}$, $I_n = \{k \in \boldsymbol{Z} \mid 2^{n-1} \leqslant |k| < 2^n\}$, and define for $\nu = (\nu_1, \ldots, \nu_d) \in \boldsymbol{N}_0^d$

$$\Delta_\nu = I_{\nu_1} \times \ldots \times I_{\nu_d}.$$

From the papers [Bur] and [B 1] we will need the following theorem. For the notion of a Banach-space-valued martingale see [D/U].

THEOREM 0.1. *For a Banach space X the following are equivalent*:

(i) *For all $p \in (1, \infty)$ there exists a constant $C_p < \infty$ such that for any martingale $(f_n)_{n \in \boldsymbol{N}_0}$ and any choice of signs $(\varepsilon_n)_{n \in \boldsymbol{N}} \subseteq \{+1, -1\}$ and any $N \in \boldsymbol{N}$*

$$\Big\| f_0 + \sum_{n=1}^{N} \varepsilon_n (f_n - f_{n-1}) \Big\|_{L_p(\Omega, \Sigma, \mu; X)} \leqslant C_p \|f_N\|_{L_p(\Omega, \Sigma, \mu; X)}.$$

(ii) *For all $p \in (1, \infty)$ the sequence $(a_\varkappa)_{\varkappa \in \boldsymbol{Z}^d}$ with*

$$a_\varkappa = \begin{cases} 1 & \text{if } \varkappa \geqslant 0 \text{ coordinatewise}, \\ 0 & \text{otherwise} \end{cases}$$

is an $\tilde{L}_p(X)$-Fourier multiplier.

Remarks. (i) is just the definition of an UMD-space from [Bur]. Formally both (i) and (ii) depend on p, but it is shown that one gets equivalent properties if one replaces "for all $p \in (1, \infty)$" by "for one $p \in (1, \infty)$". Also note that (ii) but not (i) depends on the dimension d. Well-known examples of UMD-spaces are reflexive Lebesgue spaces L_p, $1 < p < \infty$, reflexive Lorentz spaces L_{pq}, $1 < p, q < \infty$, reflexive Orlicz sequence spaces and reflexive Schatten classes S_p, $1 < p < \infty$. The operator $T_{(a_\varkappa)_{\varkappa \in \boldsymbol{Z}^d}}$ which corresponds to the sequence in (ii) is the d-dimensional *Riesz projection* and will be denoted by R.

A Banach space X is said to have *local unconditional structure* (*l.u.st.*) if there exists a constant $C < \infty$ such that for any finite-dimensional subspace Y of X there exists a finite-dimensional space Z with an unconditional basis such that the natural embedding $j\colon Y \to X$ factors as $j = AB$ with $B\colon Y \to Z$, $A\colon Z \to X$ and $\|A\| \|B\| \leqslant C$. All Banach lattices (e.g. L_p, $L_{p,q}$, Orlicz spaces, $C[0, 1]$) have l.u.st. The Schatten classes S_p, $p \neq 2$, do not have l.u.st. (see [Pis]).

We are now able to state our first result:

PROPOSITION 1 (vector-valued Littlewood–Paley theorem). *Let X be an UMD-space* (*resp. UMD-space with l.u.st.*) *and $1 < p < \infty$. Then for any choice of signs $(\varepsilon_k)_{k \in \boldsymbol{N}_0}$* (*resp. $(\varepsilon_\nu)_{\nu \in \boldsymbol{N}_0^d}$*) *the sequence $(a_\varkappa)_{\varkappa \in \boldsymbol{Z}^d}$ with $a_\varkappa = \varepsilon_k$ for $\varkappa \in D_k$* (*resp. $a_\varkappa = \varepsilon_\nu$ for $\varkappa \in \Delta_\nu$*) *is an $\tilde{L}_p(X)$-Fourier multiplier.*

Remarks. For $d = 1$ the claim of both versions of the proposition coincides and is contained in [B 2]. [McC] uses different methods to obtain similar results on $\boldsymbol{R}^d$. Other partial results are contained in the papers [Fer] and [R/T]. In the case $X = \boldsymbol{C}$ this is the classical Littlewood–Paley theorem. This can be seen by averaging over all possible choices of signs and using Khinchin's inequality. For details see [E/G].

Since the D_k are unions of the $\Delta_\varkappa$, the claim of the bracketed version is stronger, but we need the additional assumption of l.u.st. It will be seen that this assumption cannot be dropped unconditionally. Corresponding to the Littlewood–Paley theorem there are analogues of Marcinkiewicz's multiplier theorem.

DEFINITION 0.1. Let $\alpha, \beta \in \boldsymbol{Z}^d$ with $\alpha \leqslant \beta$ (coordinatewise). Let $[\alpha:\beta] = \{\varkappa \in \boldsymbol{Z}^d \mid \alpha \leqslant \varkappa \leqslant \beta\}$ (the D_k and the $\Delta_\varkappa$ are finite unions of such "cuboids"). For $\gamma = (\gamma_1, \ldots, \gamma_d) = \sum_{j=1}^{d} \gamma_j e_j$ we consider the difference operators Δ^γ: for $a = (a_\varkappa)_{\varkappa \in [\alpha:\beta]}$ let $\Delta^0 a = a$ and

$$(\Delta^{e_j} a)_\varkappa = \begin{cases} a_\varkappa - a_{\varkappa - e_j}, & \varkappa_j \neq \alpha_j, \\ 0, & \varkappa_j = \alpha_j, \end{cases} \qquad \Delta^\gamma a = \Delta^{\gamma_1 e_1} \ldots \Delta^{\gamma_d e_d} a.$$

The *variation* of a on $[\alpha:\beta]$ is defined by

$$\operatorname*{var}_{[\alpha:\beta]} a = \sum_{\varkappa \in [\alpha:\beta]} |(\Delta^{\gamma_\varkappa} a)_\varkappa| \quad \text{where}$$

$$\gamma_\varkappa = (\gamma_{\varkappa,1}, \ldots, \gamma_{\varkappa,d}), \qquad \gamma_{\varkappa,j} = \begin{cases} 1, & \varkappa_j \neq \alpha_j, \\ 0, & \varkappa_j = \alpha_j. \end{cases}$$

For $D = D_k$ (resp. $\Delta_\varkappa$), which is naturally the union of $s = 2$ (resp. $s = 2^d$) such intervals $[\alpha_i:\beta_i]$, we let

$$\operatorname*{Var}_D a = \sum_{i=1}^{s} \operatorname*{var}_{[\alpha_i:\beta_i]} a.$$

PROPOSITION 2 (vector-valued Marcinkiewicz theorem). *Let X be an UMD-space* (*resp. UMD-space with l.u.st.*) *and $1 < p < \infty$. Then there is a constant $C < \infty$ such that for any sequence $a = (a_\varkappa)_{\varkappa \in \boldsymbol{Z}^d}$*

$$\|(a_\varkappa)_{\varkappa \in \boldsymbol{Z}^d}\|_{\tilde{M}_p(X)} \leqslant C \sup_{k \in \boldsymbol{N}_0} \operatorname*{Var}_{D_k} a \qquad \Big(\textit{resp. } \|(a_\varkappa)_{\varkappa \in \boldsymbol{Z}^d}\|_{\tilde{M}_p(X)} \leqslant C \sup_{\nu \in \boldsymbol{N}_0^d} \operatorname*{Var}_{\Delta_\nu} a\Big).$$

Let us now turn to the case of $\boldsymbol{R}^d$. Let L_p and $L_p(X)$ denote the Lebesgue space on $\boldsymbol{R}^d$ with the normalized Lebesgue measure. We will work with the set of rapidly decreasing functions

$$\mathscr{S}(X) = \{\varphi\colon \boldsymbol{R}^d \to X \mid \varphi \text{ is infinitely often differentiable and}$$
$$\forall N \in \boldsymbol{N} \ \sup\{(1 + |\xi|^N) \|D^\gamma \varphi(\xi)\|_X \mid \xi \in \boldsymbol{R}^d,\ |\gamma| \leqslant N\} < \infty\}.$$

For the Fourier transform and its inverse we have as usual

$$(\mathscr{F}\varphi)(\eta) = \int_{\boldsymbol{R}^d} \varphi(\xi)\, e^{-2\pi i\langle\eta,\xi\rangle}\, d\xi, \qquad (\mathscr{F}^{-1}\varphi)(\eta) = \int_{\boldsymbol{R}^d} \varphi(\xi)\, e^{2\pi i\langle\eta,\xi\rangle}\, d\xi.$$

Now let $\psi\colon \boldsymbol{R}^d \to \boldsymbol{C}$ be any bounded function. ψ is called an $L_p(X)$-*Fourier multiplier* if there is a constant $C < \infty$ such that

$$\forall \varphi \in \mathscr{S}(X) \qquad \|\mathscr{F}^{-1}(\psi\mathscr{F}\varphi)\|_{L_p(X)} \leqslant C\|\varphi\|_{L_p(X)}. \tag{0.2}$$

The set of all $L_p(X)$-Fourier multipliers will be denoted by $M_p(X)$ and the smallest constant C for which (0.2) holds by $\|\psi\|_{M_p(X)}$.

PROPOSITION 3 (vector-valued Mikhlin theorem). *Let X be an UMD-space (resp. UMD-space with l.u.st.). Then for any $p \in (1, \infty)$ there is a constant $C < \infty$ such that for all bounded functions $\psi\colon \boldsymbol{R}^d \to \boldsymbol{C}$ whose distributional derivatives $D^\gamma\psi$ of order $\gamma \leqslant (1, \ldots, 1)$ are represented on $\boldsymbol{R}^d\setminus\{0\}$ by functions, we have*

$$\|\psi\|_{M_p(X)} \leqslant C\sup\{|\xi|^{|\gamma|}|D^\gamma\psi(\xi)| \mid \xi \in \boldsymbol{R}^d\setminus\{0\},\ \gamma \leqslant (1, \ldots, 1)\}$$
$$(\textit{resp. } \|\psi\|_{M_p(X)} \leqslant C\sup\{|\xi^\gamma D^\gamma\psi(\xi)| \mid \xi \in \boldsymbol{R}^d\setminus\{0\},\ \gamma \leqslant (1, \ldots, 1)\}).$$

(Here γ is a multiindex and $|\gamma| = \gamma_1 + \ldots + \gamma_d$ and $\xi^\gamma = \xi_1^{\gamma_1}\ldots\xi_d^{\gamma_d}$.)

Observe that the weight function $|\xi|^{|\gamma|}$ of the first version is larger than the second, $|\xi^\gamma|$. A weaker result is contained in [McC] and [Fer].

From this we are able to deduce a Littlewood–Paley theorem for $\boldsymbol{R}^d$ which corresponds to the following decompositions of $\boldsymbol{R}^d\setminus\{0\}$:

1) A coarse decomposition: for $n = dr + j$, $r \in \boldsymbol{Z}$, $j \in \{1, \ldots, d\}$ let

$$\boldsymbol{D}_n = \{\xi = (\xi_1, \ldots, \xi_d) \in \boldsymbol{R}^d \mid |\xi_i| < 2^{r+1} \text{ for } i \in \{1, \ldots, j-1\},$$
$$2^r \leqslant |\xi_j| < 2^{r+1},\ |\xi_i| < 2^r \text{ for } i \in \{j+1, \ldots, d\}\}.$$

2) A fine decomposition: for $\nu = (\nu_1, \ldots, \nu_d) \in \boldsymbol{Z}^d$

$$\Delta_\nu = \{\xi \in \boldsymbol{R}^d\setminus\{0\} \mid 2^{\nu_i - 1} \leqslant |\xi_i| < 2^{\nu_i} \text{ for } i \in \{1, \ldots, d\}\}.$$

PROPOSITION 4 (vector-valued Littlewood–Paley theorem for $\boldsymbol{R}^d$). *Let X be an UMD-space (resp. UMD-space with l.u.st.) and $1 < p < \infty$. Then for any choice of signs $(\varepsilon_k)_{k\in\boldsymbol{Z}}$ (resp. $(\varepsilon_\nu)_{\nu\in\boldsymbol{Z}^d}$) the function $\psi\colon \boldsymbol{R}^d \to \boldsymbol{C}$ with $\psi(\xi) = \varepsilon_k$ for $\xi \in \boldsymbol{D}_k$ (resp. $\psi(\xi) = \varepsilon_\nu$ for $\xi \in \Delta_\nu$) is an $L_p(X)$-Fourier multiplier.*

Our last multiplier theorem is a common generalization of Propositions 3 and 4. The notion of variation is defined analogously to the periodic case. For the precise definition see Section 2.

PROPOSITION 5. *Let X be an UMD-space (resp. UMD-space with l.u.st.) and $1 < p < \infty$. There is a constant $C < \infty$ such that for any ψ which has distributional derivatives $D^\gamma\psi$ of order $\gamma \leqslant (1, \ldots, 1)$ in L_1 in $\boldsymbol{D}_n^\circ$ (resp. Δ_ν°)*

$$\|\psi\|_{M_p(X)} \leqslant C\sup\{\operatorname*{Var}_{\boldsymbol{D}_n^\circ}\psi \mid n \in \boldsymbol{Z}\} \qquad (\textit{resp. } \|\psi\|_{M_p(X)} \leqslant C\sup\{\operatorname*{Var}_{\Delta_\nu^\circ}\psi \mid \nu \in \boldsymbol{Z}^d\}).$$

(Here $\boldsymbol{D}_n^\circ$ and Δ_ν° stand for the interior of $\boldsymbol{D}_n$ and Δ_ν.)

It should be noted that for functions with bounded derivatives

$$\sup\{\operatorname*{Var}_{\boldsymbol{D}_n^\circ}\psi \mid n \in \boldsymbol{Z}\} \leqslant C\sup\{|\xi|^{|\gamma|}|D^\gamma\psi(\xi)| \mid n \in \boldsymbol{Z},\ \gamma \leqslant (1, \ldots, 1),\ \xi \in \boldsymbol{D}_n^\circ\},$$

$$\sup\{\operatorname*{Var}_{\Delta_\nu^\circ}\psi \mid \nu \in \boldsymbol{Z}^d\} \leqslant C\sup\{|\xi^\gamma D^\gamma\psi(\xi)| \mid \nu \in \boldsymbol{Z}^d,\ \gamma \leqslant (1, \ldots, 1),\ \xi \in \Delta_\nu^\circ\}.$$

Finally, we give a negative result:

PROPOSITION 6. *In none of the propositions 1 through 5 does the bracketed claim hold for the Schatten classes S_p, $1 < p < \infty$ and $p \neq 2$.*

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1. The periodic case. The idea of the proof of the first version of Proposition 1 is taken from [B 2]. Corresponding to the coarse decomposition of $\boldsymbol{Z}^d$ we consider an increasing sequence of σ-algebras:

For $r \in \boldsymbol{N}_0$ the dyadic intervals $\{I_{r,q} = [0, 2^{-r}) + q2^{-r} \mid q \in \{0, \ldots, 2^r - 1\}\}$ generate a σ-algebra which we denote by $\mathscr{D}_r$. Furthermore, for $n = rd + j$, $r \in \boldsymbol{N}_0$, $j \in \{1, \ldots, d\}$, we introduce the σ-algebra $\mathscr{H}_n = \mathscr{H}_{rd+j}$ which is generated by

$$\underbrace{\mathscr{D}_{r+1} \times \ldots \times \mathscr{D}_{r+1}}_{j \text{ factors}} \times \underbrace{\mathscr{D}_r \times \ldots \times \mathscr{D}_r}_{d-j \text{ factors}}$$

For notational reasons we also define $\mathscr{H}_0 = \{\emptyset, [0, 1)^d\}$.

The conditional expectation with respect to $\mathscr{H}_n$ will be denoted by $E(\cdot \mid \mathscr{H}_n)$. Let $\tau_\vartheta\colon \tilde{L}_p(X) \to \tilde{L}_p(X)$, $f \mapsto f(\cdot - \vartheta)$, denote the translation with respect to the group $[0, 1)^d$ when the operation is componentwise addition modulo 1.

LEMMA 1.1. *For any UMD-space X and $p \in (1, \infty)$ there exists a $C < \infty$ such that for all $N \in \boldsymbol{N}$, all signs $(\varepsilon_n)_{n\in\boldsymbol{N}_0}$ and all $f \in \tilde{L}_p(X)$*

$$\Big\|\sum_{n=0}^{N} \varepsilon_n(h_n * f)\Big\|_{\tilde{L}_p(X)} \leqslant C\|f\|_{\tilde{L}_p(X)} \tag{1.1}$$

where $h_0 \equiv 1$ and for $n \geqslant 1$, $h_n = h_{rd+j} \in \tilde{L}_1(\boldsymbol{R}^d)$, $r \in \boldsymbol{N}_0$, $j \in \{1, \ldots, d\}$, have the Fourier coefficients

$$\hat{h}_n(\varkappa) = \Big(\prod_{l=1}^{j-1} \frac{\sin^2(\pi\varkappa_l/2^{r+1})}{(\pi\varkappa_l/2^{r+1})^2}\Big) \frac{\sin^4(\pi\varkappa_j/2^{r+1})}{(\pi\varkappa_j/2^{r+1})^2} \Big(\prod_{l=j+1}^{d} \frac{\sin^2(\pi\varkappa_l/2^r)}{(\pi\varkappa_l/2^r)^2}\Big) \tag{1.2}$$

where $\varkappa = (\varkappa_1, \ldots, \varkappa_d)$.

Note that the Fourier coefficients of h_{n+d} are away from zero on D_n. Our aim will be to replace h_{n+d} by a function whose Fourier coefficients are 1 on D_n and 0 elsewhere. This will be done by a generalization of the contraction principle for Rademacher averages using a convexity argument.

Proof. The operator $f \mapsto \int_{[0,1)^d} \tau_\vartheta E(\tau_{-\vartheta} f \mid \mathscr{H}_n)\, d\vartheta$ commutes with translations and is hence a convolution operator. To compute the kernel we write the conditional expectation as an integral operator. Let $n = rd+j$ and $\xi = (\xi_1, \ldots, \xi_d)$, $\eta = (\eta_1, \ldots, \eta_d)$. Then

$$E(f \mid \mathscr{H}_n)(\eta) = \int_{[0,1)^d} f(\xi) \underbrace{\Big(\prod_{l=1}^{j} 2^{r+1} \sum_{q=0}^{2^{r+1}-1} \chi_{J_{r+1,q}}(\xi_l, \eta_l)\Big)\Big(\prod_{l=j+1}^{d} 2^r \sum_{q=0}^{2^r-1} \chi_{J_{r,q}}(\xi_l, \eta_l)\Big)}_{= H_n(\xi,\eta)} d\xi.$$

Here $\chi_{J_{r,q}}$ stands for the characteristic function of the set $J_{r,q} = I_{r,q} \times I_{r,q}$. A simple computation using translation invariance and Fubini's theorem shows

$$\int_{[0,1)^d} \tau_\vartheta E(\tau_{-\vartheta} f \mid \mathscr{H}_n)\, d\vartheta = f * G_n \tag{1.3}$$

where

$$\begin{aligned} G_n(\xi) &= \int_{[0,1)^d} H_n(\xi+\vartheta, \vartheta)\, d\vartheta \\ &= \Big(\prod_{l=1}^{j} 2^{r+1} \int_{[0,1)} \sum_{q=0}^{2^{r+1}-1} \chi_{J_{r+1,q}}(\xi_l+\vartheta_l, \vartheta_l)\, d\vartheta_l\Big) \\ &\quad \times \Big(\prod_{l=j+1}^{d} 2^r \int_{[0,1)} \sum_{q=0}^{2^r-1} \chi_{J_{r,q}}(\xi_l+\vartheta_l, \vartheta_l)\, d\vartheta_l\Big) \\ &= \Big(\prod_{l=1}^{j} 4^{r+1} \int_{[0,1)} \chi_{J_{r+1,0}}(\xi_l+t, t)\, dt\Big)\Big(\prod_{l=j+1}^{d} 4^r \int_{[0,1)} \chi_{J_{r,0}}(\xi_l+t, t)\, dt\Big) \\ &= \Big(\prod_{l=1}^{j} 4^{r+1} \chi_{I_{r+1,0}} * \chi_{-I_{r+1,0}}(\xi_l)\Big)\Big(\prod_{l=j+1}^{d} 4^r \chi_{I_{r,0}} * \chi_{-I_{r,0}}(\xi_l)\Big). \end{aligned}$$

The Fourier coefficients of a factor of the above product satisfy

$$(4^r \chi_{I_{r,0}} * \chi_{-I_{r,0}})\hat{}\,(k) = 4^r (\sin(2^{-r}\pi k)/\pi k)^2.$$

If we furthermore use the fact that $\sin^2(x) - \frac{1}{4}\sin^2(2x) = \sin^4(x)$, we see that

the functions $h_n = G_n - G_{n-1}$ satisfy (1.2). To prove (1.1) we insert (1.3) and get

$$\begin{aligned} &\Big\|\sum_{n=0}^{N} \varepsilon_n (h_n * f)\Big\|_{\tilde{L}_p(X)} \\ &= \Big\| \int_{[0,1)^d} \tau_\vartheta\Big(\varepsilon_0 E(\tau_{-\vartheta} f) + \sum_{n=1}^{N} \varepsilon_n\big(E(\tau_{-\vartheta} f \mid \mathscr{H}_n) - E(\tau_{-\vartheta} f \mid \mathscr{H}_{n-1})\big)\Big) d\vartheta \Big\|_{\tilde{L}_p(X)} \\ &\leqslant \int_{[0,1)^d} \Big\|\varepsilon_0 E(\tau_{-\vartheta} f) + \sum_{n=1}^{N} \varepsilon_n\big(E(\tau_{-\vartheta} f \mid \mathscr{H}_n) - E(\tau_{-\vartheta} f \mid \mathscr{H}_{n-1})\big)\Big\|_{\tilde{L}_p(X)} d\vartheta, \end{aligned}$$

and by the UMD-property (Theorem 0.1) this is

$$\leqslant C \int_{[0,1)^d} \|E(\tau_{-\vartheta} f \mid \mathscr{H}_n)\|_{\tilde{L}_p(X)}\, d\vartheta = C\|f\|_{\tilde{L}_p(X)}. \ \blacksquare$$

The main tool to derive Proposition 1 from Lemma 1.1 is a generalization of the contraction principle for Rademacher averages (see [Kah]). This technique is well known in connection with square functions but as [B 2] we prefer to work with Rademacher averages, since we do not want to use a lattice structure in the space X. The set of all operators on a Banach space X will be denoted by $L(X)$.

Definition 1.1. Let T be a fixed element of $L(L_p(\Omega, \Sigma, \mu; X))$ and

$$M(T) = \{S \in L(L_p(\Omega, \Sigma, \mu; X)) \mid \exists \psi, \varphi\colon \Omega \to \mathbf{C} \text{ measurable}, \ |\psi|, |\varphi| \leqslant 1 \text{ a.e. such that } \forall f \in L_p(\Omega, \Sigma, \mu; X)\colon S(f) = \psi T(\varphi f)\}.$$

We define for an arbitrary $S \in L(L_p(\Omega, \Sigma, \mu; X))$

$$\|S\|_T = \|T\|_{L(L_p(\Omega,\Sigma,\mu;X))} \inf\{\lambda > 0 \mid S \in \lambda \overline{\operatorname{conv} M(T)}\}$$

where $\overline{\operatorname{conv} M(T)}$ denotes the closure of the convex hull of $M(T)$ in the weak operator topology.

Lemma 1.2 (generalized contraction principle). *Let (Ω, Σ, μ) be a measure space and X a Banach space and $1 \leqslant p \leqslant \infty$. For all operators T, $T_n \in L(L_p(\Omega, \Sigma, \mu; X))$ and for all finitely nonzero sequences $(f_n) \subseteq L_p(\Omega, \Sigma, \mu; X)$ we have*

$$\Big(\int_0^1 \Big\|\sum_n r_n(t)\, T_n(f_n)\Big\|^p_{L_p(\Omega,\Sigma,\mu;X)} dt\Big)^{1/p} \leqslant 2 \sup_n \|T_n\|_T \Big(\int_0^1 \Big\|\sum_n r_n(t)\, f_n\Big\|^p_{L_p(\Omega,\Sigma,\mu;X)} dt\Big)^{1/p}.$$

Proof. 1) We apply Fubini's theorem to get for any system $(\varphi_n) \subseteq L_\infty(\Omega, \Sigma, \mu; \mathbf{C})$

$$\Big(\int_0^1 \Big\|\sum_n r_n(t)\, \varphi_n f_n\Big\|^p_{L_p(\Omega,\Sigma,\mu;X)} dt\Big)^{1/p} = \Big(\int_\Omega \int_0^1 \Big\|\sum_n r_n(t)\, \varphi_n(\omega) f_n(\omega)\Big\|^p_X dt\, d\mu(\omega)\Big)^{1/p}. \tag{1.4}$$

For fixed $\omega \in \Omega$ the contraction principle and again Fubini's theorem yield that (1.4) is

$$\leqslant 2 \sup_n \|\varphi_n\|_{L_\infty} \Big(\int_\Omega \int_0^1 \Big\|\sum_n r_n(t) f_n(\omega)\Big\|_X^p dt\, d\mu(\omega)\Big)^{1/p}$$
$$= 2 \sup_n \|\varphi_n\|_{L_\infty} \Big(\int_0^1 \Big\|\sum_n r_n(t) \varphi_n f_n\Big\|_{L_p(\Omega,\Sigma,\mu;X)}^p dt\Big)^{1/p}.$$

2) First assume that $T_n \in M(T)$. Thus there exist measurable ψ_n, φ_n with modulus less than 1 almost everywhere such that for all f, $T_n(f) = \psi_n T(\varphi_n f)$. Applying the estimate for (1.4) twice we get

$$\Big(\int_0^1 \Big\|\sum_n r_n(t)\, T_n(f_n)\Big\|_{L_p(\Omega,\Sigma,\mu;X)}^p dt\Big)^{1/p} = \Big(\int_0^1 \Big\|\sum_n r_n(t)\, \psi_n T(\varphi_n f_n)\Big\|_{L_p(\Omega,\Sigma,\mu;X)}^p dt\Big)^{1/p}$$
$$\leqslant 2\Big(\int_0^1 \Big\|\sum_n r_n(t)\, T(\varphi_n f_n)\Big\|_{L_p(\Omega,\Sigma,\mu;X)}^p dt\Big)^{1/p} \leqslant 4\|T\| \Big(\int_0^1 \Big\|\sum_n r_n(t) f_n\Big\|_{L_p(\Omega,\Sigma,\mu;X)}^p dt\Big)^{1/p}.$$

The general result follows by convexity and by the lower semicontinuity of the norm in the weak topology. ■

In the application of Lemma 1.2, T will be the Riesz projection R (see Theorem 0.1) and the operators T_n will be the convolution operators with the trigonometric polynomials

$$g_n = \begin{cases} 1, & n = 0, \\ 0, & 1 \leqslant n \leqslant d, \\ \sum\limits_{\varkappa \in D_{n-d}} \hat{h}(\varkappa)^{-1} e^{2\pi i \langle \varkappa, \cdot \rangle}, & n \geqslant d+1, \end{cases} \tag{1.5}$$

i.e. $T_n(f) = g_n * f$. We then have to estimate $\|T_n\|_R$.

To estimate $\|T_n\|_R$ we need some linear algebra. For $\alpha, \beta \in \boldsymbol{Z}$, $\alpha \leqslant \beta$ (coordinatewise), the finite-dimensional vector space of finite sequences $a = (a_\varkappa)_{\varkappa \in [\alpha:\beta]}$ has the basis $\{a^\nu = (a_\varkappa^\nu)_{\varkappa \in [\alpha:\beta]} \mid \nu \in [\alpha:\beta]\}$ where

$$a_\varkappa^\nu = \begin{cases} 1, & \varkappa \geqslant \nu, \\ 0, & \text{otherwise.} \end{cases}$$

An arbitrary element can easily be expanded with respect to this basis:

Lemma 1.3. (i) *For $\varkappa \in [\alpha:\beta]$ take $\gamma_\varkappa$ as in the introduction. Then for all $\nu \in [\alpha:\beta]$*

$$(\Delta^{\gamma_\varkappa} a^\nu)_\varkappa = \begin{cases} 1, & \varkappa = \nu, \\ 0, & \textit{otherwise}. \end{cases}$$

(ii) *For any $a = (a_\varkappa)_{\varkappa \in [\alpha:\beta]} \subseteq \boldsymbol{C}$ we have*

$$a = \sum_{\varkappa \in [\alpha:\beta]} (\Delta^{\gamma_\varkappa} a)_\varkappa a^\varkappa.$$

Sketch of proof. (i) If for a $q \in \{1, \ldots, d\}$, $\varkappa_q \neq \nu_q$, then $(\Delta^{\gamma_\varkappa, q^{e_q}} a^\nu)_\varkappa = 0$. Hence also $(\Delta^{\gamma_\varkappa} a^\nu)_\varkappa = 0$. This proves the otherwise case. Suppose that $q_1, \ldots, q_r$ are all coordinate directions q for which $\varkappa_q \neq \alpha_q$. Then

$$(\Delta^{\gamma_\varkappa} a^\varkappa)_\varkappa = (\Delta^{e_{q_1}} \ldots \Delta^{e_{q_r}} a^\varkappa)_\varkappa = (\Delta^{e_{q_2}} \ldots \Delta^{e_{q_r}} a^\varkappa)_\varkappa - \underbrace{(\Delta^{e_{q_2}} \ldots \Delta^{e_{q_r}} a^\varkappa)_{\varkappa - e_{q_1}}}_{=0}$$
$$= \ldots = a_\varkappa^\varkappa = 1.$$

(ii) holds because it holds on all elements of the basis. ■

For $\nu \in \boldsymbol{Z}^d$ let us consider $c = (c_\varkappa)_{\varkappa \in \boldsymbol{Z}^d}$ given by

$$c_\varkappa = \begin{cases} 1, & \varkappa \geqslant \nu, \\ 0, & \text{otherwise} \end{cases}$$

(i.e. c is the characteristic function of the positive cone translated by ν). It is clear that for all $f \in \tilde{L}_p(X)$

$$T_c(f) = e^{2\pi i \langle \nu, \cdot \rangle} R(e^{-2\pi i \langle \nu, \cdot \rangle} f).$$

This implies

$$\|T_c\|_R = \|R\|_{L(\tilde{L}_p(X))}. \tag{1.6}$$

Extending our previous notation we consider for any $a = (a_\varkappa)_{\varkappa \in G}$, $G \subseteq \boldsymbol{Z}$, an operator T_a which acts on the trigonometric monomials as

$$T_a(e^{2\pi i \langle \varkappa, \cdot \rangle} x) = \begin{cases} a_\varkappa e^{2\pi i \langle \varkappa, \cdot \rangle} x, & \varkappa \in G, \\ 0, & \varkappa \notin G. \end{cases}$$

For the basis elements a^ν of Lemma 1.3 the norm $\|T_{a^\nu}\|_R$ can be estimated by

$$\|T_{a^\nu}\|_R \leqslant 2^d \|R\|_{L(\tilde{L}_p(X))}.$$

This can be seen by writing the cuboid $[\nu:\beta]$ as a "signed sum" of 2^d translated positive cones. (For $d = 2$ and $d = 3$ drawing a figure is sufficient, but also the general case is elementary and we therefore omit the proof.) We can then use (1.6) and the triangle inequality. A second application of the triangle inequality to Lemma 1.3(ii) gives:

Lemma 1.4. *Let X be an UMD-space and $1 < p < \infty$. Then for any $n \in N_0$ and any sequence $a = (a_\varkappa)_{\varkappa \in D_n}$*

$$\|T_a\|_R \leqslant 2^d \|R\|_{L(\tilde{L}_p(X))} \operatorname*{Var}_{D_n} a.$$

Now we need a convenient tool to estimate the variation of a sequence. For $\alpha, \beta \in \boldsymbol{R}^d$, $\alpha \leqslant \beta$, let $[\alpha, \beta] = \{\xi \in \boldsymbol{R}^d \mid \alpha \leqslant \xi \leqslant \beta \text{ coordinatewise}\}$.

Lemma 1.5. *Let $\alpha, \beta \in \boldsymbol{Z}^d$, $\alpha \leqslant \beta$, and let φ: $[\alpha, \beta] \to \boldsymbol{C}$ be a sufficiently* *smooth function. If the sequence* $a = (a_\varkappa)_{\varkappa\in[\alpha:\beta]}$ *is given by* $a_\varkappa = \varphi(\varkappa)$ *then*

$$\operatorname*{var}_{[\alpha:\beta]} a \leqslant \sum_{\gamma\leqslant(1,\ldots,1)} \max\{|(D^\gamma\varphi)(\xi)|(\beta-\alpha)^\gamma \mid \xi\in[\alpha,\beta]\}$$

(for the notation see Proposition 3).

Proof. We rearrange the sum in the definition of the variation:

$$\operatorname*{var}_{[\alpha:\beta]} a = \sum_{\gamma\leqslant(1,\ldots,1)} \sum_{\varkappa\mid\gamma_\varkappa=\gamma} |(\Delta^\gamma a)_\varkappa|.$$

If $\gamma = (0,\ldots,0)$ then we have only one $\varkappa$ for which $\gamma_\varkappa = \gamma$, namely $\varkappa = \alpha$. Hence

$$\sum_{\varkappa\mid\gamma_\varkappa=\gamma} |(\Delta^\gamma a)_\varkappa| = |a_\alpha| \leqslant \max\{|\varphi(\xi)| \mid \xi\in[\alpha,\beta]\}$$
$$= \max\{|D^\gamma\varphi(\xi)|(\beta-\alpha)^\gamma \mid \xi\in[\alpha,\beta]\}.$$

Now take any multiindex γ with $0\neq\gamma\leqslant(1,\ldots,1)$ and a $\varkappa\in[\alpha:\beta]$ such that $\gamma_\varkappa=\gamma$. Let $q_1,\ldots,q_r$ be the coordinate directions q for which $\gamma_{\varkappa,q}\neq 0$. We write $1'$ for $(1,\ldots,1)\in\boldsymbol{R}^r$ and $\varkappa'$ for $(\varkappa_{q_1},\ldots,\varkappa_{q_r})\in\boldsymbol{R}^r$. By the fundamental theorem of calculus we have

$$(\Delta^{\gamma_\varkappa} a)_\varkappa = \int_{[\varkappa'-1',\varkappa']} D^\gamma\varphi(\xi)\,d\xi' \quad\text{where}\quad \xi_j = \begin{cases}\xi'_s, & j = q_s,\\ \alpha_j, & j\notin\{q_1,\ldots,q_r\}.\end{cases}$$

This implies

$$\sum_{\varkappa\mid\gamma_\varkappa=\gamma} |(\Delta^\gamma a)_\varkappa| \leqslant \sum_{\varkappa\mid\gamma_\varkappa=\gamma} \max\{|D^\gamma\varphi(\xi)| \mid \xi'\in[\varkappa'-1',\varkappa']\}$$
$$\leqslant \sum_{\varkappa\mid\gamma_\varkappa=\gamma} \max\{|D^\gamma\varphi(\xi)| \mid \xi\in[\alpha,\beta]\} = (\beta-\alpha)^\gamma \max\{|D^\gamma\varphi(\xi)| \mid \xi\in[\alpha,\beta]\}.$$

The last equality holds because we have exactly $(\beta-\alpha)^\gamma$ different $\varkappa$ with $\gamma_\varkappa=\gamma$. Summation of the inequalities gives the result. ∎

We are now able to prove the desired inequalities for the operators T_n of (1.5):

LEMMA 1.6. *For any UMD-space X and any* $p\in(1,\infty)$ *the operators* T_n *from* (1.5) *satisfy*

$$\sup\{\|T_n\|_R \mid n\in\boldsymbol{N}_0\} < \infty.$$

Proof. Of course $\|T_1\|_R = \ldots = \|T_d\|_R = 0$ and $\|T_0\|_R \leqslant 2^d\|R\|_{L(\tilde L_p(X))}$. For $n = dr+j > d$ the set D_{n-d} is the disjoint union of two cuboids. Since the estimate for the "negative" part is the same, we only treat the "positive" part.

Its edges are given by

$$\alpha_n = (\underbrace{-2^r+1,\ldots,-2^r+1}_{j-1\text{ times}}, 2^{r-1}, \underbrace{-2^{r-1}+1,\ldots,-2^{r-1}+1}_{m-j\text{ times}}),$$
$$\beta_n = (\underbrace{2^r-1,\ldots,2^r-1}_{j-1\text{ times}}, 2^r-1, \underbrace{2^{r-1}-1,\ldots,2^{r-1}-1}_{m-j\text{ times}}).$$

To simplify notation we introduce the functions

$$s_{2,r}(x) = \sin^2(2^{-r}\pi x)/(2^{-r}\pi x)^2, \quad s_{4,r}(x) = \sin^4(2^{-r}\pi x)/(2^{-r}\pi x)^2,$$
$$\varphi_{j,r}(\xi) = \Big(\prod_{l=1}^{j-1} s_{2,r+1}(\xi_l)\Big)s_{4,r+1}(\xi_j)\Big(\prod_{l=j+1}^{d} s_{2,r}(\xi_l)\Big).$$

Thus we have for a constant C that depends only on d

$$\|T_n\|_R \leqslant C\|R\|_{L(\tilde L_p(X))}\max\{|D^\gamma\varphi_{r,j}(\xi)|(\beta_n-\alpha_n)^\gamma \mid \gamma\leqslant(1,\ldots,1),\ \xi\in[\alpha_n,\beta_n]\}$$
$$= C\|R\|_{L(\tilde L_p(X))}\big(\max\{\max\{s_{2,r+1}^{-1}(x),(2^{r+1}-2)(s_{2,r+1}^{-1})'(x)\} \mid x\in[-2^r+1,2^r-1]\}\big)^{j-1}$$
$$\times\max\{\max\{s_{4,r+1}^{-1}(x),(2^{r-1}-1)(s_{4,r+1}^{-1})'(x)\} \mid x\in[2^{r-1},2^r-1]\}$$
$$\times\big(\max\{\max\{s_{2,r}^{-1}(x),(2^r-2)(s_{2,r}^{-1})'(x)\} \mid x\in[-2^{r-1}+1,2^{r-1}-1]\}\big)^{m-j},$$

and changing the variables gives the bound

$$\leqslant C\|R\|_{L(\tilde L_p(X))}\big(\max\{\max\{s_{2,0}^{-1}(x),(s_{2,0}^{-1})'(x)\} \mid x\in[-1/2,1/2]\}\big)^{m-1}$$
$$\times\max\{\max\{s_{4,0}^{-1}(x),(s_{4,0}^{-1})'(x)\} \mid x\in[1/4,1/2]\}$$
$$= C_1\|R\|_{L(\tilde L_p(X))}.\ ∎$$

Proof of the 1st version of Proposition 1. We write S_n for the operator which acts on the trigonometric polynomials as

$$S_n(e^{2\pi i\langle\varkappa,\cdot\rangle}x) = \begin{cases}e^{2\pi i\langle\varkappa,\cdot\rangle}x, & \varkappa\in D_n,\\ 0, & \text{otherwise},\end{cases} \quad\text{for all } \varkappa\in\boldsymbol{Z}^d \text{ and } x\in X. \tag{1.7}$$

Note that for $n\geqslant d$ we have $S_{n-d}(f) = T_n(h_n * f)$. Averaging in Lemma 1.1 over all possible choices of signs we can deduce

$$\Big(\int_0^1\Big\|\sum_{n=1}^N r_n(t)\,h_n * f\Big\|_{\tilde L_p(X)}^p dt\Big)^{1/p} \leqslant C\|f\|_{\tilde L_p(X)}.$$

Because of Lemma 1.4 the generalized contraction principle (Lemma 1.2) can be used with the operators (1.5) to give with a different constant C

$$\Big(\int_0^1\Big\|\sum_{n=1}^N r_n(t)\,S_n(f)\Big\|_{\tilde L_p(X)}^p dt\Big)^{1/p} \leqslant C\|f\|_{\tilde L_p(X)}. \tag{1.8}$$

Note that the same estimate holds in the dual space $\tilde{L}_{p'}(X')$. Because of the orthogonality of the trigonometric functions we have for all X-valued trigonometric polynomials f and all X'-valued trigonometric polynomials g

$$\langle S_n(f), g\rangle = \langle S_n(f), S_n(g)\rangle, \qquad \langle S_k(f), S_n(g)\rangle = 0 \quad \text{for } k \neq n.$$

Here $\langle\cdot,\cdot\rangle$ is the duality bracket of the dual pair $(\tilde{L}_p(X), \tilde{L}_{p'}(X'))$. If for a fixed trigonometric polynomial f, N is large enough, we have $f = \sum_{n=1}^{N} S_n(f)$ and hence for fixed $t \in [0, 1]$

$$\langle f, g\rangle = \Big\langle \sum_{n=1}^{N} S_n(f), g\Big\rangle = \sum_{n=1}^{N} \langle S_n(f), S_n(g)\rangle = \sum_{n=1}^{N} \langle r_n(t) S_n(f), r_n(t) S_n(g)\rangle$$

$$= \Big\langle \sum_{n=1}^{N} r_n(t) S_n(f), \sum_{n=1}^{N} r_n(t) S_n(g)\Big\rangle.$$

By integration over t we get

$$|\langle f, g\rangle| \leqslant \int_0^1 \Big|\Big\langle \sum_{n=1}^{N} r_n(t) S_n(f), \sum_{n=1}^{N} r_n(t) S_n(g)\Big\rangle\Big|\, dt$$

$$\leqslant \Big(\int_0^1 \Big\|\sum_{n=1}^{N} r_n(t) S_n(f)\Big\|_{\tilde{L}_p(X)}^p dt\Big)^{1/p} \Big(\int_0^1 \Big\|\sum_{n=1}^{N} r_n(t) S_n(g)\Big\|_{\tilde{L}_{p'}(X')}^{p'} dt\Big)^{1/p'}$$

$$\leqslant C\Big(\int_0^1 \Big\|\sum_{n=1}^{N} r_n(t) S_n(f)\Big\|_{\tilde{L}_p(X)}^p dt\Big)^{1/p} \|g\|_{\tilde{L}_{p'}(X')}.$$

Since the X'-valued trigonometric polynomials are norming for $\tilde{L}_p(X)$ we get

$$\|f\|_{\tilde{L}_p(X)} \leqslant C\Big(\int_0^1 \Big\|\sum_{n=1}^{N} r_n(t) S_n(f)\Big\|_{\tilde{L}_p(X)}^p dt\Big)^{1/p}. \tag{1.9}$$

Now let $(a_\varkappa)_{\varkappa \in \mathbf{Z}^d}$ be the sequence which is determined by the sequence of signs $(\varepsilon_n)_{n \in \mathbf{N}_0}$ as in Proposition 1. For an arbitrary trigonometric polynomial f and N large enough

$$\|T_a(f)\|_{\tilde{L}_p(X)} = \Big\|\sum_{n=1}^{N} \varepsilon_n S_n(f)\Big\|_{\tilde{L}_p(X)} \overset{(1.9)}{\leqslant} C\Big(\int_0^1 \Big\|\sum_{n=1}^{N} r_n(t) \varepsilon_n S_n(f)\Big\|_{\tilde{L}_p(X)}^p dt\Big)^{1/p}$$

$$= C\Big(\int_0^1 \Big\|\sum_{n=1}^{N} r_n(t) S_n(f)\Big\|_{\tilde{L}_p(X)}^p dt\Big)^{1/p} \overset{(1.8)}{\leqslant} C^2 \|f\|_{\tilde{L}_p(X)}. \ \blacksquare$$

Proof of the 2nd version of Proposition 1. We need the local unconditional structure to apply a result from [Pis]. Actually, the results of this paper also hold if l.u.st. is replaced by the somewhat weaker assumptions of [Pis]. Let $S_{n,j}$ resp. S_ν denote the operators which act on the trigonometric monomials as

$$S_{n,j}(e^{2\pi i\langle\varkappa,\cdot\rangle}x) = \begin{cases} e^{2\pi i\langle\varkappa,\cdot\rangle}x, & \varkappa_j \in I_n \setminus I_{n-1},\\ 0, & \text{otherwise,}\end{cases}$$

$$S_\nu(e^{2\pi i\langle\varkappa,\cdot\rangle}x) = \begin{cases} e^{2\pi i\langle\varkappa,\cdot\rangle}x, & \varkappa \in \Delta_\nu,\\ 0, & \text{otherwise.}\end{cases}$$

Note that for UMD-spaces $S_{n,j}$ can be extended to a continuous operator on $\tilde{L}_p(X)$ if $1 < p < \infty$. Furthermore, $S_\varkappa = S_{\varkappa_1,1} \circ \ldots \circ S_{\varkappa_d,d}$. Let $(r_\varkappa)_{\varkappa \in \mathbf{N}_0^d}$ be a d-dimensional renumeration of the Rademacher functions. Suppose X has l.u.st. It was proved in [Pis] that there exists a constant $C < \infty$ such that

$$\Big(\int_0^1 \Big\|\sum_{\varkappa \in \mathbf{N}_0^d} r_\varkappa(t) S_\varkappa(f)\Big\|_{\tilde{L}_p(X)}^p dt\Big)^{1/p}$$

$$\leqslant C\Big(\int_{[0,1)^d} \Big\|\sum_{\varkappa \in \mathbf{N}_0^d} \Big(\prod_{j=1}^{d} r_{\varkappa_j}(t_j)\Big) S_\varkappa(f)\Big\|_{\tilde{L}_p(X)}^p dt_1 \ldots dt_d\Big)^{1/p} = (*).$$

We abbreviate t' for $(t_2, \ldots, t_d)$, $\varkappa'$ for $(\varkappa_2, \ldots, \varkappa_d)$ and $S_{\varkappa'}$ for $S_{\varkappa_2,2} \circ \ldots \circ S_{\varkappa_d,d}$. Then

$$(*) = C\Big(\int_{[0,1)^{d-1}} \int_{[0,1)} \Big\|\sum_{\varkappa_1 \in \mathbf{N}_0} r_{\varkappa_1}(t_1) S_{\varkappa_1,1}\Big(\sum_{\varkappa' \in \mathbf{N}_0^{d-1}} \prod_{j=2}^{d} r_{\varkappa_j}(t_j) S_{\varkappa'}(f)\Big)\Big\|_{\tilde{L}_p(X)}^p dt_1\, dt'\Big)^{1/p},$$

and by applying the first version for $d = 1$ (Bourgain's result) d times this is

$$\leqslant C_1\Big(\int_{[0,1)^{d-1}} \Big\|\sum_{\varkappa' \in \mathbf{N}_0^{d-1}} \prod_{j=2}^{d} r_{\varkappa_j}(t_j) S_{\varkappa'}(f)\Big\|_{\tilde{L}_p(X)}^p dt'\Big)^{1/p}$$

$$\leqslant \ldots \leqslant C_d \|f\|_{\tilde{L}_p(X)}.$$

Again a simple duality argument can be used to finish the proof. $\blacksquare$

Actually, Proposition 2 is an easy corollary of Proposition 1 if one uses the generalized contraction principle. Indeed, if a is as in Proposition 2 and S_n are the operators (1.7), then for any trigonometric polynomial f

$$T_a(f) = \sum_{n \in \mathbf{N}_0} T_{a|_{D_n}} \circ S_n(f).$$

Here $a|_{D_n}$ is the sequence which coincides with a on D_n and is 0 elsewhere. Thus

$$\|T_a(f)\|_{\tilde{L}_p(X)} = \Big\|\sum_{n \in \mathbf{N}_0} T_{a|_{D_n}} \circ S_n(f)\Big\|_{\tilde{L}_p(X)}$$

$$\leqslant C\Big(\int_0^1 \Big\|\sum_{n \in \mathbf{N}_0} r_n(t)\, T_{a|_{D_n}} \circ S_n(f)\Big\|_{\tilde{L}_p(X)}^p\Big)^{1/p}$$

$$\leqslant C_1 \sup\{\|T_{a|_{D_n}}\|_R \mid n\in N\}\Big(\int_0^1 \Big\|\sum_{n\in N_0} r_n(t)\,S_n(f)\Big\|^p_{\tilde L_p(X)}\Big)^{1/p} \quad \text{(by Lemma 1.2)}$$

$$\leqslant C_2 \sup\{\operatorname*{Var}_{D_n} a|_{D_n} \mid n\in N_0\}\,\|f\|_{\tilde L_p(X)}.$$

The proof for the second version of Proposition 2 is the same ■

2. The case R^d. To translate the results of Section 1 to R^d, we will make use of the well-known Poisson summation formula in the following vector-valued form:

(2.1) $\forall\,\varphi\in\mathscr{S}_n(X)$, $a\in R\setminus\{0\}$ and $\xi\in R^d$

$$\sum_{\varkappa\in Z^d}\varphi(\xi+a\varkappa) = a^{-d}\sum_{\varkappa\in Z^d}(\mathscr{F}\varphi)(\varkappa/a)\,e^{(2\pi i/a)\langle\varkappa,\xi\rangle}.$$

The translation is done by the following lemma:

LEMMA 2.1. *For $\varphi\in\mathscr{S}(X)$, $p<\infty$ and $k\in N$ we introduce*

$$\varphi^{\sim}_{k,p} = 2^{-dk/p'}\sum_{\varkappa\in Z^d}(\mathscr{F}\varphi)(\varkappa/2^k)\,e^{2\pi i\langle\varkappa,\cdot\rangle} \qquad (1/p+1/p'=1).$$

Then

$$\|\varphi\|_{L_p(X)} = \lim_{k\to\infty}\|\varphi^{\sim}_{k,p}\|_{\tilde L_p(X)}. \tag{2.2}$$

Proof.

$$\lim_{k\to\infty}\|\varphi^{\sim}_{k,p}\|_{\tilde L_p(X)} = \lim_{k\to\infty}2^{-dk/p'}\Big(\int_{[-1/2,1/2)^d}\Big\|\sum_{\varkappa\in Z^d}(\mathscr{F}\varphi)(\varkappa/2^k)\,e^{2\pi i\langle\varkappa,\xi\rangle}\Big\|^p_X d\xi\Big)^{1/p}.$$

By a change of variable this is

$$= \lim_{k\to\infty}2^{-dk}\Big(\int_{[-2^{k-1},2^{k-1})^d}\Big\|\sum_{\varkappa\in Z^d}(\mathscr{F}\varphi)(\varkappa/2^k)\,e^{(2\pi i/2^k)\langle\varkappa,\xi\rangle}\Big\|^p_X d\xi\Big)^{1/p}$$

$$\overset{(2.1)}{=} \lim_{k\to\infty}\Big(\int_{[-2^{k-1},2^{k-1})^d}\Big\|\sum_{\varkappa\in Z^d}\varphi(\xi+2^k\varkappa)\Big\|^p_X d\xi\Big)^{1/p}.$$

To calculate the above limit, we observe that for any fixed $n>d$ there is a constant $C<\infty$ such that

$$\forall\,\xi\in R^d,\ |\xi|_\infty\geqslant 1:\quad \|\varphi(\xi)\|_X\leqslant C|\xi|_\infty^{-n} \quad (\text{here } |\xi|_\infty=\max\{|\xi_1|,\ldots,|\xi_d|\}).$$

With this we have

$$\Big|\Big(\int_{[-2^{k-1},2^{k-1})^d}\Big\|\sum_{\varkappa\in Z^d}\varphi(\xi+2^k\varkappa)\Big\|^p_X d\xi\Big)^{1/p}-\Big(\int_{[-2^{k-1},2^{k-1})^d}\|\varphi(\xi)\|^p_X d\xi\Big)^{1/p}\Big|$$

$$\leqslant \sum_{\varkappa\in Z^d\setminus\{0\}}\Big(\int_{[-2^{k-1},2^{k-1})^d}\|\varphi(\xi+2^k\varkappa)\|^p_X d\xi\Big)^{1/p}$$

$$\leqslant 2^{dk/p}C\sum_{\varkappa\in Z^d\setminus\{0\}}\big(2^k(|\varkappa|_\infty-1/2)\big)^{-n}\leqslant C2^{k(d/p-n)}\underbrace{\sum_{\varkappa\in Z^d\setminus\{0\}}(|\varkappa|_\infty-1/2)^{-n}}_{<\infty}.$$

We see that for large k this tends to zero, so that the above equalities can be continued:

$$= \lim_{k\to\infty}\Big(\int_{[-2^{k-1},2^{k-1})^d}\|\varphi(\xi)\|^p_X d\xi\Big)^{1/p} = \|\varphi\|_{L_p(X)}. \ ■$$

We need a well-known convergence theorem for Fourier multipliers. It is a simple consequence of Lebesgue's dominated convergence theorem and Fatou's lemma.

LEMMA 2.2. *Let $(\psi_n)_{n\in N}$ be a sequence of $L_p(X)$-Fourier multipliers that converges almost everywhere to ψ. Then*

$$\|\psi\|_{M_p(X)}\leqslant\sup\{\|\psi_n\|_{M_p(X)}\mid n\in N\}.$$

Proof of Proposition 3. First we will assume that $\psi\colon R^d\to C$ is a rapidly decreasing function. Then for any $\varphi\in\mathscr{S}(X)$ we have $\mathscr{F}^{-1}(\psi\mathscr{F}\varphi)=(\mathscr{F}^{-1}\psi)*\varphi\in\mathscr{S}(X)$. For this function we can use Lemma 2.1:

$$\|(\mathscr{F}^{-1}\psi)*\varphi\|_{L_p(X)} = \lim_{k\to\infty}\|((\mathscr{F}^{-1}\psi)*\varphi)^{\sim}_{p,k}\|_{\tilde L_p(X)}$$

$$= \lim_{k\to\infty}\Big\|2^{-dk/p'}\sum_{\varkappa\in Z^d}\psi(\varkappa/2^k)(\mathscr{F}\varphi)(\varkappa/2^k)\,e^{2\pi i\langle\varkappa,\cdot\rangle}\Big\|_{\tilde L_p(X)}$$

$$\leqslant \sup\{\|(\psi(\varkappa/2^k))_{\varkappa\in Z^d}\|_{\tilde M_p(X)}\mid k\in N\}\underbrace{\lim_{k\to\infty}\|\varphi^{\sim}_{k,p}\|_{\tilde L_p(X)}}_{=\|\varphi\|_{L_p(X)}}.$$

To estimate $\|(\psi(\varkappa/2^k))_{\varkappa\in Z^d}\|_{\tilde M_p(X)}$ we will use the two versions of Proposition 2 to get the two versions of Proposition 3. In the case of the first version we get

$$\|(\psi(\varkappa/2^k))_{\varkappa\in Z^d}\|_{\tilde M_p(X)}\leqslant C\sup\{\operatorname*{Var}_{D_n}(\psi(\varkappa/2^k))_{\varkappa\in Z^d}\mid n\in N_0\}=(*).$$

To estimate this variation we will apply Lemma 1.4. Since the sizes of the edges of the two subcubes of D_n are proportional to 2^n we get

$$(*)\leqslant C_1\sup\{2^{n|\gamma|}|(D^\gamma\psi(\cdot/2^k))(\xi)|\mid\xi\in D_n,\ \gamma\leqslant(1,\ldots,1),\ n\in N_0\}$$

$$= C_1\sup\{2^{(n-k)|\gamma|}|(D^\gamma\psi)(\xi/2^k)|\mid\xi\in D_n,\ \gamma\leqslant(1,\ldots,1),\ n\in N_0\}.$$

By changing the variable ξ to $\xi/2^k$, this is

$$\leqslant C_2\sup\{|\xi|^{|\gamma|}|(D^\gamma\psi)(\xi)|\mid\xi\in R^d\setminus\{0\},\ \gamma\leqslant(1,\ldots,1)\}.$$

In the case of the second version we have

$$\|(\psi(\varkappa/2^k))_{\varkappa\in \mathbf{Z}^d}\|_{\tilde{M}_p(X)} \leqslant C \sup\{\operatorname*{Var}_{\Delta_\nu}(\psi(\varkappa/2^k))_{\varkappa\in\mathbf{Z}^d} \mid \nu\in N_0^d\} = (*).$$

Now the sizes of the edges of the 2^d subcuboids of Δ_ν in the jth coordinate direction are proportional to 2^{ν_j}. Writing 2^ν for the vector $(2^{\nu_1}, \ldots, 2^{\nu_d})$ and taking exponentials coordinatewise we find

$$(*) \leqslant C_1 \sup\{(2^\nu)^\gamma |(D^\gamma\psi(\cdot/2^k))(\xi)| \mid \nu\in N_0^d,\ \gamma\leqslant(1,\ldots,1),\ \xi\in\Delta_\nu\}$$

$$\leqslant C_2 \sup\{|\xi^\gamma D^\gamma\psi(\xi)| \mid \xi\in \mathbf{R}^d\setminus\{0\},\ \gamma\leqslant(1,\ldots,1)\}.$$

This proves the proposition in the case that $\psi\in\mathcal{S}$.

As a second step suppose that ψ is infinitely often differentiable. Since both versions can be treated in the same way, we just consider the first. Fix an infinitely often differentiable ϱ with compact support such that $\varrho(0)=1$. Define for all $\varepsilon>0$, $\varrho_\varepsilon = \varrho(\varepsilon\cdot)$. Then $\varrho_\varepsilon\psi$ is in $\mathcal{S}$ and converges pointwise to ψ as ε goes to 0. By Lemma 2.2 we have

$$\begin{aligned}
\|\psi\|_{M_p(X)} &\leqslant \sup\{\|\varrho_\varepsilon\psi\|_{M_p(X)} \mid \varepsilon>0\}\\
&\leqslant \sup\{|\xi|^{|\gamma|}|D^\gamma(\varrho_\varepsilon\psi)(\xi)| \mid \xi\in\mathbf{R}^d\setminus\{0\},\ \gamma\leqslant(1,\ldots,1),\ \varepsilon>0\}\\
&\leqslant \sup\Big\{\sum_{\gamma=\alpha+\beta} C(\alpha,\beta)|\xi|^{|\alpha|}|D^\alpha\psi(\xi)|\,|\xi|^{|\beta|}|D^\beta\varrho_\varepsilon(\xi)| \mid \xi\in\mathbf{R}^d\setminus\{0\},\\
&\qquad \gamma\leqslant(1,\ldots,1),\ \varepsilon>0\Big\} \quad \text{(by Leibniz's formula)}\\
&\leqslant C\sup\{|\xi|^{|\alpha|}|D^\alpha\psi(\xi)| \mid \xi\in\mathbf{R}^d\setminus\{0\},\ \alpha\leqslant(1,\ldots,1)\}\\
&\qquad \underbrace{\times\sup\{\underbrace{|\xi|^{|\beta|}|D^\beta\varrho_\varepsilon(\xi)|}_{=|\varepsilon\xi|^{|\beta|}|D^\beta\varrho(\varepsilon\xi)|} \mid \xi\in\mathbf{R}^d\setminus\{0\},\ \beta\leqslant(1,\ldots,1),\ \varepsilon>0\}.}_{<\infty}
\end{aligned}$$

Finally, take ψ arbitrary as in the proposition. Choose ϱ infinitely often differentiable such that $\varrho\geqslant 0$, $\|\varrho\|_{L_1}=1$ and $\operatorname{supp}\varrho\subseteq[-1,1]^d$ and define $\varrho_\varepsilon = \varepsilon^{-d}\varrho(\cdot/\varepsilon)$. Then $\psi*\varrho_\varepsilon$ is infinitely often differentiable and converges almost everywhere to ψ as ε converges to 0. Again we have to apply the convergence lemma:

$$\|\psi\|_{M_p(X)} \leqslant \sup\{\|\psi*\varrho_\varepsilon\|_{M_p(X)} \mid \varepsilon>0\} = (*).$$

For the first version we find

$$(*) \leqslant C\sup\{|\xi|^{|\gamma|}|(D^\gamma\psi*\varrho_\varepsilon)(\xi)| \mid \varepsilon>0,\ \xi\in\mathbf{R}^d\setminus\{0\},\ \gamma\leqslant(1,\ldots,1)\}.$$

We distinguish two cases:

If $|\xi|_\infty\leqslant 2\varepsilon$ then

$$|\xi|_\infty^{|\gamma|}|D^\gamma(\psi*\varrho_\varepsilon)(\xi)| \leqslant (2\varepsilon)^{|\gamma|}|(\psi*D^\gamma\varrho_\varepsilon)(\xi)| \leqslant (2\varepsilon)^{|\gamma|}\|\psi\|_{L_\infty}\|D^\gamma\varrho_\varepsilon\|_{L_1} = 2^{|\gamma|}\|\psi\|_{L_\infty}\|D^\gamma\varrho\|_{L_1}.$$

If $|\xi|_\infty>2\varepsilon$ then since ϱ_ε has its support in $[-\varepsilon,\varepsilon]^d$

$$\begin{aligned}
|\xi|_\infty^{|\gamma|}|D^\gamma(\psi*\varrho_\varepsilon)(\xi)| &\leqslant |\xi|_\infty^{|\gamma|}\|\varrho_\varepsilon\|_{L_1}\sup\{|D^\gamma\psi(\eta)| \mid |\eta-\xi|_\infty\leqslant\varepsilon\}\\
&\leqslant \underbrace{\sup\{(|\xi|/|\eta|)^{|\gamma|} \mid |\xi-\eta|_\infty<\varepsilon\}}_{\leqslant 2^{(d/2+1)|\gamma|}}\|\varrho\|_{L_1}\sup\{|\eta|^{|\gamma|}|D^\gamma\psi(\eta)| \mid \eta\in\mathbf{R}^d\setminus\{0\}\}.
\end{aligned}$$

That means in any case we have

$$(*) \leqslant C\sup\{|\xi|^{|\gamma|}|D^\gamma\psi(\xi)| \mid \xi\in\mathbf{R}^d\setminus\{0\},\ \gamma\leqslant(1,\ldots,1)\}.$$

For the second version we need to argue a little differently:

$$(*) \leqslant C\sup\{|\xi^\gamma D^\gamma(\psi*\varrho_\varepsilon)(\xi)| \mid \varepsilon>0,\ \xi\in\mathbf{R}^d\setminus\{0\},\ \gamma\leqslant(1,\ldots,1)\}.$$

Again we distinguish two cases:

If $|\xi|_\infty\leqslant\varepsilon$ then

$$|\xi^\gamma D^\gamma(\psi*\varrho_\varepsilon)(\xi)| \leqslant \varepsilon^{|\gamma|}|(\psi*D^\gamma\varrho_\varepsilon)(\xi)| \leqslant \varepsilon^{|\gamma|}\|\psi\|_{L_\infty}\|D^\gamma\varrho_\varepsilon\|_{L_1} = \|\psi\|_{L_\infty}\|D^\gamma\varrho\|_{L_1}.$$

If $|\xi|_\infty>\varepsilon$ then we use the binomial formula to get

$$|\xi^\gamma D^\gamma(\psi*\varrho_\varepsilon)(\xi)| = \Big|\sum_{\gamma=\alpha+\beta} C(\alpha,\beta)\big((\eta^\alpha D^\alpha\psi(\eta))*(\eta^\beta D^\beta\varrho_\varepsilon(\eta))\big)(\xi)\Big|.$$

Since $|\xi|_\infty>\varepsilon$ and $\operatorname{supp}\varrho_\varepsilon\subseteq[-\varepsilon,\varepsilon]^d$ the above convolution can be written as an integral as follows:

$$\begin{aligned}
&= \Big|\sum_{\gamma=\alpha+\beta} C(\alpha,\beta)\int_{[-\varepsilon,\varepsilon]^d}(\xi-\eta)^\alpha D^\alpha\psi(\xi-\eta)\,\eta^\beta D^\beta\varrho_\varepsilon(\eta)\,d\eta\Big|\\
&\leqslant C\sup\{|\xi^\alpha D^\alpha\psi(\xi)| \mid \xi\in\mathbf{R}^d\setminus\{0\},\ \alpha\leqslant(1,\ldots,1)\}\\
&\qquad\times\sup\{\|\eta^\beta D^\beta\varrho_\varepsilon(\eta)\|_{L_1} \mid \varepsilon>0,\ \beta\leqslant(1,\ldots,1)\}.
\end{aligned}$$

The last supremum equals actually $\sup\{\|\eta^\beta D^\beta\varrho(\eta)\|_{L_1} \mid \beta\leqslant(1,\ldots,1)\}$. Therefore also the second version is proved. ∎

For the proof of Proposition 4 we need the following fact:

Lemma 2.3. *For any Banach space X and any $1<p<\infty$ we have*

$$\{\varphi\in\mathcal{S}(X) \mid \operatorname{supp}\mathcal{F}\varphi \text{ compact},\ 0\notin\operatorname{supp}\mathcal{F}\varphi\} \underset{\text{dense}}{\subset} L_p(X).$$

Sketch of proof. Fix an infinitely often differentiable function ϱ: $\mathbf{R}^d\to\mathbf{C}$ with compact support which equals 1 on a neighbourhood of 0. For $\vartheta>0$ put $\varrho_\vartheta=\varrho(\vartheta\cdot)$. We first show that $\{\varphi\in\mathcal{S}(X) \mid \mathcal{F}\varphi$ has compact support} is dense in $L_p(X)$ for $1 \leqslant p < \infty$. For this it suffices to show that for every infinitely often differentiable $\varphi\colon \mathbf{R}^d \to X$ with compact support

(2.1) $$\lim_{\vartheta\to 0} (\mathcal{F}^{-1}\varrho_\vartheta) * \varphi = \varphi \quad \text{in the } L_p(X)\text{-norm}.$$

Note that the Fourier transforms of the functions on the left-hand side have compact support. To prove (2.1) one first uses Lebesgue's theorem to prove almost everywhere convergence and then appeals to Lebesgue's theorem a second time to get convergence in the norm. To finally prove Lemma 2.3 one notes that for $1 < p < \infty$

$$\limsup_{\vartheta\to\infty} \|(\mathcal{F}^{-1}\varrho_\vartheta) * \varphi\|_{L_p(X)} \leqslant \limsup_{\vartheta\to\infty} \|\varphi\|_{L_1(X)} \|\mathcal{F}^{-1}\varrho_\vartheta\|_{L_p}$$
$$= \limsup_{\vartheta\to\infty} \|\varphi\|_{L_1(X)} \vartheta^{-d/p'} \|\mathcal{F}^{-1}\varrho\|_{L_p(X)} = 0 \quad (1/p+1/p' = 1).$$

This means

(2.2) $$\lim_{\vartheta\to\infty} \varphi - (\mathcal{F}^{-1}\varrho_\vartheta) * \varphi = \varphi \quad \text{in the } L_p(X)\text{-norm}.$$

Note that the origin is not in the support of the Fourier transform of the left-hand side. Altogether (2.1) and (2.2) imply the lemma. ∎

As in the periodic case an $L_p(X)$-Fourier multiplier ψ induces an operator T_ψ on $L_p(X)$. It is easy to prove that the characteristic function of the positive cone $\mathbf{R}^d_+$ is an element of $M_p(X)$ for UMD-spaces X and $1 < p < \infty$. The corresponding operator is also called the *Riesz projection* and will be denoted by $\mathbf{R}$. We abbreviate S_n for the operator $T_{\chi_{D_n}}$. Similarly to the periodic case we have

(2.3) $$\|S_n\|_{\mathbf{R}} \leqslant 2\cdot 2^d \|\mathbf{R}\|_{L(L_p(X))}.$$

Of course a similar result holds for the Δ_ν.

Proof of Proposition 4. We choose an infinitely often differentiable ϱ with compact support. Furthermore, we require that $\varrho|_{D_1} \equiv 1$ (resp. $\varrho|_{\Delta(0,\ldots,0)} \equiv 1$ for the second version) and $\operatorname{supp}\varrho \subseteq \bigcup_{|n-1|\leqslant d} D_n$ (resp. $\operatorname{supp}\varrho \subseteq \bigcup_{|\nu|_\infty\leqslant 1}\Delta_\nu$). For $n = rd+j \in \mathbf{Z}$ (resp. $\nu \in \mathbf{Z}^d$) and $\xi = (\xi_1, \ldots, \xi_d)$ put

$$\varrho_n(\xi) = \varrho(2^r\xi_j, 2^{r+1}\xi_1, \ldots, 2^{r+1}\xi_{j-1}, 2^r\xi_{j+1}, \ldots, 2^r\xi_d)$$
$$(\text{resp. } \varrho_\nu(\xi) = \varrho(2^{\nu_1}\xi_1, \ldots, 2^{\nu_d}\xi_d)).$$

These functions have the following properties:

1) $\varrho_n|_{D_n} \equiv 1$ (resp. $\varrho_\nu|_{\Delta_\nu} \equiv 1$).

2) $\operatorname{supp}\varrho_n \subseteq \bigcup_{|k-n|\leqslant d} D_k$ (resp. $\operatorname{supp}\varrho_\nu \subseteq \bigcup_{|\varkappa-\nu|_\infty\leqslant 1}\Delta_\varkappa$).

3) $$\sup\{|\xi|^{|\gamma|}|D^\gamma\varrho_n(\xi)| \mid \xi \in \mathbf{R}^d\setminus\{0\},\ \gamma \leqslant (1, \ldots, 1)\}$$
$$\leqslant 2^d \sup\{|\xi|^{|\gamma|}|D^\gamma\varrho(\xi)| \mid \xi \in \mathbf{R}^d\setminus\{0\},\ \gamma \leqslant (1, \ldots, 1)\}$$
$$(\text{resp. } \sup\{|\xi^\gamma D^\gamma\varrho_\nu(\xi)| \mid \xi \in \mathbf{R}^d\setminus\{0\},\ \gamma \leqslant (1, \ldots, 1)\}$$
$$\leqslant \sup\{|\xi^\gamma D^\gamma\varrho(\xi)| \mid \xi \in \mathbf{R}^d\setminus\{0\},\ \gamma \leqslant (1, \ldots, 1)\}).$$

For simplicity we only treat the first version. The second can be handled analogously.

For $t \in [0, 1]$ put $\psi_t = \sum_{n\in\mathbf{Z}} r_n(t)\varrho_n$. Because of 2) this sum is locally finite on $\mathbf{R}^d\setminus\{0\}$ and an application of Proposition 3 will show

$$\|\psi_t\|_{M_p(X)} \leqslant C\sup\{|\xi|^{|\gamma|}|D^\gamma\psi_t(\xi)| \mid \xi \in \mathbf{R}^d,\ \gamma \leqslant (1, \ldots, 1)\}$$
$$\leqslant C_1\sup\{|\xi|^{|\gamma|}|D^\gamma\varrho_n(\xi)| \mid \xi \in \mathbf{R}^d,\ \gamma \leqslant (1, \ldots, 1),\ n \in \mathbf{Z}\} = C_2 < \infty.$$

Because of Lemma 2.3 it suffices to show the inequality (0.2) for all $\varphi \in \mathcal{S}(X)$ whose Fourier transforms have compact support and do not contain the origin. Observe that for such φ the sum

$$T_{\psi_t}(\varphi) = \sum_{n\in\mathbf{Z}} r_n(t)(\mathcal{F}^{-1}\varrho_n) * \varphi$$

is finite. Furthermore, because of property 1) we have $S_n(T_{\varrho_n}(f)) = S_n(f)$. An application of the generalized contraction principle shows

$$\Big(\int_0^1 \Big\|\sum_{n\in\mathbf{Z}} r_n(t)S_n(f)\Big\|^p_{L_p(X)} dt\Big)^{1/p} \leqslant \Big(\int_0^1 \Big\|\sum_{n\in\mathbf{Z}} r_n(t)S_n(T_{\varrho_n}(f))\Big\|^p_{L_p(X)} dt\Big)^{1/p}$$
$$\leqslant C\sup\{\|S_n\|_{\mathbf{R}} \mid n \in \mathbf{Z}\}\Big(\int_0^1 \Big\|\sum_{n\in\mathbf{Z}} r_n(t)T_{\varrho_n}(f)\Big\|^p_{L_p(X)} dt\Big)^{1/p}$$
$$= C_1\Big(\int_0^1 \|(\mathcal{F}^{-1}\psi_t) * f\|^p_{L_p(X)} dt\Big)^{1/p} \leqslant C_2\|f\|_{L_p(X)}.$$

The same duality argument as at the end of the proof of the first version of Proposition 1 can now be used to prove the full statement of Proposition 4. ∎

We turn to Proposition 5. For simplicity we will define the variation only for functions with continuous derivatives. The reader may convince himself that this notion can be extended to the functions described in Proposition 5. We need some additional notation. Let $\xi \in \mathbf{R}^d$ and let γ be a multiindex with $0 \neq \gamma \leqslant (1, \ldots, 1)$. Let $q_1, \ldots, q_r$ be the coordinate directions q for which $\gamma_q = 1$. We put $\xi_\gamma = (\xi_{q_1}, \ldots, \xi_{q_r})$. Furthermore, let $(\alpha, \beta) = \{\xi \in \mathbf{R}^d \mid \alpha < \xi < \beta\}$ and $[\alpha, \beta) = \{\xi \in \mathbf{R}^d \mid \alpha \leqslant \xi < \beta\}$.

For $\sigma \in (\alpha, \beta)$ and $\xi \in (\alpha_\gamma, \beta_\gamma)$ define $\xi_\gamma^\sigma = ((\xi_\gamma^\sigma)_1, \ldots, (\xi_\gamma^\sigma)_d)$ by

$$(\xi_\gamma^\sigma)_j = \begin{cases} \sigma_j, & j \notin \{q_1, \ldots, q_r\}, \\ \xi_s, & j = q_s. \end{cases}$$

DEFINITION 2.1. Let $\alpha, \beta \in \boldsymbol{R}^d$, $\alpha \leqslant \beta$, and let ψ be a function with continuous derivatives $D^\gamma \psi$, $\gamma \leqslant (1, \ldots, 1)$, on (α, β). We put

$$\operatorname*{var}_{(\alpha,\beta)} \psi = \inf \{\psi(\sigma) + \sum_{0 \neq \gamma \leqslant (1,\ldots,1)} \int_{(\alpha_\gamma,\beta_\gamma)} |D^\gamma \psi(\xi_\gamma^\sigma)| \, d\xi \mid \sigma \in (\alpha, \beta)\}.$$

Again $\boldsymbol{D}_n^\circ$ and Δ_ν° are naturally unions of such cuboids (α_i, β_i) and we put

$$\operatorname*{Var}_{\boldsymbol{D}_n^\circ} \psi = \sum_i \operatorname*{var}_{(\alpha_i,\beta_i)} \psi, \qquad \operatorname*{Var}_{\Delta_\nu^\circ} \psi = \sum_i \operatorname*{var}_{(\alpha_i,\beta_i)} \psi.$$

Now Proposition 5 can be deduced from Proposition 4 in the same way as Proposition 2 was deduced from Proposition 1. We just need to establish

LEMMA 2.4. *Let ψ, α, β be as in Definition 2.1. Then*

$$\|T_\psi\|_{\boldsymbol{R}} \leqslant C \operatorname*{var}_{(\alpha,\beta)} \psi.$$

Proof. Split the cuboid (α, β) into 2^d subcuboids all having the new vertex σ. We only treat the part $[\sigma, \beta)$, since the other can be handled in the same way. Now the lemma is a consequence of the elementary formula

$$\psi = \sum_{\gamma \leqslant (1,\ldots,1)} \psi_\gamma \quad \text{where}$$

$$\psi_\gamma(\xi) = \int_{[\sigma_\gamma,\beta_\gamma)} \chi_{[\eta_\gamma^\sigma,\beta)}(\xi) \, D^\gamma \psi(\eta_\gamma^\sigma) \, d\eta, \qquad \xi \in [\sigma, \beta).$$

Observe the difficulty arising by the fact that the mapping $\eta \mapsto T_{\chi_{[\eta_\gamma^\sigma,\beta)}}$ is not Bochner integrable. Nevertheless, for all $f \in L_p(X)$ the pointwise evaluated mappings

$$\eta \mapsto T_{\chi_{[\eta_\gamma^\sigma,\beta)}} f$$

are continuous and hence Bochner integrable. Thus T_{ψ_γ} can be approximated in the strong operator topology by Riemann sums (see [H/P]) so that we still have

$$\|T_{\psi_\gamma}\|_{\boldsymbol{R}} \leqslant 2^d \int_{[\sigma_\gamma,\beta_\gamma)} |D^\gamma \psi(\eta_\gamma^\sigma)| \, d\eta. \ \blacksquare$$

3. A counterexample. To prove Proposition 6 it suffices to show that the bracketed version of Proposition 4 fails for S_p, $1 < p < \infty$ and $p \neq 2$. Furthermore, we restrict to the case $d = 2$. Fix a nonzero function $\varphi \colon \boldsymbol{R} \to \boldsymbol{C}$, $\varphi \in \mathscr{S}$, such that $\operatorname{supp} \mathscr{F}\varphi \subseteq [1/2, 1]$. Let $(e_n) \subseteq l_2$ be any orthogonal system.

For $N \in \boldsymbol{N}$ we define

$$\psi\colon \boldsymbol{R}^2 \to S_p, \qquad \psi(\vartheta, \tau) = \{\sum_{n=1}^N e^{2\pi i 2^n \vartheta} \varphi(\vartheta) e_n\} \otimes \{\sum_{k=1}^N e^{2\pi i 2^k \tau} \varphi(\tau) e_k\}.$$

The norm can be computed easily:

$$\begin{aligned} \|\psi\|_{L_p(S_p)} &= \Big(\int_{\boldsymbol{R}}\int_{\boldsymbol{R}} \Big\|\{\sum_{n=1}^N e^{2\pi i 2^n \vartheta} \varphi(\vartheta) e_n\} \otimes \{\sum_{k=1}^N e^{2\pi i 2^k \tau} \varphi(\tau) e_k\}\Big\|_{S_p}^p \, d\vartheta \, d\tau\Big)^{1/p} \\ &= \Big(\int_{\boldsymbol{R}}\int_{\boldsymbol{R}} \Big\|\sum_{n=1}^N e^{2\pi i 2^n \vartheta} \varphi(\vartheta) e_n\Big\|_{l_2}^p \Big\|\sum_{k=1}^N e^{2\pi i 2^k \tau} \varphi(\tau) e_k\Big\|_{l_2}^p \, d\vartheta \, d\tau\Big)^{1/p} \\ &= N \|\varphi\|_{L_p}^2. \end{aligned}$$

On the other hand, we have

$$(T_{\chi_{\Delta(n,k)}} \psi)(\vartheta, \tau) = \begin{cases} e^{2\pi i (2^n \vartheta + 2^k \tau)} \varphi(\vartheta) \varphi(\tau) e_n \otimes e_k & \text{if } 1 \leqslant n, k \leqslant N, \\ 0 & \text{otherwise}. \end{cases}$$

If the bracketed version of Proposition 4 would hold, we would have

$$\begin{aligned} N \|\varphi\|_{L_p}^2 = \|\psi\|_{L_p(S_p)} &\leqslant C\Big(\int_0^1 \Big\|\sum_{n,k \in \boldsymbol{Z}} r_{(n,k)}(t) \, T_{\chi_{\Delta(n,k)}} \psi\Big\|_{L_p(S_p)}^p dt\Big)^{1/p} \\ &= C\Big(\int_{\boldsymbol{R}^2} \int_0^1 \Big\|\sum_{n,k=1}^N r_{(n,k)}(t) \, e^{2\pi i (2^n \vartheta + 2^k \tau)} \varphi(\vartheta) \varphi(\tau) e_n \otimes e_k\Big\|_{S_p}^p dt \, d\vartheta \, d\tau\Big)^{1/p} \end{aligned}$$

and by the contraction principle for Rademacher averages

$$\begin{aligned} &\leqslant 2C\Big(\int_{\boldsymbol{R}^2} \int_0^1 \Big\|\sum_{n,k=1}^N r_{(n,k)}(t) \, \varphi(\vartheta) \varphi(\tau) e_n \otimes e_k\Big\|_{S_p}^p dt \, d\vartheta \, d\tau\Big)^{1/p} \\ &= 2C \|\varphi\|_{L_p}^2 \Big(\int_0^1 \Big\|\sum_{n,k=1}^N r_{(n,k)}(t) \, e_n \otimes e_k\Big\|_{S_p}^p dt\Big)^{1/p}. \end{aligned}$$

In the case $p = 2$ the last expression is just N. If $p = \infty$ the average can be estimated by $C_1 N^{1/2}$ using Chevet's inequality (see [Che]). Then for general $p \in [2, \infty)$ by using the interpolation inequality $\|\cdot\|_{S_p} \leqslant \|\cdot\|_{S_2}^{2/p} \|\cdot\|_{S_\infty}^{1-2/p}$ one obtains the bound $C_2 N^{1/2+1/p}$. This shows that Proposition 4 cannot hold for $p > 2$. By duality the proposition also fails for $p < 2$. $\blacksquare$

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Quotients and interpolation spaces of stable Banach spaces

by

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Abstract. We show the stability (in the sense of Krivine–Maurey) of the quotients of several reflexive or superreflexive stable atomic spaces. We show that the space $L(E)$ is stable provided L is a stable r.i. space and E a stable Banach space. We study the stability of interpolation spaces $[L_1, L_2]_{\theta,X}$ when L_1, L_2 are stable r.i. function spaces and X a stable atomic lattice.

Introduction. In this paper we study the stability of quotients and interpolation spaces of stable Banach spaces. The notion of stability for a Banach space was introduced by Maurey and Krivine [KM]; recall that a Banach space E is *stable* if for any bounded sequences (x_n), (y_m) and ultrafilters $\mathcal{U}$, $\mathcal{V}$ we have

$$\lim_{n,\mathcal{U}}\lim_{m,\mathcal{V}}\|x_n+y_m\| = \lim_{m,\mathcal{V}}\lim_{n,\mathcal{U}}\|x_n+y_m\|.$$

This property is clearly hereditary but does not behave well with respect to other standard operations on Banach spaces. Quotients and duals of stable Banach spaces, even reflexive, may not be stable, as was shown in [G]. Similarly, no interesting result about interpolation of stable Banach spaces was known (except the stability of Lorentz spaces, cf. [R]). In fact, if E, F form an interpolation pair of stable Banach spaces, the space $E+F$ (with norm $\|x\| = \inf\{\|e\|+\|f\| \mid e \in E,\ f \in F,\ x = e+f\}$) is a quotient of the direct sum $E \oplus_1 F$ and therefore may probably not be stable. If E is a stable Banach space, it is known that $L_p(E)$, $1 \leqslant p < \infty$, is also stable ([KM]). In fact, in all cases where a lattice L is known to be stable, the same is true for $L(E)$ if E is a stable Banach space. This is the case for L_p spaces, Orlicz spaces, Lorentz spaces. But there was no general result in this direction, except in the case of atomic lattices (i.e. spaces with 1-unconditional basis, cf. [BM] and [B]).

Here we present some positive results in these three directions. For the

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