ANNALES
POLONICI MATHEMATICI
L (1989)

# The connection between number and form of bifurcation points and properties of the nonlinear perturbation of Berestycki type

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**Abstract.** In this paper we give a certain description of the bifurcation points in a bifurcation interval; in particular, we prove the sufficient condition for bifurcation interval to degenerate to point.

**I.** The definition of bifurcation interval was introduced by H. Berestycki in [1]. By a *bifurcation interval* is understood the interval which contains at least one bifurcation point. This definition is too general for formulation any abstract theory. The purpose of this paper is to give the connection between number and form of bifurcation points and properties of the nonlinear perturbation of the Berestycki type.

**II.** Consider the equation

$$\mathscr{L}u \equiv -(pu')' + qu = \lambda u + F(\cdot, u, u', \lambda) \quad \text{in } (0, \pi) \tag{1}$$

together with the separated boundary conditions

$$\alpha_1 u(0) - \alpha_2 u'(0) = 0, \quad \beta_1 u(\pi) + \beta_2 u'(\pi) = 0, \tag{2}$$

where $\alpha_i, \beta_i \geqslant 0$ and $(\alpha_1^2+\alpha_2^2)(\beta_1^2+\beta_2^2) \neq 0$. As usual, we assume that $p$ is positive and $p \in C^1[0, \pi]$, $q \in C[0, \pi]$. The nonlinear function $F$ has the form $F = f+g$, where $f, g \in C([0, \pi] \times R^3)$ and satisfy the following conditions:

(3) $\exists M > 0 \, \forall x \in [0, \pi] \, \forall w, s \in R,$

$$0 < |w| \leqslant 1, \ |s| \leqslant 1 \ \forall \lambda \in R \ \left|\frac{f(x, w, s, \lambda)}{w}\right| \leqslant M.$$

(4) $g(x, w, s, \lambda) = o(|w| + |s|)$ near $(w, s) = (0, 0)$ uniformly in $x \in [0, \pi]$ and in $\lambda \in \Lambda$ for every bounded interval $\Lambda \subset R$.

If $F = 0$ equation (1) becomes a linear Sturm-Liouville problem

(5) $\mathcal{L}u = \lambda u$ in $(0, \pi)$

together with the boundary conditions (2).

As is well known (5), (2) has an increasing sequence of simple eigenvalues $\mu_1 < \mu_2 < \ldots < \mu_k \to \infty$. Any eigenfunction $v_k$ corresponding to $\mu_k$ has exactly $k-1$ simple nodal zeros in $(0, \pi)$.

Let $E = C^1[0, \pi] \cap (2)$ under the usual maximum norm

$$\|u\|_1 = \max_{[0,\pi]} |u(x)| + \max_{[0,\pi]} |u'(x)|.$$

By $S_k^\nu$ (where $\nu \in \{+, -\}$) we denote the set of those functions in $E$ which have exactly $k-1$ interior nodal zeros in $(0, \pi)$ and have fixed sign $\nu$ in a certain neighbourhood of 0 (eventual zeros in 0 or $\pi$ must also be nodal). Let $S_k^- := -S_k^+$. Let $\mathcal{S}$ denote the closure in $R \times E$ of the set of nontrivial solutions of (1), (2).

In this notation we can give the following theorem.

THEOREM 1 ([1]). *For every $k \in N$ and $\nu \in \{+, -\}$ there exists at least one unbounded continuum of $\mathcal{S}$ bifurcating from $[\mu_k - M, \mu_k + M] \times \{0\}$ and contained in $(R \times S_k^\nu) \cup ([\mu_k - M, \mu_k + M] \times \{0\})$.*

**III.** Look at some examples.

EXAMPLE 1. Consider

$(*)$ $\quad -u'' = \lambda u - u \cdot \cos(u^2 + u'^2)^{1/2}$ in $(0, \pi)$, $u(0) = u(\pi) = 0$.

Notice that

$$\lim_{(w,s) \to 0} \frac{f(x, w, s, \lambda)}{w} = -1.$$

Let $k = 1$. The equation has the family of solutions $(1 + \cos|\gamma|, u_\gamma)$, where $u_\gamma(x) = \gamma \sin x$, $\gamma \in R$. It is clear that $(2, 0)$ is the bifurcation point for $(*)$. Proceeding analogously for $k = 2, 3, \ldots$, we obtain all bifurcation points for $(*)$. They have the form $(k^2 + 1, 0)$. In relation to the linearizable problem of Rabinowitz, they are translated by $\lim_{(w,s) \to 0} (f(x, w, s, \lambda)/w)$.

EXAMPLE 2. Consider

$(**)$ $\quad -u'' = \lambda u + |u|$ in $(0, \pi)$, $u(0) = u(\pi) = 0$.

Notice that

$$\sim \left(\exists \lim_{(w,s) \to 0} \frac{f(x, w, s, \lambda)}{w}\right).$$

Let $k = 1$. We have the family of solutions $(1-\operatorname{sgn}\gamma, u_\gamma)$, where $u_\gamma(x) = \gamma \sin x$. There are two bifurcation points $(0, 0)$ and $(2, 0)$ connected with the accumulation points of $f(x, w, s, \lambda)/w$ in $(w, s) \to 0$.

**IV.** Consider problem (1), (2) again with additional assumption on perturbation of the Berestycki type $f$:

$$\lim_{(w,s)\to 0} \frac{f(x, w, s, \lambda)}{w} = m \quad \forall x \in [0, \pi] \quad \forall \lambda \in R. \tag{3'}$$

Before giving Theorem 2, we require the following two lemmas ([1]).

LEMMA 1. *Let $j, k$ be integers, $j \geqslant k \geqslant 2$. Suppose that there exist two families of real numbers*

$$0 = \xi_0 < \xi_1 < \ldots < \xi_k = \pi, \qquad 0 = \eta_0 < \eta_1 < \ldots < \eta_j = \pi.$$

*Then, if $\xi_1 \leqslant \eta_1$, there exist integers $l$ and $m$ having the same parity, such that $\xi_l \leqslant \eta_m \leqslant \eta_{m+1} \leqslant \xi_{l+1}$ $(1 \leqslant l \leqslant k-1,\ 1 \leqslant m \leqslant j-1)$.*

The proof is trivial (induction on $j$).

LEMMA 2 (Sturm comparison theorem). *Let $[\xi, \eta] \subset [0, \pi]$ and $w, v$ $(\not\equiv 0)$ be two functions satisfying*

$$\begin{matrix} \mathscr{L}w = \lambda w \\ \mathscr{L}v = \mu v \end{matrix} \quad in(\xi, \eta),$$

*where $\lambda > \mu$. Suppose, moreover, that either*

(a) $$v(\xi) = v(\eta) = 0,$$

*or*

(b) $$\alpha_1 w(\xi) - \alpha_2 w(\xi) = 0, \quad \alpha_1 v(\xi) - \alpha_2 v(\xi) = 0, \quad v(\eta) = 0,$$

*or*

(c) $$\beta_1 w(\eta) + \beta_2 w(\eta) = 0, \quad \beta_1 v(\eta) + \beta_2 v(\eta) = 0, \quad v(\xi) = 0,$$

*or*

(d) $$\begin{matrix} \alpha_1 w(\xi) - \alpha_2 w(\xi) = 0, & \alpha_1 v(\xi) - \alpha_2 v(\xi) = 0, \\ \beta_1 w(\eta) + \beta_2 w(\eta) = 0, & \beta_1 v(\eta) + \beta_2 v(\eta) = 0. \end{matrix}$$

*Then there exists $\zeta \in (\xi, \eta)$ such that $w(\zeta) = 0$.*

THEOREM 2. *Under assumption (3′) bifurcation interval $[\mu_k - M, \mu_k + M] \times \{0\}$ degenerates to point $(\mu_k - m, 0)$ for every $k \in N$.*

Proof. Fix $k$. Let $(\lambda, 0) \in [\mu_k - M, \mu_k + M] \times \{0\}$ be a bifurcation point for (1), (2). It means that there exists a sequence of nontrivial solutions (1), (2), $(\lambda_n, u_n) \in R \times S_k^{\nu}$ such that $(\lambda_n, u_n) \xrightarrow{R \times E} (\lambda, 0)$. Dividing (1) by $\|u_n\|_1$ and setting $u_n/\|u_n\|_1 = w_n$ yields the equation

$$\mathscr{L} w_n = \lambda_n w_n + \frac{f(x, u_n, u_n', \lambda_n)}{\|u_n\|_1} + \frac{g(x, u_n, u_n', \lambda_n)}{\|u_n\|_1}. \tag{6}$$

It is easy to see that $w_n$ is bounded in $C^2[0, \pi]$. Therefore, by the Arzela–Ascoli theorem, we may assume that $w_n \xrightarrow{C^1[0,\pi]} w$, $\|w\|_1 = 1$. Notice that $w \in \bar{S}_k^{\nu}$. Applying Gronwall's inequality we can prove that $w \in S_k^{\nu}$ ([1]).

Letting $n \to \infty$, (6) yields

$$\mathscr{L} w = (\lambda + m) w.$$

At the same time we consider

$$\mathscr{L} v_k = \mu_k v_k.$$

Now we must compare the eigenvalues $\mu_k$ and $\lambda + m$. We may assume without loss of generality that the first zero of $wv_k$ to occur in $(0, \pi]$ is a zero of $v_k$. Let $v_k(\eta_1) = 0$. We apply Lemma 2 for functions $w$ and $v_k$ in $[0, \eta_1]$, with the boundary condition (b). If $\lambda + m > \mu_k$, then it must exist $\zeta \in (0, \eta_1)$ such that $w(\zeta) = 0$ contrary to the choice of $\eta_1$. So, we obtain $\lambda + m \leqslant \mu_k$. Now we use Lemma 1. We may choose an interval $[\xi_1, \xi_2]$ such that $w$, $v_k$ have the same sign in $(\xi_1, \xi_2)$ and $w(\xi_1) = w(\xi_2) = 0$ or $w(\xi_1) = 0$, $\xi_2 = \pi$ or $\xi_1 = 0$, $\xi_2 = \pi$ (in case $k = 1$). For this interval we apply again Lemma 2 ($w$, $v_k$ fulfil the boundary conditions (a) or (c) or (d)). We obtain the inequality $\lambda + m \geqslant \mu_k$ and, finally, $\lambda = \mu_k - m$. The proof is complete.

Assumption (3′) guarantees maintenance of the bifurcation points. In relation to the linearizable problem of Rabinowitz type ($f = 0$) we observe only the translation of the bifurcation points by limiting value $m$. Only lack of (3′) implicates the need of considerations of bifurcation from interval rather than bifurcation points.

V. Let us return to inquiry of problem (1), (2) without additional condition (3′). Also in this case we can prove a theorem about connection between a perturbation of the Berestycki type and the form of bifurcation points (1), (2).

Assume that

$$\forall (\tilde{w}_n, \tilde{s}_n) \to 0 \; \exists \tilde{f} \in C[0, \pi] \; \lim_{n \to \infty} \frac{f(x, \tilde{w}_n, \tilde{s}_n, \lambda)}{\tilde{w}_n} = \tilde{f}(x).$$

THEOREM 3. *Let $(\lambda, 0) \in [\mu_k - M, \mu_k + M] \times \{0\}$ be a bifurcation point for (1), (2). Then there exists an $f_0$-"accumulation function" of $f(x, w, s, \lambda)/w$ for*

$(w, s) \to 0$ *such that $\lambda$ is k-th eigenvalue of the Sturm–Liouville operator $\tilde{\mathscr{L}}$ defined by $\tilde{\mathscr{L}}u := \mathscr{L}u - f_0 u$.*

Proof. Analogously as in the proof of Theorem 2, there exists a sequence $(\lambda_n, u_n) \xrightarrow{R \times E} (\lambda, 0)$ fulfilling (6). Write

$$f_n(x) = \frac{f(x, u_n(x), u_n'(x), \lambda_n)}{\|u_n\|_1}.$$

Let

$$f_0(x) = \lim_{n \to \infty} \frac{f(x, u_n(x), u_n'(x), \lambda_n)}{u_n(x)}.$$

Hence $\lim_{n \to \infty} f_n = f_0 \cdot w$.

Letting $n \to \infty$, (6) yields

$$\tilde{\mathscr{L}}w := \mathscr{L}w - f_0 w = \lambda w.$$

Applying the same methods as these used in Theorem 2 we obtain the equality $\lambda = \tilde{\mu}_k$, where $\tilde{\mu}_k$ is $k$-th eigenvalue of "translated by $f_0$" Sturm–Liouville problem.

CONCLUSION 1. If $f(x, w, s, \lambda)/w$ for $(w, s) \to 0$ has a finite number of accumulation functions, then the bifurcation set of (1), (2) is finite.

CONCLUSION 2. If we assume additionally that all accumulation points of $f(x, w, s, \lambda)/w$ for $(w, s) \to 0$ are constant for every $x \in [0, \pi]$, then Theorem 3 receive the following form:

Every bifurcation point of (1), (2) has the form $(\mu_k - m, 0)$, where $m$ is an accumulation point of $f(x, w, s, \lambda)/w$ for $(w, s) \to 0$ independent on $x$. In this case, existence of a whole interval of bifurcation points is determined by existence of a whole interval of accumulation points of $f/w$.

**VI.** The methods used to obtain Theorem 2 and Theorem 3 can also be applied to nonlinear eigenvalue problems with elliptic partial differential operators.

Consider

$$\mathscr{M}u \equiv -\sum_{i,j=1}^{n} \frac{\partial}{\partial x_i}\left(a_{ij}(x)\frac{\partial u}{\partial x_j}\right) = \lambda u + F(x, u, Du, \lambda) \quad \text{in } \Omega, \tag{7}$$

$$u = 0 \quad \text{on } \partial\Omega, \tag{8}$$

where $\Omega$ is a bounded domain in $R^n$ with smooth boundary $\partial\Omega$. We assume that $\mathscr{M}$ is uniformely elliptic in $\bar{\Omega}$ and the coefficients of $\mathscr{M}$ are in $C^1(\bar{\Omega})$. The nonlinear term $F$ is equal to $f+g$, where the functions $f, g$ belong $C(\bar{\Omega} \times R^{n+2})$ and satisfy conditions (3), (4) with $x \in \bar{\Omega}$ and $s \in R^n$.

Let $E = \{u \in C^{1,\alpha}(\bar{\Omega}),\ u = 0 \text{ on } \partial\Omega\}$. A couple $(\lambda, u) \in R \times E$ is said to be a *solution* of (7), (8) if $u \in W^{2,p}(\Omega)$ and $(\lambda, u)$ satisfies (7), (8) ($\alpha \in (0, 1)$ is given, $p > n$ must fulfil the inequality $\alpha < 1 - n/p$).

We define $P^\nu := \{u \in E,\ u$ has a constant sign $\nu$ and $\partial u/\partial n < 0$ on $\partial\Omega\}$.

The linear eigenvalue problem

(9) $\mathcal{A} u = \lambda u$ in $\Omega$ with boundary condition (8) has a smallest eigenvalue $\mu_1$ which is simple and such that the corresponding eigenvector $v_1 \in P^\nu$.

Let $K^\nu = \bar{P}^\nu \setminus \{0\}$.

We have the following result ([1]).

THEOREM 4. *For every $\nu \in \{+, -\}$ there exists unbounded continuum of solutions* (7), (8) *bifurcating from $[\mu_1 - M, \mu_1 + M] \times \{0\}$ and contained in $(R \times K^\nu) \cup [\mu_1 - M, \mu_1 + M] \times \{0\}$.*

Now we give analogical theorems as in Sections IV and V.

THEOREM 5. *Under assumption* (3') (*with $x \in \bar{\Omega}$*), *the bifurcation interval $[\mu_1 - M, \mu_1 + M] \times \{0\}$ degenerates to point $(\mu_1 - m, 0)$.*

Proof. Let $(\lambda, 0) \in [\mu_1 - M, \mu_1 + M] \times \{0\}$ be a bifurcation point of (7), (8). It means that exists $(\lambda_n, u_n) \xrightarrow{R \times E} (\lambda, 0)$ satisfying (7), $u_n \in K^\nu$. Set $w_n = u_n / \|u_n\|_{C^{1,\alpha}}$.

We have the equation

$$(10) \qquad \mathcal{A} w_n = \lambda_n w_n + \frac{f(x, u_n, Du_n, \lambda_n)}{\|u_n\|_{C^{1,\alpha}}} + \frac{g(x, u_n, Du_n, \lambda_n)}{\|u_n\|_{C^{1,\alpha}}}.$$

The right-hand side of (10) is bounded in $C(\bar{\Omega})$, $\|w_n\|_{C^{1,\alpha}} = 1$. By the $L^p$ estimate, $w_n$ is bounded in $W^{2,p}(\Omega)$. Since $W^{2,p}(\Omega)$ is compactly embedded in $C^{1,\alpha}(\bar{\Omega})$, $\{w_n\}$ is relatively compact. Hence after extraction of a subsequence, we may assume that $w_n \xrightarrow{E} w$ with $\|w\|_{C^{1,\alpha}} = 1$.

Since $\mathcal{A}$ is self-adjoint, one has

$$0 = \int_\Omega (v_1 \mathcal{A} w_n - w_n \mathcal{A} v_1)\, d\mu.$$

Letting $n \to \infty$ we obtain

$$0 = \int_\Omega (\lambda + m - \mu_1) w v_1 \, d\mu.$$

Thus $\lambda = \mu_1 - m$. This completes the proof.

Without condition (3') we formulate the next theorem.

THEOREM 6. *Let $(\lambda, 0) \in [\mu_1 - M, \mu_1 + M] \times \{0\}$ be a bifurcation point for* (7), (8). *Then $\lambda$ is the first eigenvalue of the operator $\tilde{\mathcal{A}}$ defined by $\tilde{\mathcal{A}} w := \mathcal{A} w - f_0 w$, where $f_0$ is accumulation function of $f(x, w, s, \lambda)/w$ for $(w, s) \to 0$.*

The proof of Theorem 6 follows the same lines as that of Theorem 4.

Remark. All conclusions from Section V formulated for the Sturm–Liouville operator may be applicable to $\mathcal{M}$ without any changes.

**VII.** The proven theorems and following from them conclusions may be automatically transferred to the case of bifurcation from infinity.

Consider the equation

$$\mathcal{L}u = \lambda u + F_1(\cdot, u, u', \lambda) \quad \text{in } (0, \pi) \tag{11}$$

with the boundary conditions (2).

We assume that $F_1 \in C([0, \pi] \times R^3)$ and $F_1$ has the form $F_1 = f_1 + g_1$. The functions $f_1$ and $g_1$ satisfy the following conditions:

$$\exists\, M > 0 \ \forall x \in [0, \pi] \ \forall w, s \in R, \tag{12}$$

$$|w| \geqslant 1,\ |s| \geqslant 1, \quad \left|\frac{f_1(x, w, s, \lambda)}{w}\right| \leqslant M,$$

$$g_1(x, w, s, \lambda) = o(|w| + |s|) \quad \text{at } (w, s) = \infty \tag{13}$$

uniformly in $x \in [0, \pi]$ and on bounded $\lambda$ intervals.

Definition 1. We say that $(\mu, \infty)$ is a *bifurcation point* for (11), (2) if every neighbourhood of $(\mu, \infty)$ contains solution of (11), (2), i.e. there exists a sequence $(\lambda_n, u_n)$ of solutions of (11), (2) such that $\lambda_n \to \mu$ and $\|u_n\|_1 \to \infty$.

Let $\mathcal{T}$ denote the set of solutions of (11), (2).

Theorem 7 ([2]). *For every $k \in N$ and $\nu \in \{+, -\}$ there exists at least one unbounded continuum of $\mathcal{T}$, $\mathcal{D}_k^\nu$, bifurcating from $[\mu_k - M, \mu_k + M] \times \{\infty\}$. Moreover, there exists on open, bounded set $\mathcal{O}$ inclusive $[\mu_k - M, \mu_k + M] \times \{\infty\}$ such that $\mathcal{D}_k^\nu \cap \mathcal{O} \subset (R \times S_k^\nu) \cup [\mu_k - M, \mu_k + M] \times \{\infty\}$.*

Considering problem (11), (2) with additional assumption

$$\exists \lim_{(w,s) \to \infty} \frac{f_1(x, w, s, \lambda)}{w} = m \quad \forall x \in [0, \pi] \ \forall \lambda \in R \tag{12'}$$

we obtain theorem analogous to that of Section IV.

Theorem 8. *Under assumption (12′), the bifurcation interval $[\mu_k - M, \mu_k + M] \times \{\infty\}$ degenerates to point $(\mu_k - m, \infty)$ for every $k \in N$.*

Without this assumption we formulate the next theorem.

Theorem 9. *Let $(\lambda, \infty) \in [\mu_k - M, \mu_k + M] \times \{\infty\}$ be a bifurcation point for (11), (2). Then there exists $f_0$-"accumulation function" of $f_1(x, w, s, \lambda)/w$ for $(w, s) \to \infty$ such that $\lambda$ is k-th eigenvalue of the Sturm–Liouville operator $\tilde{\mathcal{L}}$ defined by $\tilde{\mathcal{L}}w := \mathcal{L}w - f_0 w$.*

Remark. The above results may be transferred to operator $\mathcal{M}$.

**References**

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*Reçu par la Rédaction le 02.04.1987*