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THE CENTRAL LIMIT THEOREM FOR DYNAMICAL SYSTEMS

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We give a survey about problems involving central limit theorems for dynamical systems, i.e. results ensuring that suitable normalized sums $f+f\circ T+\ldots+f\circ T^{n-1}$ converge weakly to the normal distribution with mean zero and variance 1. In particular, we are interested in properties of the subspace in L^2 consisting of those functions for which the central limit theorem holds. On the other hand, some applications of the central limit theorem are given, which deal with isomorphisms, Hausdorff measures and statistical analysis.

§ 0. Introduction

Throughout this paper, let $(\Omega, \mathfrak{F}, \mu, T)$ denote a measure preserving dynamical system with normalized measure μ on the σ-algebra $\mathfrak{F}$ and (not necessarily invertible) μ-preserving transformation T. The main object of our discussion will be the central limit theorem: Let f be a square integrable function ($f\in L^2$). We shall denote by $S_n = S_n f = \sum_{j=0}^{n-1} f\circ T^j$, $n\geqslant 1$, the partial sums of f, by $E(f)=\mu(f)=\int f\,d\mu$ the expectation of f and by

$$\sigma_n^2 = E(S_n^2)-n^2(E(f))^2$$

the variance of $S_n (n\geqslant 1)$. We shall say that the function f satisfies the *central limit theorem* (CLT) if

$$\text{(0.1)}\quad \lim_{n\to\infty}\sup_{x\in\mathbb{R}}\left|\mu\left(\left\{\omega\in\Omega:\ \frac{S_n(\omega)-nE(f)}{\sigma_n}\leqslant x\right\}\right)-\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x}\exp\left(-\frac{1}{2}t^2\right)dt\right| = 0.$$

If the process $(f\circ T^n)_{n\geqslant 0}$ is independent, f satisfies (0.1) by the classical central limit theorem. However, this situation is hardly met for functions on a fixed dynamical system, so that one has to look for different conditions to ensure the CLT. It is one of the important problems in probability theory to show the CLT in quite different situations. It turns out that two of these are useful for dynamical systems: the approximation by ergodic martingale differences and the approximation by weakly dependent processes. In § 1 we will sketch both methods to an extent which is needed in the present context. When applied to dynamical systems it will be clear that the results are of no interest to probabilists. Indeed, the problem of determining the subspace of L^2 of those functions f satisfying the CLT is not a probabilistic question. Henceforth we also denote by CLT this subspace. In § 2 we discuss a special feature of this question: when does CLT contain a sufficiently rich class of functions? Can CLT be empty? The general problem seems to be very hard, and nothing is known about it.

An isomorphism φ between two dynamical systems transforms the corresponding CLT spaces into each other, but in general it does not follow that the spaces of continuous functions (or Hölder continuous functions) are contained in CLT if this is true for one system. Additional properties of the isomorphism are needed and it follows then that new invariants can be formulated. As a motivation for this problem we may think of some analysis carried out using a certain function f (like pressure) for a dynamical system whose properties are described by an isomorphism. There are cases (e.g. Hausdorff measures, statistical analysis of symmetry, mean, median...) when it has to be verified that $f \in$CLT. This kind of questions will be discussed in Section 3. In statistical analysis very often one has to use functions in more than one variable (i.e. $f\colon \Omega^m \to \boldsymbol{R}$). The problems we are faced here become rather technical and it is not possible to explain the results within a reasonable length. Therefore we can only give a rough idea of it at the end.

The normalizing constants σ_n^2 in (0.1) may be replaced by any sequence a_n satisfying $\lim_{n\to\infty}(a_n/\sigma_n^2) = 1$. We shall make use of it frequently without explicitly mentioning. Moreover, one might say that f satisifes the CLT if (0.1) holds where σ_n^2 is replaced by some other sequence a_n. There are theorems of this type (e.g. when f has no finite second moment, or the mixing rate is too bad) but these results do not have any implication to dynamical systems, so that we are not dealing with this case here.

On the other hand, there are stronger versions of (0.1) which are of interest for dynamical system, the first one being the *functional central limit theorem* (FCLT). Let $D([0, 1])$ denote the space of all functions $h\colon [0, 1] \to \boldsymbol{R}$ which are right continuous and have left limits. On $D([0, 1])$ we consider the topology of uniform convergence (in general it is necessary to take a stronger topology but for our purpose it will be sufficient). Given a function $f \in L^2$, consider the random element Z_n defined by

$$Z_n(t) = \sigma_n^{-1}\left(S_{[nt]} - [nt]\,E(f)\right) \quad (0 \leqslant t \leqslant 1). \tag{0.2}$$

Then f is said to satisfy the FCLT if the random functions Z_n converge weakly to the standard Wiener process W (i.e. the sequence $Z_n\mu$ of measures on $D([0, 1])$ converge weakly to the Wiener measure, which is carried by $C([0, 1]) \subset D([0, 1])$). Recall that W is a Gaussian process with independent increments, continuous trajectories and variances $EW(t)^2 = t$. If f satisifies the FCLT, then it also satisfies the CLT. This is immediate observing that π: $D([0, 1]) \to \mathbf{R}$, $\pi(h) = h(1)$ is continuous, so that $\pi Z_n = \sigma_n^{-1}(S_n - nE(f))$ converges weakly to $W(1) = \pi W$, and this is equivalent to (0.1), because $W(1)$ is standard normal. Conversely, if (0.1) holds (0.2) does not hold in general, but using an additional argument for tightness, (0.2) can be verified from (0.1) in many cases which are of interest to us.

Weak convergence may be strengthened to almost sure convergence. Skorohod's theorem says that if $Y_n \to Y_0$ weakly, then there exist random variables $Y_n'(n \geqslant 0)$ such that $Y_n' \to Y_0'$ a.s. and Y_n and Y_n' have the same distribution for each $n \geqslant 0$. It does not say that the common distributions of $(Y_n)_{n \geqslant 0}$ and $(Y_n')_{n \geqslant 0}$ are the same. This is the content of an *almost sure invariance principle* (ASIP). The FCLT is often called the *weak invariance principle.*

Let $f \in L^2$. We say that f satisfies an ASIP if there exists a probability space $(\Omega', \mathfrak{F}', \mu')$, a function $f' \in L^2((\Omega', \mathfrak{F}', \mu'))$ and a standard Brownian motion $\{B(t)\colon t \geqslant 0\}$ defined on Ω' such that the joint distributions of $\{f \circ T^k\colon k \geqslant 0\}$ and $\{f' \circ T'^k\colon k \geqslant 0\}$ are the same and

$$(0.3) \qquad \sum_{k \leqslant n} f' \circ T'^k - nE(f') - B(\sigma_n^2) = O(\sigma_n^{1-\lambda}) \text{ a.s.}$$

$$\text{for some } \lambda > 0, \text{ as } n \to \infty$$

or (at least)

$$(0.4) \qquad \sum_{k \leqslant n} f' \circ T'^k - nE(f') - B(\sigma_n^2) = o(\sqrt{\sigma_n^2 \log\log \sigma_n^2}) \quad \text{a.s.} \quad \text{as } n \to \infty.$$

It is not hard to see that f satisfies the FCLT if it satisfies the ASIP with condition (0.3) and if

$$\sigma_k^2/\sigma_n^2 = k/n(1 + o(1)) \quad (0 \leqslant k \leqslant n).$$

Indeed, $\{n^{-1/2} B(t)\colon 0 \leqslant t \leqslant n\}$ has the same distribution as $\{W(t)\colon 0 \leqslant t \leqslant 1\}$ and $n^{-1/2} B(k) = \sigma_n^{-1} B(\sigma_k^2)(1 + o(1))$ a.s. Thus, dividing (0.3) by σ_n, it follows that

$$\sigma_n^{-1}\Big(\sum_{j \leqslant k} f' \circ T'^j - kE(f')\Big) - n^{-1/2} B(k) = \sigma_n^{-\lambda}(1 + o(1)) = o(1) \text{ a.s.}$$

(0.4) is of course not sufficient. It turns out that the ASIP with (0.3) holds only under more restrictive assumptions than the FCLT.

The ASIP with (0.4) implies the different forms of the law of the iterated logarithm. We say that $f \in L^2$ satisfies the *functional law of the iterated* *logarithm* (FLIL) if the random functions Y_n: $[0, 1] \to \mathbf{R}$ defined by

$$Y_n(t) = (2\sigma_n^2 \log\log \sigma_n^2)^{-1/2}\Big(\sum_{j \leqslant [tn]} f \circ T^j - [tn]\, E(f)\Big)$$

$(0 \leqslant t \leqslant 1)$, are relatively compact in $D([0, 1])$ and its set of limit points is precisely given by

$$K = \{x \in C([0, 1]):\ x(t) = \int_0^t \dot{x}(s)\, ds,\ \int \dot{x}^2(s)\, ds \leqslant 1\}.$$

It then follows that f also satisfies the law of the *iterated logarithm* (LIL) by restricting to $t = 1$, i.e.

$$\limsup_{n \to \infty} (2\sigma_n^2 \log\log \sigma_n^2)^{-1/2} (S_n - nE(f)) = 1 \text{ a.s.} \tag{0.5}$$

In Section 3 we shall make use of all these generalizations of the CLT. From a dynamical viewpoint it is not necessary to give the probabilistic background for these results; what will be said about the CLT in § 1 is sufficient to describe the general approach to these questions. The same kind of definitions can be given for flows replacing S_n by $\int_0^t f \circ T_\tau\, d\tau$ in (0.1)–(0.5). Probabilistically, the approach in this case will be the same again, so that a discussion in § 1 is also unnecessary.

This survey paper is based on a lecture given during the Semester on Dynamical Systems and Ergodic Theory in 86 at Warsaw. I would like to thank the organizers of the Semester for their hospitality during my stay in Warsaw. I am also indebted to R. Bradley and the referee for helpful discussions and valuable comments.

§ 1. Some results on the central limit theorem problem in probability theory

For the better understanding of many results on the CLT for dynamical systems it seems to be worthwile to go into some probabilistic details. Weakening the independence assumption in the classical CLT is a subject of research for many years. Quite different approaches can be found in literature. We are going to choose two out of them which turn out to be of particular interest when applied to dynamical systems. Certainly, we are only able to sketch the main features and results, others might become important for dynamical systems in the future. We restrict our discussion to the question concerning the CLT, though results on a.s. invariance principles are needed later on. However, the step in understanding the more advanced theorems is not so big (but the proofs are much more complicated).

In this section, let $(X_n)_{n \geqslant 0}$ be a (strictly) stationary process with $E(X_0) = \int X_0\, d\mu = 0$. We define $S_n = X_0 + \ldots + X_{n-1}$ and $\sigma_n^2 = E(S_n^2) = \int S_n^2\, d\mu$. We also shall use $||X_0||_p$ for the p-norm in L^p. As before we shall say that $(X_n)_{n\geq 0}$ satisfies the CLT, if S_n/σ_n converges weakly to the standard normal distribution. Similarly, $(X_n)_{n\geq 0}$ satisfies the FCLT if the random functions $t \to \sigma_n^{-1} S_{[nt]}$ converge weakly in $D([0, 1])$ to the Wiener process. We note that $(X_n)_{n\geq 0}$ always can be represented by a transformation T, i.e.

$$X_n = X_0 \circ T^n \quad (n \geq 0).$$

Thus the results of this section immediately apply to functions defined on a dynamical system.

1.1. Approximation by ergodic martingale differences

The classical CLT for stationary ergodic martingale difference sequences applies to dynamical systems, but in general it is hard to check the martingale property. It turns out, however, that some processes (on dynamical systems) can be approximated by ergodic martingale differences and this can be verified for concrete examples. This approach will be discussed first.

A process $(X_n)_{n\geq 0}$ is called a *martingale difference sequence with respect to the filtration* $\{\mathfrak{F}_n: n \geq 0\}$ of σ-algebras $\mathfrak{F}_n \subset \mathfrak{F}$ if for each $n \geq 0$

(1) $\mathfrak{F}_n \subset \mathfrak{F}_{n+1}$;

(2) X_n is $\mathfrak{F}_n$-measurable and $X_n \in L^1$;

(3) The conditional expectation of X_{n+1} given $\mathfrak{F}_n$ vanishes, i.e. $E(X_{n+1} | \mathfrak{F}_n) = 0$ a.s.

The classical CLT for ergodic martingale difference sequences is this (see Billingsley 1968, Ibragimov 1963).

THEOREM 1.1.1. *Let* $(X_n)_{n\geq 0}$ *be a stationary, ergodic martingale difference sequence with respect to the filtration* $\{\mathfrak{F}_n: n \geq 0\}$. *If* $X_0 \in L^2$, *then* $\sigma_n^2 = n||X_0||_2^2$ *and the* CLT *holds.*

A short proof of this can be found in Holewijn, Meilijson (1983). Once having established Theorem 1.1.1, we obtain the CLT for stationary processes having a "good" approximation by ergodic martingale differences. This observation was first made by Gordin (1969), and this direction was investigated by him and others in subsequent years. The book of Hall and Heyde (1980) gives a good idea of it up to that time. Recently, Dürr and Goldstein (1985) among others took up the subject again. At this point I should make a comment on Corollary 5.3 and the proof of Theorem 5.4 in the book of Hall and Heyde (1980). Both do not seem to be correct. For example the corollary makes this statement:

If for some $\delta \geq 0$ $||X_0||_{2+\delta} < \infty$, $n^{-1}\sigma_n^2 \to \sigma^2 > 0$ and if $(X_n)_{n\geq 0}$ is strongly mixing with mixing coefficients $\alpha(n)$ satisfying $\sum \alpha(n)^{\frac{1+\delta}{2+\delta}} < \infty$, then then the CLT holds. (For the definition of strong mixing see section 1.2, (1.2.2).)

Given the stationary sequence $(X_m)_{m\geqslant 0}$, we define sub-σ-algebras $\mathfrak{F}_k^l = \sigma(X_u\colon k \leqslant u \leqslant l)$ for $0 \leqslant k \leqslant l \leqslant \infty$, and consider the coefficients

$$(1.2.1) \qquad \alpha_{s,t}(n) = \sup\left\{\frac{|\mu(A\cap B)-\mu(A)\mu(B)|}{(\mu(A))^s(\mu(B))^t}\colon A\in\mathfrak{F}_0^k,\ B\in\mathfrak{F}_{k+n}^{\infty},\ k\geqslant 0\right\}$$

for $0 \leqslant s, t \leqslant 1$ and $n \geqslant 1$.

The process $(X_m)_{m\geqslant 0}$ is called:

(1.2.2) *strong mixing* (α-*mixing*), if $\lim_{n\to\infty} \alpha_{0,0}(n) = 0$;

in this case we shall write $\alpha(n) = \alpha_{0,0}(n)$; this notion is due to Rosenblatt (1956);

(1.2.3) *uniform mixing* (φ-*mixing*), if $\lim_{n\to\infty} \alpha_{1,0}(n) = 0$;

in this case we put $\varphi(n) = \alpha_{1,0}(n)$; this notion is due to Ibragimov (1959);

(1.2.4) *reversed uniform mixing*, if $\lim_{n\to\infty} \alpha_{0,1}(n) = 0$.

We put $\varphi^*(n) = \alpha_{0,1}(n)$; this notion has been first used by Halverson and Wise (1980) (to my knowledge) and later by Denker, Keller (1983) and Peligrad (1983);

(1.2.5) ψ-*mixing*, if $\lim_{n\to\infty} \alpha_{1,1}(n) = 0$.

We put $\psi(n) = \alpha_{1,1}(n)$.

It is immediately clear that for $t \leqslant t'$, $s \leqslant s'$ and all n

$$(1.2.6) \qquad \alpha_{t,s}(n) \leqslant \alpha_{t',s'}(n);$$

consequently, every ψ-mixing sequence is φ- and φ^*-mixing and every φ- or φ^*-mixing sequence is α-mixing as well. But all four notions (1.2.2)–(1.2.5) are not equivalent. Counterexamples are given in Bradley (1982) and the literature given there.

Another type of weak dependence is defined by the correlation coefficient. Denote by $L^2(\mathscr{A})$ the space of square integrable, $\mathscr{A}$-measurable functions. Setting

(1.2.7)

$$\varrho(n) = \sup\left\{\frac{|\mu(fg)-\mu(f)\mu(g)|}{\|f-\mu(f)\|\,\|g-\mu(g)\|};\ f\in L^2(\mathfrak{F}_0^k),\ g\in L^2(\mathfrak{F}_{k+n}^{\infty}),\ k\geqslant 0\right\},$$

we shall say that $(X_m)_{m\geqslant 0}$ is ϱ-*mixing* if $\lim_{n\to\infty}\varrho(n) = 0$.

The correlation coefficient was studied as early as 1935 by Hirschfeld, the notion of ϱ-mixing is due to Kolmogorov and was first studied by Rozanov (1960, 1985).

Due to a theorem of Bradley (1983) one knows that $\varrho(n) \to 0$ if and only if $\alpha_{1/2,1/2}(n) \to 0$, and Theorem 4.1 (ii) in Bradley, Bryc (1985) even implies that $\varrho(n) \to 0$ if and only if $\alpha_{t,1-t}(n) \to 0$ for any $0 < t < 1$. In the same paper it is also shown that for $t, s \geqslant 0$, $s+t < 1$ $\alpha(n) \to 0$ if and only if $\alpha_{t,s}(n) \to 0$. Thus in the lower triangle $\{(t, s)\colon 0 \leqslant s, t \leqslant 1, t+s \leqslant 1\}$ there are only four equivalence classes of dependence notions: α-, ϱ-, φ- and φ^*-mixing. For $s+t > 1$ the situation is not at all as nice (Bradley, Bryc, Janson 1985), fortunately these notions, except for $s = t = 1$, do not play any role in our further discussion.

There is another dependence notion to be discussed here. (Certainly a few more are known in probability theory, but they are of minor interest.)

Define the *coefficient of absolute regularity* (*β-mixing coefficient*) by

$$\beta(n) = \int \sup \{|\mu(B|\,\mathfrak{F}_0^k) - \mu(B)|;\, B \in \mathfrak{F}_{k+n}^{\infty},\, k \geqslant 0\}\, d\mu. \tag{1.2.8}$$

The process $(X_n)_{n\geqslant 0}$ is called *absolutely regular* if $\lim_{n\to\infty} \beta(n) = 0$. This definition goes back to Volkonskii, Rozanov (1959). This mixing concept is not equivalent to any of the $\alpha_{t,s}$-mixing conditions. (Bradley, Bryc, Janson 1985, Theorem 3.1.) On the other hand, it is immediately seen that φ- (or φ^*-) mixing implies absolute regularity and the latter implies strong mixing. From a dynamical viewpoint absolute regularity for countable state processes is equivalent to the weak Bernoulli property in the following sense. Let $P = P_1, P_2, \ldots)$ be a finite or countable partition, $P^i = P \vee T^{-1}P \ldots \vee T^{-i+1}P$. For given $n \geqslant 1$ let $\omega(n)$ be the infimum over all $\varepsilon \geqslant 0$ with the following property: For each $k, l \geqslant 1$ there exists a collection $\mathfrak{C}$ of atoms of P^k such that

$$\mu\Big(\bigcup_{A\in\mathfrak{C}} A\Big) > 1-\varepsilon \tag{1.2.9a}$$

and for all $A \in \mathfrak{C}$ one has

$$\sum_{B\in T^{-n-k}P^l} |\mu(B|A) - \mu(B)| < \varepsilon. \tag{1.2.9b}$$

If $\omega(n) \to 0$, as $n \to \infty$, P is called a *weak Bernoulli process* (cf. Ornstein 1974) and $\omega(n)$ is its rate. It is not hard to see that $\omega(n) = 2\beta(n)$.

The situation for very weak Bernoulli partitions P is in general unknown. Denote by $\mu(\ |A)$ the conditional measure given A and by $d(v, v')$ the Wasserstein distance of the two measures v and v'. Let $\omega^*(n)$ denote the infimum over all $\varepsilon \geqslant 0$ with the following property:

For each $k \geqslant 0$, there exists a collection $\mathfrak{C}$ of atoms in P^k, such that

$$\mu\Big(\bigcup_{A\in\mathfrak{C}} A\Big) > 1-\varepsilon \tag{1.2.10a}$$

and for every $A \in \mathfrak{C}$

(1.2.10b) $$d(\mu_n(\cdot|A), \mu_n) < \varepsilon$$

where $\mu_n = \mu | T^{-k-1} P$ denotes the restriction of μ on the σ-field generated by $T^{-k-1} P$.

If $\omega^*(n) \to 0$ then P is called *very weak Bernoulli* (cf. Ornstein 1974). Generalizations of this notion to general stationary processes are due to Gray, Neuhoff and Shields (1975), Eberlein (1983) and Dehling, Denker, Philipp (1984).

Certainly weak Bernoulli implies very weak Bernoulli. On the other hand, if $\liminf n\omega^*(n) = 0$, then the partitions $\{T^{-i}P\colon i \geqslant 0\}$ are independent (i.e. $\psi(n) = 0$ for all $n \geqslant 1$) (Dehling, Denker, Philipp 1984). It is also shown in this paper that mixing Markov shifts have the rate $\omega^*(n) = O\left(\frac{1}{n}\right)$ and if $\omega^*(n) = O\left(\frac{1}{n}\right)$, then $\beta(n) \to 0$, provided P is a finite partition. In general, the very weak Bernoulli property need not imply even strong mixing (Bradley 1984).

Finally, to complete this discussion of mixing concepts, let us mention that strong mixing implies the K-property. It seems to be unknown whether a transformation admitting a strong mixing generator (any rate) is Bernoulli (see Martin 1974).

The needs of ergodic theory concerning the CLT problem are covered by the monograph of Philipp and Stout (1975), where the martingale approach discussed in Section 1.1 is used to prove various results for mixing processes. Also the book and articles of Ibragimov and Linnik (1971), Iosifescu, Teodorescu (1969), Iosifescu (1980) or Ibragimov and Rozanov (1978) may serve as a general reference. Nevertheless it seems to be necessary to sketch the border line at which the CLT fails to hold. What will be said below also holds in most cases for weak and a.s. invariance principles and the law of the iterated logarithm. In order to keep this survey within a reasonable length we will omit these generalization, though we will need some of them in later sections.

The CLT problem for weakly dependent stationary sequences can be stated as follows: Under which mixing rates and moment condition does the CLT hold?

The first theorems are due to Rosenblatt (1956) and Ibragimov (1959, 1962, 1975). (Sinai's theorem (1960) will be discussed in § 2.)

THEOREM 1.2.1 (Ibragimov). *Let $(X_n)_{n \geqslant 0}$ be a stationary process.*

(a) *If $\sum \alpha(n) < \infty$ and $\|X_0\|_\infty < \infty$, then $\sigma^2 = \|X_0\|_2^2 + 2\sum_{t=1}^{\infty} E(X_0 X_t)$ exists and if $\sigma^2 > 0$ then $(X_n)_{n \geqslant 0}$ satisfies the CLT with $\sigma_n^2 = n\sigma^2(1+o(1))$.*

(b) *The assertion of* (a) *remains true if* $\sum \alpha(n)^{\delta/2+\delta} < \infty$ *and* $\|X_0\|_{2+\delta} < \infty$ *for some* $\delta > 0$.

(c) *If* $\sum \varrho(2^n) < \infty$, $\sigma_n^2 \to \infty$ *and* $\|X_0\|^2 < \infty$ *then* $(X_n)_{n\geq 0}$ *satisfies the* CLT. *Moreover* $(X_n)_{n\geq 0}$ *has a continuous spectral density* f *and* $\sigma_n^2 = 2\pi f(0)\, n(1+o(1))$.

(d) *If* $\varrho(n) \to 0$, $\sigma_n^2 \to \infty$ *and* $\|X_0\|_{2+\delta} < \infty$ *for some* $\delta > 0$, *then* $(X_n)_{n\geq 0}$ *satisfies the* CLT.

The functional form of assertions (a) and (b) have been obtained by Davydov (1968) and Oodaira, Yoshihara (1972). Slight generalizations are due to Herrndorf (1983). The conditions in Theorem 1.2.1 (a) and (b) are nearly sharp. Davydov (1969, 1973) showed that the imposed rate on $\alpha(n)$ essentially cannot be weakened: If for some $\varepsilon > 0$ $\bar{\alpha}(n) = O(n^{-\frac{2+\delta}{\delta}(1-\varepsilon)})$ (resp. $\bar{\alpha}(n) = O(n^{-1+\varepsilon})$), then there exists a stationary strongly mixing sequence (X_n) with α-mixing coefficients $\leq \bar{\alpha}(n)$ and with $\mu(|X_1|^{2+\delta}) < \infty$ such that $\sigma_n^2 \to \infty$ and the CLT does not hold. The assumption $\sigma^2 > 0$ in (a)–(c) means, of course, that X_0 is no coboundary. If one needs to avoid this assumption in (a) or (b), just assuming $\sigma_n^2 \to \infty$, it is necessary to strengthen the mixing rates to $\sum n\alpha(n) < \infty$ (resp. $\sum n\alpha(n)^{\delta/2+\delta} < \infty$). This has been shown by Bradley (1985). He also showed that these conditions are almost optimal.

The statement in (c) and (d) hold in particular in the φ-mixing case. In (c) the condition $\sigma_n^2 \to \infty$ can be replaced by $\sigma^2 > 0$ (Bradley 1981, Peligrad 1982). Examples showing that (c) and (d) are almost sharp are known (Bradley 1980) but not in the φ-mixing case. In fact Ibragimov conjectured that $\varphi(n) \to 0$, $\sigma_n^2 \to \infty$ and $\|X_0\|_2 < \infty$ suffice to ensure the CLT. (The functional form of this conjecture has been posed by Iosifescu.) Recently, some progress has been made towards a solution, proving equivalent conditions.

THEOREM 1.2.2 (Denker (1986), Dehling, Denker, Philipp (1986)). *Let* $(X_n)_{n\geq 0}$ *be a strongly mixing stationary sequence of random variables such that* $\|X_0\|_2 < \infty$ *and* $\sigma_n^2 \to \infty$.

Then the following are equivalent

(1) $(X_n)_{n\geq 0}$ *satisfies the* CLT;

(2) $\{S_n^2/\sigma_n^2 \colon n \geq 0\}$ *is uniformly integrable*;

(3) $\limsup_{n\to\infty} \sigma_n/\|S_n\|_1 \leq (\pi/2)^{1/2}$.

Condition (3) can be understood from an interesting characterization of normality. Let Y have an infinitely divisible distribution and assume that Y is symmetric. Then $\|Y\|_2/\|Y\|_1 \geq (\pi/2)^{1/2}$ and equality holds if and only if Y is normal.

In the uniform mixing case Peligrad (1985) obtained

THEOREM 1.2.3. *Let $(X_n)_{n\geq 0}$ be a uniformly mixing stationary sequence of random variables such that $\sigma_n^2 \to \infty$, $\varphi(1) < 1$ and $\|X_0\|_2 < \infty$. Then the following are equivalent*

(1) *(X_n) satisfies the* FCLT;

(2) *(X_n) satisfies the Lindeberg condition, i.e.*

$$\lim_{n\to\infty} \frac{n}{\sigma_n^2} \int_{\{|X_0|>\varepsilon\sigma_n\}} X_0^2\, d\mu = 0 \quad \text{for every } \varepsilon > 0.$$

In particular, a uniformly mixing sequence satisfies the CLT *if* $\liminf \sigma_n^2/n > 0$.

Compared to the results on the CLT in Section 1.1 it should be noted that parts of Theorem 1.2.1 can be proved by martingale approximation (Theorem 1.1.2), but not all of it. This is essentially true, if the mixing rate is fast enough.

In case of a very weak Bernoulli process the CLT is known only in a very special case, due to Eberlein (1983).

THEOREM 1.2.4. *Suppose that $(X_n)_{n\geq 0}$ is a very weak Bernoulli process with a compact state space and at a rate of $O(n^{-1})$. If the exceptional sets in the definition of very weak Bernoulliness are empty, then $\sigma^2 = \|X_0\|^2 + 2\sum_{t\geq 1} E(X_0 X_t)$ exists and if $\sigma^2 > 0$ the* CLT *holds with $\sigma_n^2 = n\sigma^2(1+o(1))$.*

(In fact it is shown that even the FCLT and an a.s. invariance principle hold.)

D. Fiebig (1988) recently showed that the rate $O(1/n)$ already implies that the exceptional set is empty.

It is certainly an interesting problem to prove a CLT for very weak Bernoulli processes in general, in order to apply it to dynamical systems. One way might be to recode finitarily the vwB process into an absolutely regular one such that the code has finite expected code length and such that the β-mixing coefficients are decreasing fast enough.

Finally, we need to discuss the class of functions for which one can obtain the CLT out of the mixing rates. The existence of

$$(1.2.11) \quad \sigma^2 = \|X_0\|^2 + 2\sum_{t\geq 1} E(X_0 X_t) = \int f^2\, d\mu + 2\sum_{t\geq 1} \int f\cdot(f\circ T^t)\, d\mu$$

if $X_0 = f$ and $X_t = f\circ T^t$ $(t \geq 1)$ turns out to be essential. Certainly σ^2 does not always exist, but Theorem 1.2.1 (a), (b) says that σ^2 exist if $\sum \alpha(n) < \infty$ (resp. $\sum \alpha(n)^{\delta/2+\delta} < \infty$). In the ϱ-mixing case this is true if $\sum \varrho(n) < \infty$ and $\|X_0\|^2 < \infty$, since

$$(1.2.12) \qquad E(X_0 X_t) \leq \varrho(t)\|X_0\|^2$$

and since $\varrho(n) \leq \varphi(n)^{1/2}$, $\varrho(n) \leq \varphi^*(n)^{1/2}$ and $\varrho(n) \leq \max(\varphi(n), \varphi^*(n))$ we deduce also the necessary condition in the uniform mixing case. (For completeness let us note that if $\|X\|_{2+\delta} < \infty$, then by Davydov's lemma (Davydov 1969)

$$E(X_0 X_t) \leqslant \alpha(t)^{\delta/2+\delta} \|X_0\|^2_{2+\delta}.) \tag{1.2.13}$$

Now let $f \in L^2$, $\int f d\mu = 0$. In the ϱ-mixing case (strong mixing is similar) let us assume that

$$\sum_{n \geqslant 1} [E(f - E(f \mid \mathfrak{F}_0^n))^2]^{1/2} < \infty. \tag{1.2.14}$$

Using (1.2.12), it is not hard to show that

$$E(f \cdot (f \circ T^n)) = O\left(\left(\varrho\left(\left[\frac{n}{3}\right]\right) + \left(E(f - E(f \mid \mathfrak{F}_0^{[\frac{n}{3}]}))^2\right)^{1/2}\right)\|f\|_2\right)$$

so that σ^2 in 1.2.11 still exists for $(f \circ T^n)_{n \geqslant 0}$. The same holds true with f replaced by $f - E(f \mid \mathfrak{F}_0^m)$, so that

$$\lim_{m \to \infty} \limsup_{n \to \infty} n^{-1/2} \left\| \sum_{k=0}^{n=1} (f \circ T^k - E(f \mid \mathfrak{F}_0^m) \circ T^k) \right\|_2 = 0.$$

These two basic observations together with the CLT for each function $E(f \mid \mathfrak{F}_0^m)$ permit to prove the following theorem (cf. Ibragimov, Linnik 1971).

THEOREM 1.2.5. (a) *Let $(X_n)_{n \geqslant 0}$ be ϱ-mixing satisfying $\Sigma\, \varrho(n) < \infty$. Assume that $f \in L^2$ satisfies $\int f d\mu = 0$ and* (1.2.14). *Then σ^2 in* (1.2.11) *exists and if $\sigma^2 > 0$ the* CLT *holds for f with $\sigma_n^2 = n\sigma^2(1 + o(1))$.*

(b) *Let $(X_n)_{n \geqslant 0}$ be strongly mixing satisfying $\sum \alpha(n)^{\delta/2+\delta} < \infty$ for some $\delta < 0$. Assume that $f \in L^{2+\delta}$ satisfies $\int f d\mu = 0$ and*

$$\sum_{n \geqslant 1} \|f - E(f \mid \mathfrak{F}_0^n)\|_{2+\delta} < \infty.$$

Then σ^2 in (1.2.11) *exists and f satisfies the* CLT *with $\sigma_n^2 = n\sigma^2(1 + o(1))$ if $\sigma^2 > 0$.*

This theorem permits to obtain CLT for large classes of functions (cf. the following section). In case of a.s. invariance principles (or FCLT) the reader is referred to Philipp and Stout (1975). Concluding this section we should mention investigations on the CLT for subadditive processes, initiated by Ishitani in 1977. Sufficiently strong results are derived by Wacker (1983), including ASIP. The mixing concepts for these processes can be easily adapted from the previous definitions observing that the time delay plays the important role. Using an additional approximation by a stationary process Wacker obtained the following result.

THEOREM 1.2.6. *Let $\{X_{n,m}\colon m \geqslant n \geqslant 0\}$ be a subadditive strongly mixing process such that $h(n) = n^{-1}(EX_{0,n}^2 - (EX_{0,n})^2)$ is a slowly varying function.*

Assume that there exists a stationary sequence $\{Y_n\colon n \geqslant 1\}$ *such that* $\|X_{0,n} - \sum_{i=0}^{n-1} Y_i\|_1 = o((nh(n))^{1/2})$. *If the sequence* $\{(nh(n))^{-1}(X_{0,n} - EX_{0,n})^2\colon n \geqslant 0\}$ *is uniformly integrable, then* $\{X_{n,m}\}$ *satisfies the* CLT, *i.e. for any* $x \in \boldsymbol{R}$

$$\lim_{n\to\infty} \mu\big((nh(n))^{-1/2}(X_{0,n} - EX_{0,n}) \leqslant x\big) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp(-t^2/2)\,dt.$$

§ 2. Central limit theorems for some dynamical systems

From a probabilistic point of view the important mixing conditions are uniform, correlation and strong mixing. This was discussed in some detail in the last section. From a dynamic viewpoint, however, ψ-mixing and very weak Bernoulliness play the essential role. Thus CLT's for φ- and α-mixing processes apply in these cases (provided the mixing rates are fast enough). This will be one of the main ideas of this section.

2.1. The CLT for geodesic flows

The first (non-trivial) CLT for a dynamical system was proved by Sinai in 1960 for geodesic flows on manifolds of constant negative curvature. Later proofs of CLT's also use the essential ideas of Sinai's approach (1960, 1962) which is based on the K-property and the fundamental approach of Rosenblatt (1956, see Section 1).

Let $(X_t\colon t \in \boldsymbol{R})$ be a continuous time, (strictly) stationary process and put $\mathfrak{F}_s^t = \sigma(X_u\colon s \leqslant u \leqslant t)$ as in Section 1.

Definition 2.1.1. The process $(X_t\colon t \in \boldsymbol{R})$ is called *locally Rosenblatt mixing* if for any $\varepsilon > 0$ there exist $t > 0$, an integer n and a partition $\{A_0, A_1, \ldots, A_n\}$ of the probability space such that each A_i is $\mathfrak{F}_{-\infty}^0$-measurable, $P(A_0) < \varepsilon$ and for each $i = 1, \ldots, n$

$$\sup\{|\mu(B \cap C|A_i) - \mu(B|A_i)\,\mu(C|A_i)|\colon B \in \mathfrak{F}_{-\infty}^0, C \in \mathfrak{F}_t^\infty\} < \varepsilon. \tag{2.1.1}$$

We note that this definition is weaker than strong mixing. (It reduces to strong mixing if n and $A_0, \ldots, A_n$ do not depend on ε.) If the process (X_t) is mixing (ergodic theoretic sense) and locally Rosenblatt mixing, then it is regular (i.e. a K-flow): $\bigcap_{t>0} \mathfrak{F}_{-\infty}^{-t}$ is trivial. It does not imply that (X_t) is regular in the other time direction, i.e. $\bigcap_{t>0} \mathfrak{F}_t^\infty$ trivial. This was noted by Krengel (1971) (and also by Sinai). However, if (X_t) is also regular in the forward time direction then it is strong mixing.

THEOREM 2.1.2 (Sinai, 1962). *Let* $(X_t\colon t \in \boldsymbol{R})$ *be a (strictly) stationary, mixing and locally Rosenblatt mixing process with*

$$\int X_0 \, d\mu = 0 \quad \text{and} \quad \sigma_\tau^2 = \int \Big[\int_0^\tau X_t \, dt\Big]^2 d\mu \to \infty \text{ as } \tau \to \infty.$$

Then (X_t) *satisfies the* CLT

$$\lim_{\tau \to \infty} \left| \sup_{x \in \boldsymbol{R}} \mu\Big(\int_0^\tau X_t \, dt < x\sqrt{a_\tau}\Big) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} \exp -\frac{u^2}{2} du \right| = 0$$

for some sequence $a_\tau \to \infty$ $(\tau \to \infty)$ *provided* (X_t) *satisfies the Lindeberg condition: For any* $\varepsilon > 0$ *there exists a constant* $N = N(\varepsilon)$ *such that*

$$\limsup_{\tau \to \infty} \frac{1}{\sigma_\tau^2} \int_{\{|\int_0^\tau X_t dt| > N\sigma_\tau^2\}} \Big(\int_0^\tau X_t \, dt\Big)^2 d\mu < \varepsilon. \tag{2.1.2}$$

Now let $(T_t\colon t \in \boldsymbol{R})$ denote a geodesic flow on a manifold of constant negative curvature. Using the previous theorem, Sinai (1960) obtained

THEOREM 2.1.3. *The space* CLT *contains all functions* f *having the following properties:* $\|f\|_\infty < \infty$, (2.1.2), $\lim_{t \to \infty} t^{-1}\sigma_t^2 > 0$ *and:*

there exist numbers α, ε_1, ε_2, $c_1 > 0$ *and* $c_2 \geqslant 0$ *such that for all* a *(with respect to the metric* d *on the space of linear elements)*

$$\mu\left(\left\{\sup_{y:d(x,y)<a} \Big|\int_0^\alpha \big(f(T_t(y)) - f(T_t(x))\big) dt\Big| > \frac{c_1}{|\log^{1+\varepsilon_1} a|}\right\}\right) \leqslant \frac{c_2}{|\log^{1+\varepsilon_2} a|}. \tag{2.1.3}$$

Condition (2.1.3) is certainly satisfied for continuous functions with modulus of continuity $1/|\log^{1+\varepsilon} a|$ and also by certain indicator functions. In the case of dimension 2 Sinai's paper (1960) contains conditions (similar to the functions of mixing processes) to ensure (2.1.2), i.e. essentially the CLT in the spirit of Theorem 1.2.5.

The method of proof for Theorem 2.1.3 was developed by Bunimovich (1974) to show the CLT for a class of billiards. Under certain conditions billiards in a domain $Q \subset \boldsymbol{R}^2$ are at least K, so that the above method seems to be appropriate. These conditions are the following (see Bunimovich 1973 for details):

(2.1.4) the scattering components of its boundary ∂Q intersect transversally

(2.1.5) the curvature of each focusing component Γ_i $(1 \leqslant i \geqslant l)$ is constant

(2.1.6) circles $\{O_{\Gamma_i} \supset \Gamma_i\}$ do not intersect each other,

(2.1.7) the arc $O_{\Gamma_i} - \Gamma_i$ is contained in Q $(1 \leqslant i \leqslant l)$.

THEOREM 2.1.4 (Bunimovich 1974). *If* $(T_t\colon t \in \boldsymbol{R})$ *denotes the billiard flow in a domain* Q *satisfying* (2.4)–(2.7), *then* CLT *contains all functions* f *satisfying* $\|f\|_\infty < \infty$, $\lim_{t\to\infty} t^{-1}\sigma_t^2 > 0$, (2.1.2) *and* (2.1.3), *where now* x *and* y *belong to the same element of the* K*-partition for* (T_t) *and where* d *denotes the metric on the elements of the partition induced by the metric on* M.

Sinai's result has been extended by Ratner (1973) to non-constant curvature. She used a representation of the flow as a suspension over a Gibbs measure (cf. Sinai 1972).

Let Σ denote a (one-or two-sided) topologically mixing subshift of finite type over the alphabet $\{1, \ldots, a\}$. Recall that a Gibbs measure μ on Σ is specified by a constant $P \in \boldsymbol{R}$ and a function $\varphi \in C(\Sigma)$ satisfying

$$\operatorname{var}_k(\varphi) = \sup\{|\varphi(x) - \varphi(y)|\colon x_i = y_i \text{ for } |i| \leqslant k\} \leqslant C\alpha^k$$

$(\alpha < 1,\ C > 0)$. The condition

$$0 < C_1 \leqslant \frac{\mu([a_0, a_1, \ldots, a_{n-1}])}{\exp(-nP + S_n\varphi(y))} \leqslant C_2 < \infty \tag{2.1.8}$$

for all points y in the cylinder set $[a_0, \ldots, a_{n-1}] \subset \Sigma$ determines μ uniquely (cf. Bowen, 1975). Here C_1 and C_2 denote two constants. It is known that the process $X_n\colon \Sigma \to \{1, \ldots, a\}$ defined by $X_n((x_k)_{k\in\boldsymbol{Z}}) = x_n$ $(n \in \boldsymbol{Z})$ is ψ-mixing with coefficients $\psi(n)$ decreasing exponentially fast (cf. Bowen, 1975). Thus we have an immediate corollary to Theorem 1.2.5.

THEOREM 2.1.5.

$$\text{CLT} \supset \{f \in L^2(\mu)\colon \sum_{n\geqslant 0} \big(E(f - E(f \mid \mathfrak{F}_{-n}^{n}))^2\big)^{1/2} < \infty\}.$$

We should note at this point that the definition of a Gibbs measure in (2.1.8) also makes sense on other spaces than subshifts of finite type. In this case, and even if the system is mixing, the coordinate process for a Gibbs measure need not to be strongly mixing, to that in this case it is not clear whether Hölder-continuous functions satisfy the CLT (which holds in the situation of Theorem 2.1.5). Certainly, in the situation of the previous theorem one also has invariance principles and laws of the iterated logarithm (Philipp, Stout 1975, p. 81, remark). This result can be used in a randomized version to prove CLT's for flows under a function and with base (Σ, μ). This gives Ratner's result and the extension by Denker, Philipp (1984).

Let (Σ, μ) be as before and let $l\colon \Sigma \to \boldsymbol{R}_+$ be a Hölder-continuous function with $\inf_{x\in\Sigma} l(x) > 0$.

Let $\Omega = \{(x, s)\colon x \in \Sigma\ 0 \leqslant s < l(x)\}$, $dv = (\int l\,d\mu)^{-1} ds \times d\mu$ and let $(T_t)_{t\in\boldsymbol{R}}$ denote the corresponding flow under a function. This flow serves as a model for smooth invariant measures for transitive C^2-Anosov flows (even Axiom A flows) including geodesic flows on manifolds of negative curvature relative to the invariant Riemannian volume.

THEOREM 2.1.6 (Denker, Philipp 1984). *Let f be a measurable function on Ω with $\int f d\mu = 0$ and $\|f\|_{2+\delta} < \infty$ for some $0 < \delta \leqslant 1$. Suppose that*

$$\|f - E(f \mid \gamma_n)\|_{2+\delta} = O(n^{-2-7/\delta}) \tag{2.1.9}$$

where γ_n denotes the σ-algebra generated by the sets of the form $\{(x, s)\colon 0 \leqslant s < l(x),\ x \in A\}$ where $A \subset \Sigma$ is a centered cylinder of length $2n+1$. Then

$$\sigma^2 = \lim_{t\to\infty} t^{-1}\sigma_t^2$$

exists and if $\sigma^2 > 0$ then there exists a standard Brownian motion $B(u, x)$ $(x \in \Sigma,\ u \geqslant 0)$ defined on the probability space (Σ, μ) such that

$$\sup_{0 \leqslant s < l(x)} \left|\int_0^u f(T_t(x, s))\,dt - B(\sigma^2 u, x)\right| = O(u^{1/2-\lambda}) \tag{2.1.10}$$

for μ a.a. $x \in \Sigma$ and for some $0 < \lambda < {}^{\delta/588}$.

Unfortunately, (2.1.10) does not imply the CLT for f, however the LIL. In order to obtain the CLT for f also, (2.1.10) can be replaced by the same assertion when B is defined on (Σ, μ^*) where

$$d\mu^* = \left(\int l d\mu\right)^{-1} l d\mu.$$

This means that B is no longer Gaussian with respect to μ but with respect to μ^*.

In fact, considering B as a function on Ω constant on fibers, for $\lambda \in \mathbf{R}$ we have

$$\begin{aligned}
&\nu\Big(\Big\{(x, u) \in \Omega\colon \int_0^T f(T_t(x, u))\,dt \leqslant \sigma\sqrt{T}\,\lambda\Big\}\Big) \qquad (2.1.11)\\
&= \nu\Big(\Big\{(x, u) \in \Omega\colon \int_0^T f(T_t(x, u))\,dt \leqslant \sigma\sqrt{T}\,\lambda;\ \Big|\int_0^T f(T_t(x, u))\,dt\\
&\qquad B(\sigma^2 T, x)\Big| \leqslant T^{\frac{1-\lambda}{2}}\Big\}\Big)\\
&\quad + \nu\Big(\Big\{(x, u) \in \Omega\colon \int_0^T f(T_t(x, u))\,dt \leqslant \sigma\sqrt{T}\,\lambda;\ \Big|\int_0^T f(T_t(x, u))\,dt\\
&\qquad - B(\sigma^2 T, x)\Big| > T^{\frac{1-\lambda}{2}}\Big\}\Big)\\
&\leqslant \nu\Big(\Big\{(x, u) \in \Omega\colon B(\sigma^2 T, x) \leqslant \sigma\sqrt{T}\,\lambda(1 + T^{-\lambda/2}\sigma^{-1}\lambda^{-1})\Big\}\Big)\\
&\quad + \nu\Big(\Big\{(x, u) \in \Omega\colon \Big|\int_0^T f(T_t(x, u))\,dt - B(\sigma^2 T, x)\Big| > T^{\frac{1-\lambda}{2}}\Big\}\Big).
\end{aligned}$$

The second term tends to zero as $T \to \infty$ by (2.1.10). Since $B(\sigma^2 T, \cdot)$ is normal with mean zero variance $\sigma^2 T$ with respect to μ^*, the first term equals

$$\left(\int l d\mu\right)^{-1} \int_{\{x\in\Sigma: B(\sigma^2 T, x) \leqslant \sigma\sqrt{T}\lambda(1+T^{-\lambda/2}\sigma^{-1}\lambda^{-1})\}} \int_0^{l(x)} dt\mu(dx)$$

$$= \mu^*\left(\{x \in \Sigma\colon B(\sigma^2 T, x)\sigma\sqrt{T}\,\lambda(1+T^{-\lambda/2}\sigma^{-1}\lambda^{-1})\}\right)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{(\lambda(1+T^{-\lambda/2}\sigma^{-1}\lambda^{-1})} \exp\left(-\frac{v^2}{2}\right) dv.$$

Letting $T \to \infty$ and using a similar estimate to obtain a lower bound for (2.1.11), the CLT follows.

Using the existence of Markov partitions for Axiom A diffeomorphisms, it follows immediately from Theorem 2.1.5 that CLT contains all Hölder-continuous functions. Clearly they also satisfy the ASIP with remainder $n^{1/2-\lambda}$ $(\lambda > 0)$. It is also immediately clear that the correlation function for two Hölder-continuous functions decreases exponentially fast. This is not true for Axiom A flows (Ruelle, 1983). Pollicott (1985) has shown that in order to have an exponential decay it is necessary that the Zeta function of the flow has an analytic extension to a domain $\{z \in \boldsymbol{C}\colon \operatorname{Re}(z) > P(F)-\varepsilon\}$ for some $\varepsilon > 0$, where $P(F)$ denotes the pressure of the Hölder-continuous function and where F defines the Gibbs measure (equilibrium for $P(F)$). Condition (2.1.9) in Theorem 2.1.6 does not hold for all Hölder-continuous functions, since it is necessary for (2.1.9) to hold that the function f is constant for a while in the flow direction. Nevertheless the assertion of Theorem 2.1.6 holds for all Hölder-continuous functions $g\colon X \to \boldsymbol{R}$. Indeed, the function f defined by

$$f(x, t) = l(x)^{-1} \int_0^{l(x)} g(x, \tau)\, d\tau \qquad (x \in \Sigma,\ 0 \leqslant t > l(x))$$

satisfies the assumptions of 2.6, and hence satisfies the ASIP. Consequently it is sufficient to show that

$$\int_0^T g(T_t(x, u))\,dt - \int_0^T f(T_t(x, u))\,dt = O(1) \text{ as } T \to \infty \text{ for all } (x, u) \in \Omega.$$

For fixed $(x, u) \in \Omega$ and $T > 0$ write

$$T = \sum_{k=0}^{n-1} l(S^k(x)) - u + \tau$$

where $0 \leqslant \tau < l(S^n(x))$ and where $S\colon \Sigma \to \Sigma$ denotes the shift transformation.

Then

$$\begin{aligned}\int_0^T g(T_t(x, u))\,dt &= \sum_{k=1}^{n-1} \int_0^{l(S^k(x))} g(S^k(x), t)\,dt \\ &\quad + \int_u^{l(x)} g(x, t)\,dt + \int_0^{\tau} g(S^n(x), t)\,dt \\ &= \sum_{k=1}^{n-1} \int_0^{l(S^k(x))} f(S^k(x), t)\,dt + O(1) \\ &= \int_0^T f(T_t(x, u))\,dt + O(1).\end{aligned}$$

2.2. The CLT for maps of the interval

In the previous section we discussed several examples where the ψ-mixing property has been established and hence the CLT holds using the results of § 1 about uniform mixing sequence (or at least strong mixing). For expanding maps on the interval one can only use the weak Bernoulli property in general, thus the strong mixing results of § 1 apply.

Neglecting some results in probabilistic number theory the first results on the CLT question for expanding piecewise C^2-mappings on the unit interval were established by Ishitani (1976) and more generally by Wong (1979). Wong considered transformations T: $[0, 1] \to [0, 1]$ having the following property: There exists a finite partition of $[0, 1]$, $\mathfrak{R} = \{(0, a_1), (a_1, a_2), \ldots, (a_{r-1}, 1)\}$ such that for each i $T_i = T|_{(a_{i-1}, a_i)}$ can be extended to the closed interval $[a_{i-1}, a_i]$ as a C^2-function. Moreover, $\inf\{|T'(x)|: x \in (a_{i-1}, a_i), 1 \leqslant i \leqslant r\} > 1$.

If T denotes such a transformation there exists an invariant absolutely continuous probability μ on $[0, 1]$ by the result of Lasota and Yorke (1973). Moreover, if μ is weak mixing then it is unique, and Bowen (1977) showed that in the latter case the partition $\mathfrak{R}$ is a weak Bernoulli generator. Using this result, Wong showed that for all Hölder-continuous functions f: $[0, 1] \to \boldsymbol{R}$ for which $\sigma_n^2/n \to \sigma^2 > 0$ the CLT holds provided $S_n f$ satisfies a Lindeberg condition (similar to Sinai's and Bunimovich's result). Wong's CLT is of the type of Theorem 1.2.2, and stronger results are immediate from § 1.

Boyarsky and Scarowsky (1979) have a CLT in their paper for a subclass of piecewise C^2-mappings. They observed that the results of § 1 can be applied, but incidently their Theorem 2 does not seem to be completely correct. Other special cases of maps of the interval are treated by Jabłoński, Malczak (1983).

These results are all contained in the Hofbauer–Keller theorem. A function $f\colon [0,1] \to \boldsymbol{R}$ is said to be of *bounded p-variation* if

$$\sup \sum_{i=1}^{n} |f(a_i) - f(a_{i-1})|^p < \infty \tag{2.2.1}$$

where the supremum extends over all finite collections $0 \leqslant a_0 < a_1 < \ldots < a_n \leqslant 1$.

THEOREM 2.2.1 (Hofbauer, Keller, 1983). *Assume that the expanding, piecewise C^2 map T on $[0, 1]$ is weakly mixing. Let $f\colon [0, 1] \to \boldsymbol{R}$ be a function of bounded p-variation for some $p \geqslant 1$ and $\int f\,d\mu = 0$. Then*

$$\sigma^2 = \int f^2\,d\mu + 2\sum_{k=1}^{\infty} \int f(f \circ T^k)\,d\mu$$

converges absolutely. If $\sigma^2 > 0$ then we have an ASIP *with a Brownian motion $B(t)$ such that*

$$\sigma^{-1} S_{[t]} - B(t) = O(t^{1/2-\lambda}) \ a.s.$$

for some $\lambda > 0$.

In the non-weak mixing case Ishitani (1986) has proved CLT for functions f with bounded variation together with convergence rates in this theorem up to the error term $O(n^{-1/2})$ which is best possible. It turns out that if μ is not weak mixing one obtains a convergence theorem in which the limiting distribution is a mixture of normal distributions. It is also worth mentioning that T-invariance of μ is not necessary for the CLT to hold. Therefore one can replace the measures μ by absolutely continuous probabilities.

The essential part in the proof of Theorem 2.2.1 (and assumed by Ishitani and others) is the estimate

$$\operatorname{var}(f \circ T^n) \leqslant \alpha \operatorname{var} f + \beta \|f\|_1 \qquad (f \in L_1(dx)) \tag{2.2.2}$$

for some $n \geqslant 1$, $0 < \alpha < 1$ and $\beta > 0$. Special transformations with this property are β-transformations (the CLT was obtained by Ishitani 1976), unimodal linear transformations, Wilkinson's piecewise linear transformations, continued fraction, piecewise convex mappings (defined in Lasota, Yorke (1982). CLT is due to Jabłoński, Malczak (1983)).

An elegant presentation and an extension of the Hofbauer, Keller result can be found in Rychlik (1983). The definition of the expanding, piecewise C^2-map can be extended to a countable number of intervals if one assumes in addition that only a finite number of intervals appear as images of the partition-intervals and if $1/|T'|$ is of bounded variation. The paper of Jabłoński, Malczak (1985) deals also with this type of extension as well as that of Rousseau–Egele (1983). He obtained a CLT with convergence rate $O(n^{-1/2})$ and a local limit theorem. It is remarkable that his method of proof is different. The idea goes back to Doeblin and Fortet and was further developed in a recent paper by Guivarc'h and Hardy (1986). They show the convergence of the Fourier transforms of $S_n f$ by applying the theorem of C. Ionescu–Tulcea and Marinescu directly to the operators $P_\lambda g(x) = P(e^{\lambda\varphi} g)(x) = \int e^{\lambda\varphi(x,y)} g(y) P(x, dy)$, where φ is a fixed function and P a transition kernel (this is the more general set up of Guivarc'h and Hardy).

At this point it should be noted that Keller (1980) used Gordin's approach to give a satisfactory description of functions satisfying the CLT for expanding maps of the interval. Gordin remarked that it is sufficient to show that

$$\sum_{k=0}^{\infty} \left[\int |T^k \circ T^{*k} f|^2 \, dm\right]^{1/2} < \infty$$

in order to have the CLT for $S_n f$. Then CLT contains all functions $f \in L^2$, $\int f \, d\mu = 0$ such that there exist functions f_n of bounded variation satisfying

$$\sum_{n \geq 1} \left(\int |f - f_n|^2 \, d\lambda\right)^{1/2} < \infty \quad \text{and}$$

$$\lim_{n > \infty} \operatorname{var}(f_n)(1+h)^{-n} = 0 \quad (\text{for all } h > 0).$$

The essential condition in these theorems is the bounded variation of $1/|T'|$ which ensures (2.2.2). Keller (1985) extended Theorem 2.2.1 to transformations T for which $1/|T'|$ is of bounded p-variation (see (2.2.1)), proving (2.2.2) for p-variations.

There are still important classes of transformations of the interval not covered by the previous discussion. Misiurewicz (1981) showed the existence of absolutely continuous T^k-invariant measures (and exactness) of certain maps T of non-positive Schwarzian derivative. Ziemian (1985) proved an ASIP for the maps using the basic idea of Hofbauer and Keller (1983). The assumptions on T are as follows: T is continuous and there exists a finite set $A \subset [0, 1]$ containing 0 and 1 such that T is monotone on each of the subintervals of A^c and such that T/A^c has the following 6 properties:

(M1) T is C^3;
(M2) $T \neq 0$;
(M3) $(T'''/T' - \frac{3}{2}(T''/T')^2) \leq 0$ (non-positive Schwarzian derivative);
(M4) if $T^p x = x$ then $|(T^p)'(x)| > 1$;
(M5) there exists a neighbourhood U of A such that for all $n \geq 0$ and $a \in A$ $T^n(a) \in A$ or $T^m(a) \notin U$ $(\forall m \geq n)$;

(M6) For every $a \in A$ there exist a neighbourhood U_a, constants $\alpha, \omega, \delta > 0$ and $u \geqslant 0$ such for all $x \in U_a$

(a) $\alpha |x-a|^u \leqslant |T'(x)| \leqslant \omega |x-a|^{u-1}$;

(b) $|T''(x)| \leqslant \delta |x-a|^{u-1}$.

THEOREM 2.2.2 (Ziemian 1985). *The assertion of Theorem* 2.2.1 *remains valid for T^k, where T satisfies the above conditions and where k is such that T^k is exact with respect to the absolutely continuous invariant measure.*

Concluding this section, we pose a problem. Ledrappier (1981) has shown that certain differentiable maps on the unit interval are weakly Bernoulli. The argument does not seem to permit to compute mixing rates. So, it is an interesting problem to show CLT's in this case.

2.3. The CLT for general dynamical systems

In this section we consider a general dynamical system and we want to get some insight which functions satisfy the CLT. The only question to which we can make a contribution is to show that CLT $\neq \emptyset$ and that CLT is dense if T is ergodic. If T has positive entropy then clearly, CLT $\neq \emptyset$ since there are Bernoulli factors. So the interesting case is entropy zero.

Among the entropy zero transformations let us first consider a Gaussian system. The first one has been found by Girsanov (1958). Since linear images of n-dimensional normal measures are again normal we clearly have a CLT if the variance do not vanish on a subsequence (a non-typical case). Maruyama (1967) gave a weak mixing entropy zero Gaussian system and using this result produced (1975) an example of such a system with sufficiently many functions satisfying the CLT. The idea is to use a kind of multiple stochastic integral to produce these functions (like for self-similar processes).

Conze raised the question whether an irrational rotation has a function satisfying the CLT. Since the sequence $\{S_n f\}$ is relatively compact in L^2 (for $f \in L^2$), it seems to be surprising that the answer is positive.

THEOREM 2.3.1 (Burton, Denker, 1987). *Let T be an irrational rotation of S^1. Then* CLT *is dense in L^2.*

This result cannot be a corollary to the limit theorems in § 1 because of the relative compactness property. If turns out, however, that it can be proved using the Salem–Zagmund (1947) CLT for lacunary sequences and a fine choice of the growth of the variance in this theorem. The result of Salem and Zygmund can be proved using variants of the limit theorems in § 1 (non-stationary versions). It says the following (this is not the strongest form, which has been given by Erdös, but it suffices here): Let $n_k \in \boldsymbol{N}$ be a lacunary sequence satisfying $n_{k+1} > 3n_k$ $(k = 1, 2, \ldots)$. Let $C_k, D_k \in \boldsymbol{R}$ and $a_k^2 = C_k^2 + D_k^2$. If $A_m^2 = \frac{1}{2} \sum_{j=1}^{m} C_j^2 + D_j^2 \to \infty$ and $a_m/A_m \to 0$ then (with respect to Lebesgue measure μ)

$$\mu\left(x\in[0,1]\colon \frac{1}{A_m}\sum_{k=1}^{m} C_k\cos(2\pi n_k x)+D_k\sin(2\pi n_k x)\leqslant u\right)$$

$$\to\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{u}\exp\left(-\frac{t^2}{2}\right)dt \text{ as } m\to\infty.$$

Having established this result, one obtains the idea to prove Theorem 2.9 in general.

THEOREM 2.3.2 (Burton, Denker, 1987). *Let T be an aperiodic transformation. Then* CLT $\neq \emptyset$. *If T is in addition ergodic, then* CLT *is dense in* L^2.

The first example of a non-Gaussian, entropy zero transformation, I believe, has been known to Rothstein (1984) (oral communication to Burton). He has a construction of a class of Vershik processes with CLT $\neq \emptyset$.

The density statement follows from the fact that coboundaries are dense in L^2 in the ergodic case (see Gordin, Lifšic (1978) or Bhattacharya (1982)). By a construction due to Ornstein (oral communication by A. Katok) one can show that every $f\in L^2$, $\int f\,d\mu=0$ is cohomologuous to a continuous function (if the dynamical system is compact, metric and T continuous). It is not known whether one can always obtain a Hölder-continuous function in CLT.

It remains to determine CLT, especially for irrational rotations. Is it possible, for example, to find an $f\in$CLT such that the variance σ_n^2 behaves linearly? The examples of Rothstein do have this property, while the constructions in Theorems 2.3.1 and 2.3.2 can not provide such a function.

§ 3. Some further investigations using CLT

Theorems on the central limit problem certainly are of general interest in itself. However there are also results which can be proved using information about the CLT, for example concerning entropy and the information function (cf. Philipp, Stout (1975)). Here we give three of these applications.

3.1. Isomorphism theory

Given two dynamical systems (X, T, μ) and (Y, S, ν) and a measure preserving homomorphism $\varphi\colon X\to Y$, then clearly φ (CLT(Y)) $\subset$ CLT(X), i.e. if $f\colon Y\to \boldsymbol{R}$ satisfies the CLT so does $f\circ\varphi$. It is of particular interest in various situations to show the CLT for Hölder-continuous functions on Y; one way of doing this is to show that the class H of Hölder-continuous functions satisfies $\varphi(H)\subset\mathcal{S}$ for some $\mathcal{S}\subset$ CLT(X). Candidates for $\mathcal{S}$ are given in § 1, for example if $g\colon X\to\boldsymbol{R}$ is continuous and $\{g\circ T^n\colon n\geqslant 0\}$ is strongly mixing with rate $\alpha(n) = O(n^{-1-\delta})$, then all bounded functions h on X with

$$\sum_{n\geq 0} \|h - E_\mu(h|g \circ T^j\colon\ 0 \leq j < n)\|_\infty < \infty \tag{3.1.1}$$

satisfy the CLT. Then it is easy to see that the homomorphism has to be finitary, and the problem is to find a condition to guarantee the above condition.

This is the idea behind the CLT in Denker, Keane (1979). Let X and Y be subshifts. The finitary homomorphism φ: $X \to Y$ is said to have *finite expected code length* if the random variable τ on X is integrable, where $\tau(x)$ denotes the minimal length of a (centered) cylinder around x whose image under φ is completely contained in the cylinder $\{y \in Y\colon y_0 = (\varphi(x))_0\}$. A function h: $X \to \boldsymbol{R}$ is called *sequential* if $h = \sum_{k=1}^{\infty} h_k$ (in L^1) for some functions h_k measurable w.r. to the centered cylinders of length $2k+1$, and if $\Sigma|kEh_k| < \infty$. Similarly sequential function for Y are defined. Denote by $\mathscr{S}_X$ and $\mathscr{S}_Y$ the class of these functions. Then, if T has finite expected code length and if L denotes the class of all Lipschitz functions, $\varphi(L) \subset \mathscr{S}_X$. Thus, if the coordinate process on X has a good mixing rate, the CLT holds for all $f \in L$. Using a slightly stronger notion of finite expected code length one can even show φ $(\mathscr{S}_Y) \subset \mathscr{S}_X$. For higher moments of τ one obtains the corresponding statements for the law of the iterated logarithm (Denker, Keane 1980) and even for the ASIP (no proof exists for this).

In particular, under the strong mixing assumption for example, the variances of functions in $\mathscr{S}_X$ behave linearly and hence linearity of the variances in $\mathscr{S}_Y$ resp. $\mathscr{S}_X$ is an invariant for this type of isomorphism φ. Parry (1978) has investigated a more sophisticated invariant than this one (cf. also Parry, Tuncel 1982). If the function f: $X \to \boldsymbol{R}$ is cohomologuous to a function $h \circ \varphi$ (h: $Y \to \boldsymbol{R}$) then clearly the asymptotic variances (if one of them exists) have to be the same. If α, β denote the natural partitions on X and Y, respectively, the information cocycles are defined by

$$I_T = I(\hat{\alpha}|T^{-1}\hat{\alpha}) \quad (\hat{\alpha} = \alpha \vee T^{-1}\alpha v \ldots)$$

and

$$I_S = I(\hat{\beta}|S^{-1}\hat{\beta}) \quad (\hat{\beta} = \hat{\beta} \vee T^{-1}\beta \vee \ldots).$$

Denote by I_T^0 and I_S^0 their centered forms. Their asymptotic variances are called *information variances* (Fellgett, Parry 1975).

Parry showed that the information cocycles are cohomologuous if φ is a regular isomorphism (i.e. an isomorphism φ such that for some $p \geq 1$ $\varphi^{-1}\hat{\beta} \subset T^p\hat{\alpha}$ and $\varphi\hat{\alpha} \subset S^p\hat{\beta}$).

Thus the information variance is a regular isomorphism invariant (if it exists) and the CLT holds for one information cocycle if it does for the other.

This result is also true for quasi-regular isomorphisms, block codes (Bowen 1975, Parry 1978), and finitary isomorphisms φ with finite expected code lengths for φ and φ^{-1} (Parry 1979). As a consequence certain Markov chains are seen to be non-isomorphic in any of these senses (Parry, Schmidt 1984). The idea of using the CLT as an isomorphism invariant can be found for the first time in Bowen (1975b) and Fellgett, Parry (1975).

3.2. The CLT and Hausdorff measures

Let U be a neighbourhood of a compact set X (card $X \geqslant 2$) in a differentiable manifold M. Following Ruelle (1982) a differentiable map T: $U \to M$ is called a *mixing repeller* if

(i) $T(X) = X$;

(ii) T is expanding on X (i.e. $\|DT^n(v)\| \geqslant K\lambda^n\|v\|$ for some constants $K > 0$, $\lambda > 1$ and all $v \in T_x M\,(x \in X)$, $n \geqslant 0$;

(iii) X is a repeller for T, i.e. there exists a neighbourhood V, $X \subset V \subset U$ such that $\bigcap_{n \geqslant 0} (T/V)^{-n} V = X$;

(iv) T is topologically exact, i.e. for every open non-empty $W \subset X$ there is an $n \geqslant 0$ such that $T^n W \supset X$.

If T is a mixing repeller then (X, T) is an almost one-to-one continuous factor of a one sided topologically mixing Markov chain, generated by a one sided Markov partition for T. It follows that Theorem 2.1.5 applies and one obtains the CLT, LIL and an ASIP. This situation arises especially for expanding rational maps on the Riemann sphere.

Przytycki, Urbański and Zdunik (1986) used the result for the law of the iterated logarithm to study the relations between harmonic measures and Hausdorff measures for repelling boundary domains in $\boldsymbol{C}$. They consider a holomorphic map T on a simply connected domain Ω in the Riemann sphere such that Ω is a mixing repeller. A harmonic measure ω is defined by the image of the normalized Lebesgue measure on S^1 by a Riemann map R: $\{|z| < 1\} \to \Omega$, extended to the boundaries. Every two harmonic measures are known to be equivalent. If φ: $\boldsymbol{R}_+ \to \boldsymbol{R}_+$ is an increasing continuous function with $\varphi(0) = 0$, we denote by μ_φ the corresponding Hausdorff measure.

THEOREM 3.2.1 (Przytycki, Urbański, Zdunik 1986). *For $c \geqslant 0$ denote by φ_c the function*

$$\varphi_c(t) = t \exp\left[c(\log 1/t \log\log\log 1/t)^{1/2}\right] \qquad (t \geqslant 0).$$

Then for a repelling boundary domain Ω there exists a number c_0 such that the following dichotomy holds:

If $0 < c < c_0$, then the harmonic measure ω and the Hausdorff measure μ_{φ_c} are singular.

If $c > c_0$, then the harmonic measure ω is absolutely continuous with respect to the Hausdorff measure μ_{φ_c}.

If $c_0 > 0$ and if $T/\partial\Omega$ is expanding then the first case also holds for $c = c_0$, and if $c_0 = 0$, T expanding on $\partial\Omega$, then $\partial\Omega$ must be a Jordan curve. Moreover, if $c_0 = 0$ and $T/\partial\Omega$ extends holomorphically to Ω, then $\omega \ll \mu_1$.

The transition parameter c_0 is obtained from the asymptotic variance σ^2 in the CLT; $c_0 = \frac{\sigma\sqrt{2}}{\sqrt{\chi}}$ where χ is the Lyapunov exponent of f with respect to ω

$$\sigma^2 = \lim_{n\to\infty} \frac{1}{n} \operatorname{Var} S_n \psi,$$

$$\psi = \log|(R^{-1}\circ T\circ R)'| - \log|T'\circ R|$$

and where the probability measure is given by the $R^{-1}\circ T\circ R$-invariant measure on $\{|z| = 1\}$, equivalent to Lebesgue measure. We leave the discussion at this point, indicating to what type of results CLT questions can lead. For more details and information on the subject we refer to the paper of Przytycki, Urbański and Zdunik. Recently Denker and Urbański (1987) obtained similar results for expanding maps on S^1 restricted to compact invariant subsets.

3.3. Statistical functionals

One of the motivations for the classical CLT is its statistical relevance for estimating or testing procedures. It was pointed out in Denker (1982) that these statistical procedures also make sense for dynamical systems under suitable assumptions, and that the limit theory in various situations for dynamical systems has to be carried out. There are many papers generalizing the limit theory for statistical functions to mixing sequences. Contrary to the results in § 1 it is not immediate to obtain the corresponding results for functionals of mixing sequences. This has been carried out first in Denker and Keller (1986). In this paper it is shown that the mean $S_n f$ is not sufficient to obtain optimal results. Thus it is necessary to consider more complicated functionals. One of them is called *von Mises functional* (more generally, *differentiable statistical functional*) and can be defined by "generalized" sums

$$V_n h(x) = \sum_{1\leq i_1,\ldots,i_m\leq n} h(f\circ T^{i_1}(x), f\circ T^{i_2}(x), \ldots, f\circ T^{i_m}(x))$$

where f is a measurable function with values in $[0, 1]$ and h: $[0, 1]^m \to \boldsymbol{R}$ is measurable and symmetric. The limit theory for $V_n h$ is carried out in Denker, Keller (1986) in the case where $h \in L^2(\lambda^m)$, h vanishes on all diagonals and is symmetric and degenerate. It turns out that in many cases one has a CLT. If h is degenerate, $m \leqslant 2$ the limit distributions are of different type (cf. Denker (1982), Babbel (1986)). Basic to these investigations are the results of § 1 together with extensions to ASIP for empirical processes of functionals of mixing processes. Since the details become rather technical and lengthy, we leave it at this stage referring to the above papers.

Another type of more complicated statistical functionals is given by ranking methods. They can be treated as stochastic integrals (like $V_n f$) using the formula

$$R_n f(x, y) = \int_0^1 f\big(H_{n,m}(x, y, t)\big) F_n(x, dt)$$

where f: $(0, 1) \to \boldsymbol{R}$ is measurable,

$$F_n(x, t) = \frac{1}{n} \sum_{j=0}^{n-1} c_{jn} 1_{\{g \circ T^j(x) \leqslant t\}}$$

and

$$H_{n,m}(x, y, t) = \frac{1}{n+m+1} \Big(\sum_{j=0}^{n-1} 1_{\{g \circ T^j(x) \leqslant t\}} + \sum_{l=0}^{m-1} 1_{\{g' \circ T'^l(y) \leqslant t\}} \Big).$$

Here $\{g \circ T^j\}$ and $\{g' \circ T'\}$ are two observed processes (from possibly two systems T and T') and c_{jn} constants. The asymptotic distributions in this case are obtained in Denker, Keller, Puri (1987) and again we refer to this paper for details.

The paper of Denker, Keller (1986) contains only very little on applications of these limit results. Much more has to be done, especially in finding optimal (or good) procedures for the various statistical questions arising in the study of dynamical systems, for example density estimation, symmetry, entropy, Hausdorff dimension etc.

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