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ON INVARIANT SUBMANIFOLDS IN ALMOST COSYMPLECTIC MANIFOLDS

BY

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0. Introduction. Kon [4] has proved the following result:

THEOREM. *Let $\bar{M}^{2r+1}$ be a normal contact metric manifold of constant $\bar{\phi}$-sectional curvature $\bar{K}$ and M^{2n+1} be an invariant submanifold of $\bar{M}^{2r+1}$. Then M^{2n+1} is totally geodesic if and only if*

$$S_c \geqslant n^2(\bar{K}+3)+n(\bar{K}+1).$$

The purpose of this paper is to prove a similar result in the case where $\bar{M}^{2r+1}$ is an almost cosymplectic manifold.

1. Preliminaries. Let $(\bar{M}^{2r+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ be a $(2r+1)$-dimensional almost contact Riemannian manifold, that is, $\bar{M}^{2r+1}$ is a differentiable manifold and $(\bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ an almost contact Riemannian structure on $\bar{M}^{2r+1}$, formed by tensor fields $\bar{\phi}, \bar{\xi}, \bar{\eta}$, of type (1, 1), (1, 0) and (0, 1), respectively, and a Riemannian metric $\bar{g}$ such that

$$\begin{gathered}\bar{\phi}^2 = -I+\bar{\eta}\otimes\bar{\xi}, \quad \bar{\phi}\bar{\xi} = 0, \quad \bar{\eta}\circ\bar{\phi} = 0, \quad \bar{\eta}(\bar{\xi}) = 1,\\ \bar{\eta}(\bar{X}) = \bar{g}(\bar{X}, \bar{\xi}), \quad \bar{g}(\bar{\phi}\bar{X}, \bar{\phi}\bar{Y}) = \bar{g}(\bar{X}, \bar{Y})-\bar{\eta}(\bar{X})\bar{\eta}(\bar{Y}).\end{gathered} \tag{1.1}$$

On such a manifold we may always define a 2-form $\bar{\Phi}$ by

$$\bar{\Phi}(\bar{X}, \bar{Y}) = \bar{g}(\bar{\phi}\bar{X}, \bar{Y}).$$

Then $(\bar{M}^{2r+1}, \bar{\phi}, \bar{\xi}, \bar{\eta}, \bar{g})$ is said to be an *almost cosymplectic manifold* if the forms $\bar{\Phi}$ and $\bar{\eta}$ are closed, i.e., $d\bar{\Phi} = 0$, $d\bar{\eta} = 0$, where d is the operator of exterior differentiation. In particular, if the almost contact structure of an almost cosymplectic manifold is normal, then it is said to be a *cosymplectic manifold*. As is known, an almost contact metric structure is cosymplectic if and only if both $\bar{\nabla}\bar{\eta}$ and $\bar{\nabla}\bar{\phi}$ vanish, where $\bar{\nabla}$ is the covariant differentiation with respect to $\bar{g}$. In the sequel, an almost cosymplectic manifold will be denoted simply by $\bar{M}^{2r+1}$.

Let M^{2n+1} be a $(2n+1)$-dimensional submanifold of $\bar{M}^{2r+1}$. By N_A ($A = 1, 2, \ldots, 2(r-n)$) we denote local mutually orthogonal unit vector fields normal to M^{2n+1}. Let ∇ be the Riemannian connection on M^{2n+1} determined by the induced metric g. Then the Gauss and Weingarten formulas are given by

$$(1.2)\quad \bar{\nabla}_X Y = \nabla_X Y + \sum_A h_A(X, Y) N_A, \qquad \bar{\nabla}_X N_A = -H_A X + \sum_B L_{BA}(X) N_B,$$

where h_A and H_A are the second fundamental forms, and L_{BA} is the third fundamental form of M^{2n+1}. The forms h_A and H_A satisfy

$$h_A(X, Y) = g(H_A X, Y) = g(X, H_A Y) = h_A(Y, X).$$

And for any vector fields X, Y, Z and W on M^{2n+1} the Gauss equation is given by

$$(1.3)\quad \bar{g}(\bar{R}(X, Y)Z, W) = g(R(X, Y)Z, W) - \sum_B g(H_B Y, Z) g(H_B X, W) + \sum_B g(H_B X, Z) g(H_B Y, W),$$

where $\bar{R}$ is the Riemannian curvature tensor of $\bar{M}^{2r+1}$, and R is the Riemannian curvature tensor of M^{2n+1}.

M^{2n+1} is said to be *invariant* if $\bar{\xi}$ is tangent to M^{2n+1} and $\bar{\phi}X$ is tangent to M^{2n+1} for any tangent vector field X on M^{2n+1}. Any invariant submanifold M^{2n+1} of an almost contact Riemannian manifold with the induced structure (ϕ, ξ, η, g) is also an almost contact Riemannian manifold (see, e.g., [5]). Moreover, for any vector fields X, Y and Z on M^{2n+1}, we have

$$d\eta(X, Y) = d\bar{\eta}(X, Y), \qquad d\Phi(X, Y, Z) = d\bar{\Phi}(X, Y, Z),$$

where Φ is a 2-form defined by $\Phi(X, Y) = g(\phi X, Y)$. Consequently, we see that any invariant submanifold M^{2n+1} of an almost cosymplectic manifold $\bar{M}^{2r+1}$ with induced structure (ϕ, ξ, η, g) is also an almost cosymplectic manifold. And if $\bar{M}^{2r+1}$ is a cosymplectic manifold, then M^{2n+1} is also a cosymplectic manifold (see, e.g., [7]).

If a unit vector $\bar{X} \in T_m(\bar{M}^{2r+1})$ (where $T_m(\bar{M}^{2r+1})$ denotes the tangent space at a point m of $\bar{M}^{2r+1}$) is orthogonal to $\bar{\xi}$, then the sectional curvature

$$\bar{K}(\bar{X}, \bar{\phi}\bar{X}) = \bar{g}(\bar{R}(\bar{X}, \bar{\phi}\bar{X})\bar{\phi}\bar{X}, \bar{X})$$

is called the *$\bar{\phi}$-sectional curvature* of the plane $\{\bar{X}, \bar{\phi}\bar{X}\}$. If $\bar{K}(\bar{X}, \bar{\phi}\bar{X})$ does not depend on the choice of $\bar{X} \in T_m(\bar{M}^{2r+1})$, $\bar{X} \perp \bar{\xi}$, we denote it by $\bar{K}_m$.

Define

$$s^* = \sum_{i,j=0}^{2n} g(R(\tilde{E}_i, \tilde{E}_j)\phi\tilde{E}_j, \phi\tilde{E}_i),$$

where $\{\tilde{E}_i\}$ is an orthonormal basis.

We need in the sequel the following theorem and lemmas.

THEOREM 1 ([1] and [2]). *Any invariant submanifold M^{2n+1} of an almost cosymplectic manifold $\bar{M}^{2r+1}$ is minimal.*

LEMMA 1 ((4.3) and (4.11) of [6]). *Any almost cosymplectic manifold satisfies the identities*

(1) $S_c - s^* - S(\xi, \xi) + \frac{1}{2}|\nabla\phi|^2 = 0,$

(2) $S(\xi, \xi) + |\nabla\xi|^2 = 0,$

where S is the Ricci tensor, and S_c is the scalar curvature tensor of M^{2n+1}.

LEMMA 2 ([3]). *If M^{2n+1} is an invariant submanifold of an almost cosymplectic manifold $\bar{M}^{2r+1}$, then $H_A \xi = 0$.*

LEMMA 3 ([3]). *Let M^{2n+1} be an invariant submanifold of an almost cosymplectic manifold $\bar{M}^{2r+1}$. Then*

$$\operatorname{Tr}(\phi H_B)^2 = \operatorname{Tr}(H_B^2).$$

2. Main result.

THEOREM. *Let $\bar{M}^{2r+1}$ be an almost cosymplectic manifold of pointwise constant ϕ-sectional curvature $\bar{K}$, and M^{2n+1} be an invariant submanifold of $\bar{M}^{2r+1}$. Then the scalar curvature S_c of M^{2n+1} satisfies the inequality*

$$S_c \leqslant n(n+1)\bar{K}.$$

Equality holds if and only if M^{2n+1} is totally geodesic and cosymplectic.

Proof. In virtue of our assumption, we have

$$\bar{g}(\bar{R}(X, \phi X)X, \phi X) + \bar{K}_m \|X\|^4 = 0$$

at any point $m \in M^{2n+1}$ and for any vector $X \in T_m(M^{2n+1})$, $X \perp \xi$. It is clear that this condition implies

$$\bar{g}(\bar{R}(\phi X, \phi^2 X)\phi X, \phi^2 X) + \bar{K}_m \|\phi X\|^4 = 0 \tag{2.1}$$

at any point $m \in M^{2n+1}$ and for any $X \in T_m(M^{2n+1})$. Set

$$\bar{P}(X, Y, Z, W) = \bar{g}(\bar{R}(\phi X, \phi^2 Y)\phi Z, \phi^2 W) + \bar{K}_m g(\phi X, \phi Z) g(\phi Y, \phi W).$$

Then the tensor $\bar{P}$ satisfies $\bar{P}(X, Y, Z, W) = \bar{P}(Z, W, X, Y)$. Therefore (2.1) is equivalent to

$$\begin{aligned}\bar{P}(X, Y, Z, W) &+ \bar{P}(X, Y, W, Z) + \bar{P}(Y, X, Z, W) + \bar{P}(Y, X, W, Z) \\ &+ \bar{P}(X, W, Y, Z) + \bar{P}(X, W, Z, Y) + \bar{P}(W, X, Y, Z) + \bar{P}(W, X, Z, Y) \\ &+ \bar{P}(X, Z, Y, W) + \bar{P}(X, Z, W, Y) + \bar{P}(Z, X, Y, W) + \bar{P}(Z, X, W, Y) = 0\end{aligned} \tag{2.2}$$

for arbitrary $X, Y, Z, W \in T_m(M^{2n+1})$. Choosing a ϕ-basis

$$\{E_0 = \xi, E_1, \ldots, E_n, E_{n+1} = \phi E_1, \ldots, E_{2n} = \phi E_n\}$$

in $T_m(M^{2n+1})$, taking $X = W = E_i$, $Y = Z = E_j$ into (2.2) and summing over

i and j we obtain

$$\sum_{i,j=0}^{2n} \{\bar{P}(E_i, E_j, E_j, E_i) + \bar{P}(E_i, E_j, E_i, E_j) + \bar{P}(E_i, E_i, E_j, E_j)\} = 0.$$

Hence, using the definition of $\bar{P}$, the Gauss equation (1.3), the first Bianchi identity, Theorem 1 and Lemmas 2 and 3, we find, after some lengthy computation,

$$4n(n+1)\bar{K} - 3s^* - S_c + 2S(\xi, \xi) = 4\sum_B \mathrm{Tr}(H_B^2),$$

from which, using Lemma 1, we get

$$n(n+1)\bar{K} - S_c = \tfrac{5}{4}|\nabla\xi|^2 + \tfrac{3}{8}|\nabla\phi|^2 + \sum_B \mathrm{Tr}(H_B^2), \tag{2.3}$$

which completes the proof.

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