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On Muckenhoupt's classes of weight functions

by

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Abstract. It is proved that if ω is in the weight class A_p on a cube Q, then ω^*, the nonincreasing rearrangement, lies in A_p on the interval $[0, |Q|]$. This gives another proof that ω then also lies in the more restricted class $A_{p-\varepsilon}$ for some $\varepsilon > 0$. An estimate of ε is given as well as a new characterization of A_p. The doubling condition $\omega(Q) \geqslant c\omega(2Q)$ is strictly weaker than the condition that $\omega \in A_\infty$. A new counterexample, comparatively simple, is given to demonstrate this fact.

1. Notations and introduction. The *Lebesgue measure* of a set E in $\boldsymbol{R}^n$ is denoted by $|E|$. ω will always be a locally integrable, nonnegative, real-valued function on $\boldsymbol{R}^n$ and we use the notation

$$\omega(E) = \int_E \omega(x)\,dx.$$

We denote by ω_* and ω^* the *nondecreasing* and *nonincreasing rearrangement* respectively of the function ω. E' is the *set of density points* of E. The *mean value* of a function f on a set E is written

$$⨍ f(x)\,dx = |E|^{-1} \int_E f(x)\,dx.$$

The *Muckenhoupt classes* A_p, $p > 1$, were introduced in [4]. A_p is defined as the class of locally integrable nonnegative functions ω that satisfy

$$⨍_Q \omega\,dx \Big(⨍_Q \omega^{-1/(p-1)}\,dx\Big)^{p-1} < A \tag{1}$$

for every cube Q and some constant A. We then say that *ω belongs to A_p with constant A*. If the cubes Q are restricted to lie within a fixed cube Q_0 we say that *ω belongs to A_p in Q_0*.

The paper consists of a proof that $\omega \in A_p$ in Q implies that ω^* (or equivalently ω_*) belongs to A_p on $[0, |Q|]$ (Theorem 1). Using a couple of elementary function-theoretic lemmas we proceed to prove the well-known fact that $\omega \in A_p$ implies $\omega \in A_{p-\varepsilon}$ for some $\varepsilon > 0$. An estimate of ε comes as bonus (Theorem 2).As a corollary we obtain another characterization of A_p. Theorem 3 is a strengthening of a theorem by Muckenhoupt [5], connecting A_∞ and A_p. A consequence of Theorem 3 is that A_∞ may be defined as the class of locally integrable functions for which there exist constants r and k such that

$$|E| > r|Q| \Rightarrow \omega(E) \geqslant k\omega(Q)$$

holds for any subset E of any cube Q ($r < 1$, $k > 0$).

It is an interesting fact, proved by Fefferman and Muckenhoupt in [3], that A_p is a strictly stronger condition that the doubling condition $\omega(Q) \geqslant c\omega(2Q)$, where $2Q$ is a cube with twice the side-length of Q and the same center. We end this paper by giving a simpler such counterexample.

2. Theorems and proofs. We start by proving a lemma which gives a covering of a set E by disjoint (dyadic) cubes in each of which E takes up roughly the same portion (and equally important, the complement of E takes up at least some fixed portion). This lemma has been used earlier by the author [6]. A similar lemma can be found in the work [1] by Bagby and Kurtz.

LEMMA 1. *Let E be a measurable set with finite measure and ϱ a real number in $]0, 1[$. Suppose that E is contained in a cube Q (or just $\mathbf{R}^n$) and that $|E| \leqslant \varrho|Q|$. Then there exists a sequence $\{Q_\nu\}_1^\infty$ of dyadic cubes, dyadic with respect to Q, such that the cubes Q_ν have disjoint interiors and*

1) $2^{-n}\varrho < |Q_\nu \cap E|/|Q_\nu| \leqslant \varrho$,

2) $\bigcup_{\nu=1}^{\infty} Q_\nu \supset E'$,

where E' is the set of density points of E.

Proof. Let x be a point of density of E, $x \in E'$. Then there exists a dyadic cube $Q_{x,1}$ containing x, such that $|Q_{x,1} \cap E| > \varrho|Q_{x,1}|$. We double the cube $Q_{x,1}$ dyadically and obtain $Q_{x,2}$, double that cube, etc. Since $|E|$ is finite we will after finitely many doublings reach a first cube $Q_{x,p}$ with "density" $|Q_{x,p} \cap E|/|Q_{x,p}| \leqslant \varrho$. Since $|Q_{x,p-1} \cap E|/|Q_{x,p-1}| > \varrho$ we have

$$|Q_{x,p} \cap E| > |Q_{x,p-1} \cap E| > \varrho|Q_{x,p-1}| = 2^{-n}\varrho|Q_{x,p}|, \quad \text{i.e.}$$

$$2^{-n}\varrho|Q_{x,p}| < |Q_{x,p} \cap E| \leqslant \varrho|Q_{x,p}|.$$

This procedure can be carried out for every $x \in E'$. We obtain a family of cubes $\{Q_{x,p}\}_{x\in E'}$. Two such cubes have disjoint interiors unless one of them is contained in the other. We can number the cubes by size, disregard the cubes which are subsets of larger cubes in the family and thereby obtain the sequence claimed in the lemma.

THEOREM 1. *Let $\omega(x)$ belong to A_p in Q_0 with constant A. Then $\omega^*(t)$ (or $\omega_*(t)$) belongs to A_p in $[0, |Q_0|]$ with constant $A^* = 2A\cdot 2^{(n+1)(p-1)}$.*

Proof. Let a be a positive number and put

$$E = \{x \in Q_0;\ \omega(t) < a\}.$$

Cover E', the density set of E, by a union of disjoint cubes in Q_0, $\bigcup_{\nu=1}^\infty Q_\nu$, such that

$$2^{-n-1}|Q_\nu| < |E \cap Q_\nu| \leqslant \tfrac{1}{2}|Q_\nu|.$$

($|E| \leqslant \frac{1}{2}|Q_0|$ is assumed.) This means that

$$\sum|Q_\nu| < 2^{n+1}|E|. \tag{1}$$

By the definition of A_p

$$\int_{Q_\nu} \omega(x)^{-1/(p-1)}\,dx \leqslant A^{1/(p-1)}|Q_\nu|^{p/(p-1)}\Big(\int_{Q_\nu}\omega(x)\,dx\Big)^{-1/(p-1)}.$$

Since $\omega(x) \geqslant a$ on at least half of Q_ν, we obtain

$$\int_{Q_\nu} \omega(x)^{-1/(p-1)}\,dx \leqslant (2A)^{1/(p-1)}|Q_\nu|\,a^{-1/(p-1)}.$$

A summation gives

$$\int_E \omega(x)^{-1/(p-1)}\,dx \leqslant \int_{\cup Q_\nu} \omega(x)^{-1/(p-1)}\,dx \leqslant (2A)^{1/(p-1)}a^{-1/(p-1)}\sum|Q_\nu|.$$

We use (1) to find

$$\int_E \omega(x)^{-1/(p-1)}\,dx \leqslant (2A)^{1/(p-1)}2^{n+1}a^{-1/(p-1)}|E|. \tag{2}$$

Of course $\int_E \omega(x)\,dx < a|E|$. Combined with (2) this gives

$$⨍_E \omega(x)\,dx\Big(⨍_E \omega(x)^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant 2A\cdot 2^{(n+1)(p-1)}.$$

The arguments which lead to (2) also give

$$⨍_0^\infty \omega_*(t)^{-1/(p-1)}\,dt \leqslant 2^{n+1}(2A)^{1/(p-1)}\omega_*(b)^{-1/(p-1)}.$$

Since

$$⨍_a^b \omega_*(t)^{-1/(p-1)}\,dt \leqslant ⨍_0^b \omega_*(t)^{-1/(p-1)}\,dt \quad \text{and} \quad ⨍_a^b \omega_*(t)\,dt < \omega_*(b),$$

we have

$$⨍_a^b \omega_*(t)\,dt\Big(⨍_a^b \omega_*(t)^{-1/(p-1)}\,dt\Big)^{p-1} \leqslant 2A\cdot 2^{(n+1)(p-1)},$$

which by definition means that ω_* and ω^* belong to A_p on $[0, |Q_0|]$ with constant $2A\cdot 2^{(n+1)(p-1)}$.

Lemma 2. *Let g be an integrable function on $[0, 1]$ and c a constant such that*

$$\int_0^t g(u)\,du \leqslant ctg(t), \qquad 0 < t \leqslant 1. \tag{3}$$

Then

$$\int_0^t g(u)\,du \leqslant t^{1/c}\int_0^1 g(t)\,dt.$$

Proof. Put $\sigma = c^{-1}$, multiply (3) by $t^{-1-\sigma}$ and integrate. We then obtain

$$\int_a^1 t^{-1-\sigma}\,dt\int_0^t g(u)\,du \leqslant c\int_a^1 t^{-\sigma}g(t)\,dt.$$

A change of the order of integration gives

$$\frac{a^{-\sigma}-1}{\sigma}\int_0^a g(u)\,du+\frac{1}{\sigma}\int_a^1 (u^{-\sigma}-1)g(u)\,du \leqslant c\int_a^1 t^{-\sigma}g(t)\,dt.$$

Since $\sigma c = 1$ this can be rewritten as

$$a^{-\sigma}\int_0^a g(u)\,du \leqslant \int_0^1 g(u)\,du,$$

which is the statement of the lemma.

Lemma 3. *Let f be a positive, nondecreasing function on $[0, 1]$ with the property that*

$$\int_0^t f(u)\,du\Big(\int_0^t f(u)^{-1/(p-1)}\,du\Big)^{p-1} \leqslant A_1 \tag{4}$$

holds for some $p > 1$, some A_1 and every $t \in [0, 1]$. Then

$$\int_0^t f(u)\,du \geqslant c_1 t^{p-\sigma(p-1)}\int_0^1 f(u)\,du,$$

where $c_1 = 2^{-\sigma(p-1)}A_1^{-1}$ and $\sigma = (2A_1)^{-1/(p-1)}$.

Proof. Since

$$\int_0^t f(u)\,du > \tfrac{1}{2} f(t/2), \qquad 0 \leqslant t \leqslant 1,$$

we deduce from (4)

$$\int_0^t f(u)^{-1/(p-1)}\,du \leqslant (2A_1)^{1/(p-1)} f(t/2)^{-1/(p-1)}$$

and therefore

$$\int_0^t f(u/2)^{-1/(p-1)}\,du \leqslant (2A_1)^{1/(p-1)} f(t/2)^{-1/(p-1)}.$$

The function $g(u) = f(u/2)^{-1/(p-1)}$ satisfies the assumptions of Lemma 2 and we conclude

$$\int_0^t f(u/2)^{-1/(p-1)}\,du \leqslant t^{\sigma}\int_0^1 f(u/2)^{-1/(p-1)}\,du$$

for $\sigma = (2A_1)^{-1/(p-1)}$. This means that for $0 \leqslant t \leqslant \frac{1}{2}$

$$\int_0^t f(u)^{-1/(p-1)}\,du \leqslant (2t)^{\sigma}\int_0^{1/2} f(u)^{-1/(p-1)}\,du \leqslant (2t)^{\sigma}\int_0^1 f(u)^{-1/(p-1)}\,du. \tag{5}$$

By Hölder's inequality

$$\int_0^t f(u)\,du\Big(\int_0^t f(u)^{-1/(p-1)}\,du\Big)^{p-1} \geqslant t^p.$$

Combined with (5) this gives

$$\int_0^t f(u)\,du \geqslant 2^{-\sigma(p-1)}t^{p-\sigma(p-1)}\Big(\int_0^1 f(u)^{-1/(p-1)}\,du\Big)^{-(p-1)}.$$

We finally apply assumption (4) to obtain

$$\int_0^t f(u)\,du \geqslant \frac{1}{A_1}2^{-\sigma(p-1)}\Big(\int_0^1 f(u)\,du\Big)t^{p-\sigma(p-1)}$$

and the lemma is proved.

We will now prove the old result ([2], p. 243) that if $\omega \in A_p$, then ω already belongs to $A_{p-\varepsilon}$ for some $\varepsilon > 0$.

Theorem 2. *Let ω belong to A_p with constant A. Then $\omega \in A_{p_1}$, for every $p_1 > p-\sigma(p-1)$, with constant*

$$\frac{1}{c_1}\Big(\frac{p_1-1}{p_1-p+\sigma(p-1)}\Big)^{p_1-1},$$

where $c_1 = 2^{-\sigma(p-1)}A_1^{-1}$, $\sigma = (2A_1)^{-1/(p-1)}$ and $A_1 = 2A\cdot 2^{(n+1)(p-1)}$.

Proof. Let Q be an arbitrary cube. By change of scales we may assume that $|Q| = 1$ and $\omega(Q) = 1$. Then, using Theorem 1, we get

$$\int_0^{|E|} \omega_*(t)\,dt\Big(\int_0^{|E|} \omega_*(t)^{-1/(p-1)}\,dt\Big)^{p-1} \leqslant A_1.$$

We now apply Lemma 3 to obtain

$$\int_0^t \omega_*(u)\,du \geqslant c_1 t^{p-\sigma(p-1)},$$

and using the monotonicity of ω_*

$$\omega_*(t) \geqslant c_1 t^{(p-1)(1-\sigma)}.$$

A direct computation gives

$$\Big(\int\limits_Q \omega(x)^{-1/(p_1-1)}\,dx\Big)^{p_1-1} = \Big(\int\limits_0^1 \omega_*(t)^{-1/(p_1-1)}\,dt\Big)^{p_1-1} \leqslant \frac{1}{c_1}\left(\frac{p_1-1}{p_1-p+\sigma(p-1)}\right)^{p_1-1}$$

if $p_1 > p-\sigma(p-1)$. This finishes the proof of Theorem 2.

As a consequence we obtain the following characterization of A_p.

COROLLARY 1. *A locally integrable, nonnegative function ω is in A_p, $p>1$, if, and only if, there exists a number p_1, $1<p_1<p$, and a constant A, such that for every cube Q*

(6) $$\omega(E)/\omega(Q) \geqslant A(|E|/|Q|)^{p_1}$$

for every measurable subset E of Q.

Proof. Suppose ω is in A_p. By Theorem 2 there exists p_1, $1<p_1<p$, such that $\omega\in A_{p_1}$ with constant A^{-1}. Therefore, by Hölder's inequality

$$(|E|/|Q|)^{p_1} = \Big(⨍\limits_Q \chi_E\,dx\Big)^{p_1} \leqslant ⨍\limits_Q \chi_E\omega(x)\,dx\Big(⨍\limits_Q \omega(x)^{-1/(p_1-1)}\,dx\Big)^{p_1-1} \leqslant A^{-1}\frac{\int_E \omega(x)\,dx}{\int_Q \omega(x)\,dx},$$

which proves the necessity of condition (6).

Suppose now that (6) is valid. Take an arbitrary cube Q and let $\omega_*(t)$ be the nondecreasing rearrangement of ω in Q. Then (6) can be interpreted as

$$\int\limits_0^t \omega_*(u)\,du \geqslant A(t/|Q|)^{p_1}\omega(Q),$$

which implies $\omega_*(t) \geqslant (A/|Q|^{p_1})t^{p_1-1}\omega(Q)$. Therefore

$$\omega(Q)\Big(\int\limits_Q \omega(x)^{-1/(p-1)}\,dx\Big)^{p-1} \leqslant \omega(Q)\Big(\int\limits_0^{|Q|} \omega_*(t)^{-1/(p-1)}\,dt\Big)^{p-1} \leqslant \frac{|Q|^{p_1}}{A}\Big(\int\limits_0^{|Q|} t^{-(p_1-1)/(p-1)}\,dt\Big)^{p-1} = \frac{1}{A}\left(\frac{p-1}{p-p_1}\right)^{p-1}|Q|^p.$$

This means, by definition, that $\omega\in A_p$ and we have proved the sufficiency of (6).

Muckenhoupt has in [5] defined A_∞ as the set of all nonnegative locally integrable functions, ω, on $\boldsymbol{R}^n$ which have the property that for every subset E of any cube Q

$$\omega(E)/\omega(Q) < g(|E|/|Q|)$$

for some function g on $[0,1]$ with $\lim_{t\to 0+} g(t) = 0$. In that paper he then proves that $\omega\in A_\infty$ if, and only if, $\omega\in A_p$ for some $p>1$. We will now show that the same conclusion holds true with a seemingly weaker condition.

THEOREM 3. *Suppose ω is a nonnegative, locally integrable function on $\boldsymbol{R}^n$ with the property that there exist two constants, $r<1$ and $k>0$, such that if E is any measurable subset of any cube Q then*

(7) $$|E|/|Q| \geqslant r \quad \text{implies} \quad \omega(E)/\omega(Q) \geqslant k.$$

Then ω is in A_p for every $p > \ln k \ln(2^{-n}r^2)/(\ln r)^2$.

COROLLARY 2. *If condition (7) is replaced by*

(8) $$|E|/|Q| \leqslant r \quad \text{implies} \quad \omega(E)/\omega(Q) \leqslant k,$$

then ω is in A_p for every $p > \ln(1-k)\ln(2^{-n}(1-r)^2)/(\ln(1-r))^2$.

Proof of the corollary. By taking complements with respect to Q, (8) is transformed to

$$|C(E)|/|Q| > 1-r \quad \text{implies} \quad \omega(C(E))/\omega(Q) > 1-k$$

and the conclusion follows from Theorem 3.

Proof of Theorem 3. Let Q be any cube and E an arbitrary subset of Q with $|E| < r|Q|$. Let p be the nonnegative integer determined by

(9) $$r^{p+2} \leqslant |E|/|Q| < r^{p+1}.$$

We use Lemma 1 to cover E' with disjoint, dyadic cubes, $\{Q_{\nu,1}\}_1^\infty$, such that

(10) $$r\cdot 2^{-n}|Q_{\nu,1}| < |Q_{\nu,1}\cap E| < r|Q_{\nu,1}|, \qquad \nu = 1,2,\ldots$$

Put

$$E_1 = \bigcup_{\nu=1}^{\infty} Q_{\nu,1}.$$

From the construction in Lemma 1 of the cubes $Q_{\nu,1}$ it follows that $Q_{\nu,1}$ is the union of 2^n cubes with half the side-length of $Q_{\nu,1}$ and where at least one of the subcubes, $Q_{\nu,1}^*$, has the property $|Q_{\nu,1}^*\cap E|/|Q_{\nu,1}^*| \geqslant r$. By assumption (7) we then have

$$\omega(Q_{\nu,1}^*\cap E) \geqslant k\omega(Q_{\nu,1}^*).$$

We can expand $Q^*_{\nu,1}$ to $Q_{\nu,1}$ stepwise by increasing the side-length by a factor of $r^{-1/n}$ at each step except for the last. The number of steps we have to take is m where

$$m-1 = \left[\frac{n}{\log_2(1/r)}\right]. \tag{11}$$

Using (7) we therefore obtain $\omega(Q^*_{\nu,1}) \geqslant k^m \omega(Q_{\nu,1})$ and thus

$$\omega(Q^*_{\nu,1} \cap E) \geqslant k^{m+1}\omega(Q_{\nu,1}).$$

Since $\omega(Q^*_{\nu,1} \cap E) \leqslant \omega(Q_{\nu,1} \cap E)$ a summation yields

$$\omega(E) \geqslant k^{m+1}\omega(E_1) \quad \text{or} \quad \omega(E_1) \leqslant k^{-m-1}\omega(E)$$

and from (10)

$$|E_1| > \frac{1}{r}|E|.$$

By (9)

$$r^{p+1} \leqslant |E_1|/|Q| < r^p.$$

We start anew with E_1 as our new E, increase the Lebesgue measure by a factor of at least r^{-1} and obtain E_2 with

$$r^p|Q| \leqslant |E_2| < r^{p-1}|Q|, \quad \omega(E_1) \geqslant k^{m+1}\omega(E_2).$$

After $p+1$ steps we have

$$r|Q| \leqslant |E_{p+1}| < |Q|, \quad \omega(E_{p+1}) \leqslant k^{-(m+1)(p+1)}\omega(E).$$

Since E_{p+1} is large enough for us to use assumption (7) we have

$$\omega(E) \geqslant k \cdot k^{(m+1)(p+1)}\omega(Q). \tag{12}$$

We observe that obviously k is smaller than r and put $k^{m+1} = r^\beta$. Then, $\beta = (m+1)\ln k/\ln r > 1$ and

$$\omega(E) > k(|E|/|Q|)^\beta \omega(Q).$$

Using (11) we find this to be true if e.g.

$$\beta > \frac{\ln k \ln(2^{-n}r^2)}{(\ln r)^2}.$$

The statement of the theorem now follows from Corollary 1.

We end this paper with a new example showing that A_∞ is a strictly smaller class of functions than the class of nonnegative functions, ω, satisfying the *doubling property*:

$$\omega(2Q) < c\omega(Q), \quad \text{for some } c > 0.$$

In [3] it is observed that it is sufficient to produce an example, ω, in $\boldsymbol{R}$ since this gives an example, ω_1, in $\boldsymbol{R}^n$ if we define $\omega_1(x) = \omega(x_1)$.

THEOREM 4. *There exists a function which does not belong to A_∞ but still has the doubling property.*

Construction of the function ω. Our starting point is the triangular function

$$\omega_0(x) = \begin{cases} x, & 0 \leqslant x \leqslant 1, \\ 2-x, & 1 \leqslant x \leqslant 2, \\ 0, & \text{elsewhere.} \end{cases}$$

We then construct the graph $y = \omega_1(x)$ by successively adjoining to the graph $y = \omega_0(x)$ translations of that graph diminished by a factor of $\frac{1}{2}, \frac{1}{4}, \ldots$ in both the x and y directions. We reach the point $x = 4$ and complete the graph from $x = 4$ to $x = 8$ by making it symmetric with respect to the line $x = 4$. In general ω_n is constructed from ω_{n-1} in the same way. In formulas this yields

$$\omega_{n+1}(x) = \begin{cases} \sum_{k=0}^{\infty} 2^{-k}\omega_n(2^k(x-4^{n+1})+4^{n+1}), & 0 \leqslant x \leqslant 4^{n+1}, \\ \omega_{n+1}(2\cdot 4^{n+1}-x), & 4^{n+1} \leqslant x \leqslant 2\cdot 4^{n+1}, \\ 0, & \text{elsewhere.} \end{cases}$$

Then we define

$$\omega(x) = \lim_{n\to\infty} \omega_n(x), \quad \text{for } x > 0,$$

and make ω even by $\omega(x) = \omega(-x)$, for $x < 0$. An immediate calculation gives

$$\int_0^{2\cdot 4^n} \omega_n(x)\,dx = \left(\tfrac{8}{3}\right)^n.$$

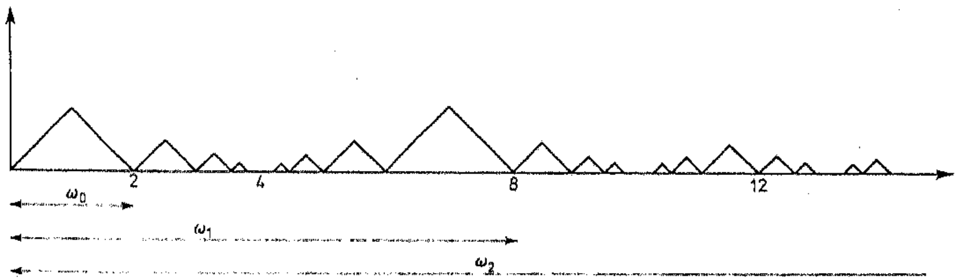

Fig. 1

We now make an estimation of $|E_n|$, where

$$E_n = \{x \in [0, 2\cdot 4^n];\ \omega(x) < 2^{-n}\}.$$

Obviously,

$$|E_0| = 2\cdot 2^{-n},$$
$$|E_1| > 4\cdot 2^{-n}\sum_{k=1}^{n} 1 = 4\cdot 2^{-n}\cdot n,$$
$$|E_2| > 8\cdot 2^{-n}\sum_{k=1}^{n} k = 8\cdot 2^{-n}\frac{n(n+1)}{2},$$
$$|E_n| > 2\frac{(2n-1)!}{(n-1)!\,n!}.$$

Using Stirling's formula, we obtain

$$|E_n| > \frac{4^n}{\sqrt{\pi n}}(1+O(1/n)). \tag{13}$$

Proof that ω does not belong to A_∞. It is sufficient to show that ω violates (6) if p_1 is large enough. Take $p_1 = \ln n$, $Q_n = [0, 2\cdot 4^n]$ and E_n as above. We use $\omega(Q_n) = (\frac{8}{3})^n$, $\omega(E_n) \leqslant 2^{-n}|E_n|$ and (13) to find

$$\frac{\omega(E_n)}{\omega(Q_n)}\cdot\frac{|Q_n|^{p_1}}{|E_n|^{p_1}} \leqslant (\tfrac{3}{4})^n\cdot 2^{p_1}\cdot(\sqrt{n\pi})^{p_1-1}\left(1+O\left(\frac{\ln n}{n}\right)\right),$$

which certainly is $o(1)$. Thus, ω does not belong to any A_p, $p \geqslant 1$, and therefore not to A_∞.

Proof that ω has the doubling property. I. Let I be an interval $[a, b]$ where $a = m\cdot 2^k$ and $b = (m+1)2^k$, for some integer k and some nonnegative integer m. We claim that

$$\omega(3I) < 6\omega(I). \tag{14}$$

If $b = 4^n$ or $a = 4^n$ we see that

(i) $\omega|_I$ consists of an infinite succession of copies of ω_{n-1}.

Otherwise one of the following cases can occur:

(ii) $\omega|_I$ consists of exactly one, or of one half of a copy of ω_{n-1}.
(iii) $\omega|_I$ consists of $\frac{1}{4}$ or $\frac{1}{8}$ or ... of a copy of ω_{n-1}.

In case (i) it is evident that $\omega(3I) = 5\omega(I)$. In case (ii) it is also evident that $\omega(3I) < 6\omega(I)$. In case (iii) we easily see that if $a = 2\cdot 4^{n-1}$ or $b = 2\cdot 4^n$, then (14) is true. In the remaining possibilities we can, without loss, instead study the corresponding subinterval of $[0, 2\cdot 4^{n-1}]$. The inequality (14) is obviously true for $I \subset [0, 8]$. The general result therefore follows by induction. Now ω is an even function. Therefore (14) is true also if $m < 0$.

II. We are now in a position to prove

$$\omega(2I) \leqslant 6^4\omega(I)$$

for any interval I.

Let the length of I, $|I|$, satisfy $2^k \leqslant |I| < 2^{k+1}$. Then I contains an interval of the form $I' = [m\cdot 2^{k-1}, (m+1)2^{k-1}]$. It is easily seen that $2I \subset 15I'$. We double I' three times in suitable directions to obtain I'' with endpoints of the right form. Then $3I'' \supset 2I$.

Using (14) four times, we obtain

$$\omega(2I) \leqslant 6\omega(I'') \leqslant 6^4\omega(I)$$

and we are through.

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