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DERIVATIONS SATISFYING POLYNOMIAL IDENTITIES

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0. Introduction. In [7] Posner proved that if d_1, d_2 are derivations in a 2-torsion free prime ring R such that the composite $d_1 d_2$ is a derivation, then $d_1 = 0$ or $d_2 = 0$. It is natural to ask whether an analogous statement is true for any finite set of derivations. One can show that it is not true in general (see the Remark in Section 3). In this paper we prove that if R is a commutative ring, then the answer to the above question is in the affirmative. Moreover, the answer is in the affirmative if we consider a derivation which is a special type polynomial function in several derivations (Corollary 3.7).

It is well known (see [1], Section 0.6, Ex. 17) that if d is a nilpotent derivation of a domain R, then $d = 0$ (under some restriction on the characteristic of R). We prove that this is also true for a derivation d such that

$$d^n(a)+r_1 d^{n-1}(a)+\ldots+r_0 a = 0 \quad \text{for any } a \in A,$$

where A is a d-ideal of an $n!$-torsion free reduced ring R and $\operatorname{Ann}_R A = 0$ (Corollary 4.4). Moreover, we get (Theorem 4.2) some generalization of a Jacobson result for differential fields (see [2]). Further, we prove that if $d^i(d^j(A)^k) = 0$ for $i+j = n$ and some k, where A is a d-ideal, then $d = 0$ (Corollary 4.5).

In the paper we study derivations in commutative rings satisfying polynomial identities for differential modules.

Let R be a commutative ring with identity, and D be a family of derivations of R. If M is a D-module (Section 1), then M is a left $R[D]$-module (Section 1), where $R[D]$ is the ring of differential polynomials over D with coefficients in R (see [8]). Denote $R[D]$ by S.

The paper is concerned with the study of ideals $\operatorname{Ann}_S M$ of the ring S. They play an important role in describing the properties of derivations satisfying polynomial identities.

Section 1 contains basic properties of S. In Section 2, for any ideal A in $R[D]$ we show some class of elements which belong to $A \cap R$ (Theorem 2.1). This result will often be applied in further parts of the paper.

The main results are contained in Section 3. We prove first (Proposition 3.1) that

$$\mathrm{Ann}_S(SA) = S\,\mathrm{Ann}_R(A),$$

where A is a D-ideal in R, and next (Theorem 3.2) that if $\mathrm{Ann}_S R = 0$ and $\mathrm{Ann}_R M = 0$, then $\mathrm{Ann}_S M = 0$. Next we study the constant term of polynomials in $\mathrm{Ann}_S A$ (Theorem 3.3, Corollary 3.4) and we generalize the theorem of Posner [7].

Section 4 gives some properties of $\mathrm{Ann}_S M$ in the case $D = \{d\}$.

1. Notation and preliminary results. Throughout this paper, R is a commutative ring with unit and D is a family of derivations of R. A *D-module* is a left R-module M together with a specified mapping

$$^{-}\colon D \to \mathrm{Hom}_Z(M, M)$$

such that

$$\bar{d}(rm) = r\bar{d}(m) + d(r)m$$

for $d \in D$, $r \in R$, $m \in M$. An ideal A in R is called a *D-ideal* iff $d(A) \subseteq A$ for all $d \in D$. Every D-ideal is a D-module by $\bar{d}(a) = d(a)$. If A, B are D-ideals, then

$$A :_R B = \{r \in R\colon rB \subseteq A\}$$

is a D-ideal.

We denote by $R[D]$ the *ring of differential polynomials over D with coefficients in R* (see [8]). This is the ring of polynomials in non-commuting indeterminates, one for each derivation in D, with coefficients from R written on the left. Here the multiplication is defined by $tr = d(r) + rt$, where t is the indeterminate corresponding to the derivation $d \in D$. For $d_1, \ldots, d_n \in D$ let $t_1, \ldots, t_n$ denote the indeterminates corresponding to $d_1, \ldots, d_n$, respectively. By an easy induction we get

LEMMA 1.1.

$$t_1 t_2 \ldots t_n r = r t_1 t_2 \ldots t_n + \sum_{k=1}^{n} \sum_{i_1 < \ldots < i_k} d_{i_1} \ldots d_{i_k}(r)\, t_1 \ldots \hat{t}_{i_1} \ldots \hat{t}_{i_k} \ldots t_n.$$

Every element U of the form $U = t_1 \ldots t_n$ will be called a *monic monomial of degree n* (if $n = 0$, then $U = 1$). Let f be a polynomial in $R[D]$. If $f \neq 0$, then f has a unique representation

$$f = r_1 U_1 + \ldots + r_k U_k,$$

where $r_1, \ldots, r_k$ are non-zero elements in R and $U_1, \ldots, U_k$ are distinct monic monomials. The *degree* of f (denoted by $\deg f$) is the maximum of the degrees of $U_1, \ldots, U_k$. For $f = 0$, $\deg f = -\infty$. A polynomial f will be called *singular* iff it has exactly one monic monomial of degree equal to $\deg f$. The coefficient at the monic monomial of degree zero in f will be denoted by $w(f)$. Further, we denote by S the ring $R[D]$.

If M is a D-module, then M together with the multiplication

$$*\colon S \times M \to M$$

defined by

$$t_1 \ldots t_n * m = \bar{d}_1 \ldots \bar{d}_n(m)$$

for monic monomials and

$$(r_1 U_1 + \ldots + r_k U_k) * m = r_1(U_1 * m) + \ldots + r_k(U_k * m)$$

for arbitrary polynomials is a left S-module.

Conversely, if M is a left S-module, then $\bar{}\,\colon D \to \mathrm{Hom}_Z(M, M)$, $\bar{d}(m) = tm$, makes M a D-module.

LEMMA 1.2. *If $f \in S$ and $r \in R$, then*

$$w(fr - rf) = f * r - rw(f).$$

Proof. We may restrict ourselves to monic monomials. If $U = 1$, then

$$w(Ur - rU) = 0 = U * r - rw(U).$$

If $U = t_1 \ldots t_n$, then Lemma 1.1 implies

$$w(Ur - rU) = w(Ur) - w(rU) = d_1 \ldots d_n(r) - 0 = U * r - rw(U).$$

For a subset X of R let $\bar{X}$ denote the set

$$\{r_1 U_1 + \ldots + r_k U_k \in S\colon r_1, \ldots, r_k \in X\}.$$

LEMMA 1.3. *Let A, B be D-ideals of R. Then*

(1) *$\bar{A}$ is an ideal in S and $\bar{A} = SA$;*

(2) *$\bar{A} \cap R = A$;*

(3) *$\overline{AB} = \bar{A}\bar{B}$;*

(4) *$\overline{A :_R B} = \bar{A} :_S \bar{B}$.*

Proof. By the definition of the multiplication in $R[D]$ we have immediately (1)–(3).

(4) The inclusion $\subseteq$ is trivial. Conversely, assume that $f \in \bar{A} : \bar{B}$. We prove, by induction on $n = \deg f$, that $f \in \overline{A : B}$. Let

$$f = r_1 U_1 + \ldots + r_k U_k + g,$$

where $U_1, \ldots, U_k$ are monic monomials of degree n and $\deg g < n$. Then, for every $b \in B$, we have $fb \in \bar{A}$ and, by Lemma 1.1,

$$fb = r_1 bU_1 + \ldots + r_k bU_k + h,$$

where $h \in S$, $\deg h < n$. This implies that $r_1 b, \ldots, r_k b \in A$, i.e., $r_1, \ldots, r_k \in A : B$, whence

$$r_1 U_1, \ldots, r_k U_k \in \overline{A:B} \subseteq \bar{A}:\bar{B}.$$

Therefore $g \in \bar{A}:\bar{B}$ and, by induction, $g \in \overline{A:B}$, so $f \in \overline{A:B}$.

Remark. Lemmas 1.1 and 1.3 hold also for non-commutative rings.

2. The operator E. In this section we prove the following

THEOREM 2.1. *Let A be an ideal in $R[D]$ and f be an element in A such that* $\deg f = n \geqslant 1$. *Assume that*

$$f = \sum_{i=1}^{k} a_i t_{i1} \ldots t_{in} + g,$$

where $t_{i1}, \ldots, t_{in}$ for $i = 1, \ldots, k$ are distinct monic monomials of degree n, and $\deg g < n$, $a_1, \ldots, a_k \in R$. *Then, for any $r_1, \ldots, r_n \in R$, the element*

$$\sum_{\sigma} \sum_{i=1}^{k} a_i d_{i\sigma(1)}(r_1) \ldots d_{i\sigma(n)}(r_n),$$

where σ runs over all permutations of the set $\{1, \ldots, n\}$, belongs to $A \cap R$.

This theorem will be often applied in the sequel. Before the proof we introduce some operator E and prove four lemmas. Let E: $S \to \mathrm{Hom}_Z(R, S)$ be an R-linear mapping such that

$$EU(r) = \begin{cases} 0 & \text{for } n = 0, \\ \sum_{i=1}^{n} d_i(r) t_1 \ldots \hat{t}_i \ldots t_n & \text{for } n > 0, \end{cases}$$

where $U = t_1 \ldots t_n$. Denote by $L^n_Z(R, S)$ the set of n-multilinear (over Z) mappings of R into S. We define the sequence $\{E^n\colon S \to L^n_Z(R, S)\}$ as follows:

$$E^1 = E, \quad E^{n+1} f(r_1, \ldots, r_{n+1}) = E(E^n f(r_1, \ldots, r_n))(r_{n+1}),$$

where $f \in S$, $r_1, \ldots, r_{n+1} \in R$. By the above definition we get immediately the following properties of the mappings E^n:

LEMMA 2.2.

(1) $\deg Ef(r) \leqslant (\deg f) - 1$ *for* $f \in S$, $r \in R$.

(2) $E^n f(r_1, \ldots, r_n) = E^{n-k}(E^k f(r_1, \ldots, r_k))(r_{k+1}, \ldots, r_n)$ *for* $1 \leqslant k \leqslant n$.

(3) $E^n f(r_1, \ldots, r_n) = 0$ *if* $n > \deg f$.

Now we give some other properties of E^n.

LEMMA 2.3. *Let $f \in S$, $r_1, \ldots, r_n \in R$. If t is the indeterminate corresponding to the derivation $d \in D$, then*

$$E^n(ft)(r_1, \ldots, r_n) = E^n f(r_1, \ldots, r_n) t + \sum_{i=1}^{n} d(r_i) E^{n-1} f(r_1, \ldots, \hat{r}_i, \ldots, r_n).$$

Proof. This result follows from the equality $E(ft)(r) = Ef(r)t + d(r)f$ and from the definition of the mappings E^n.

LEMMA 2.4. *If* $r_1, \ldots, r_n \in R$, *then*

$$E^n(t_1 \ldots t_n)(r_1, \ldots, r_n) = \sum_{\sigma \in S_n} d_{\sigma(1)}(r_1) \ldots d_{\sigma(n)}(r_n).$$

Proof (by induction on n). The case $n = 1$ is trivial. Suppose that the lemma is true for some positive integer n. Then, by Lemma 2.3, we have

$$\begin{aligned} E^{n+1}(t_1 \ldots t_{n+1})(r_1, \ldots, r_{n+1}) &= E^{n+1}(t_1 \ldots t_n)(r_1, \ldots, r_{n+1})t_{n+1} \\ &\quad + \sum_{i=1}^{n+1} d_{n+1}(r_i) E^n(t_1 \ldots t_n)(r_1, \ldots, \hat{r}_i, \ldots, r_{n+1}). \end{aligned}$$

The first summand (by Lemma 2.2 (3)) is equal to zero. The second summand, by induction, is equal to

$$\begin{aligned} &\sum_{i=1}^{n+1} d_{n+1}(r_i) \sum_{\sigma \in S_n} d_{\sigma(1)}(r_1) \ldots d_{\sigma(i-1)}(r_{i-1}) d_{\sigma(i)}(r_{i+1}) \ldots d_{\sigma(n)}(r_{n+1}) \\ &\qquad = \sum_{\sigma \in S_{n+1}} d_{\sigma(1)}(r_1) \ldots d_{\sigma(n+1)}(r_{n+1}). \end{aligned}$$

LEMMA 2.5. *Let A be an ideal in S. If $f \in A$,* $\deg f = n \geqslant 1$, *then*

$$E^n f(r_1, \ldots, r_n) \in A \cap R \quad \textit{for every } r_1, \ldots, r_n \in R.$$

Proof (by induction on n). If $n = 1$, then

$$E^1 f(r_1) = fr_1 - r_1 f \in A \cap R.$$

Let $f \in A$ and $\deg f = n$. Then $g = fr_1 - r_1 f \in A$ and, by Lemma 1.1, $g = Ef(r_1) + h$, where $h \in S$, $\deg h \leqslant n-2$. By induction we have

$$E^{n-1} g(r_2, \ldots, r_n) \in A \cap R.$$

Therefore, by Lemma 2.2,

$$\begin{aligned} E^n f(r_1, \ldots, r_n) &= E^{n-1}(Ef(r_1))(r_2, \ldots, r_n) \\ &= E^{n-1}(Ef(r_1) + h)(r_2, \ldots, r_n) = E^{n-1} g(r_2, \ldots, r_n) \in A \cap R. \end{aligned}$$

From Lemmas 2.5 and 2.4 we get the assertion of Theorem 2.1.

3. The differential annihilator of a D-module. Let M be a D-module. In this section we give some properties of the ideal $\mathrm{Ann}_S M$.

By definition

$$\mathrm{Ann}_S M = \{f \in S\colon f * m = 0 \text{ for any } m \in M\},$$

i.e., the polynomial $f = \sum_i r_i t_{i1} \ldots t_{in_i}$ belongs to $\mathrm{Ann}_S M$ iff

$$\sum_i r_i \bar{d}_{i1} \ldots \bar{d}_{in_i}(m) = 0 \quad \text{for any } m \in M.$$

Notice first that $\mathrm{Ann}_S M \cap R$ is a D-ideal in R equal to $\mathrm{Ann}_R M$. The proposition below describes $\mathrm{Ann}_S M$ in the case $M = SA$, where A is a D-ideal.

PROPOSITION 3.1. *If A is a D-ideal in R, then*

$$\mathrm{Ann}_S(SA) = S\,\mathrm{Ann}_R A.$$

Proof. By Lemma 1.3 we have

$$S\,\mathrm{Ann}_R A = \overline{\mathrm{Ann}_R A} = \bar{0}:_S \bar{A} = \mathrm{Ann}_S(\bar{A}) = \mathrm{Ann}_S(SA).$$

Now we are going to study main properties of the ideal $\mathrm{Ann}_S M$.

THEOREM 3.2. *Let R be a ring such that $\mathrm{Ann}_S R = 0$, and let M be a D-module. If $\mathrm{Ann}_R M = 0$, then $\mathrm{Ann}_S M = 0$.*

Proof. Suppose that $\mathrm{Ann}_S M \neq 0$. Let f be a non-zero polynomial in $\mathrm{Ann}_S M$ of minimal degree. Since for any $r \in R$ we have $\deg(fr - rf) < \deg f$ and $fr - rf \in \mathrm{Ann}_S M$, by the minimality of f the equality $fr = rf$ holds. Therefore, by Lemma 1.2,

$$0 = w(fr - rf) = f * r - w(f)r = (f - w(f)) * r$$

for any $r \in R$, i.e.,

$$f = w(f) \in \mathrm{Ann}_S M \cap R = \mathrm{Ann}_R M = 0,$$

which contradicts the fact that $f \neq 0$.

THEOREM 3.3. *Let A be a D-ideal. If $f \in \mathrm{Ann}_S A$, then*

$$w(f) \in \mathrm{Ann}_R(A^{n+1}), \quad \textit{where } n = \deg f.$$

Proof (by induction on n). If $\deg f = 0$, then $f \in R \cap \mathrm{Ann}_S A = \mathrm{Ann}_R A$. Let $n > 0$ and assume that the theorem is true for polynomials of degree less than n. Let $f \in \mathrm{Ann}_S A$ and $\deg f = n$. Then, for every $a \in A$,

$$fa - af \in \mathrm{Ann}_S A \quad \text{and} \quad \deg(fa - af) < n.$$

Therefore, by Lemma 1.2 and by induction we have

$$w(f)a = w(f)a - (f * a) = -w(fa - af) \in \mathrm{Ann}_R(A^n)$$

for any $a \in A$, i.e., $w(f) \in \mathrm{Ann}_R(A^{n+1})$.

COROLLARY 3.4. *Let A be a D-ideal such that $\mathrm{Ann}_R A = 0$. If $f \in \mathrm{Ann}_S A$, then $w(f) = 0$.*

We shall prove other properties of the ideal $\mathrm{Ann}_S M$ using the following

LEMMA 3.5. *Let R be an $(n-1)!$-torsion free ring without zero divisors. If $d_1, \ldots, d_n$ are derivations of R such that*

$$d_1(x)\, d_2(x) \ldots d_n(x) = 0$$

for any $x \in R$, then $d_i = 0$ for some i.

Proof. Let $A_i = \{x \in R\colon d_i(x) = 0\}$, where $i = 1, \ldots, n$. The sets $A_1, \ldots, A_n$ are subgroups of the additive group of R and have the following properties:

(1) $A_1 \cup \ldots \cup A_n = R$;

(2) if $kx \in A_i$, where $1 \leqslant k < n$, then $x \in A_i$.

By a theorem of McCoy [4] we have $R = A_i$ for some i.

THEOREM 3.6. *Let R be an $n!$-torsion free ring without zero divisors, M a D-module such that $\operatorname{Ann}_R M = 0$, and f a singular polynomial in $\operatorname{Ann}_S M$ of degree n. If $f = at_1 \ldots t_n + g$, where $\deg g < n$, $0 \neq a \in R$, and $t_1, \ldots, t_n$ are the indeterminates corresponding to the derivations $d_1, \ldots, d_n \in D$, respectively, then $d_i = 0$ for some i.*

Proof. Theorem 2.1 (for $r_1 = r_2 = \ldots = r_n = x$) and the equality

$$R \cap \operatorname{Ann}_S M = \operatorname{Ann}_R M = 0$$

imply that

$$n!\, ad_1(x) \ldots d_n(x) = 0 \quad \text{for all } x \in R.$$

Therefore $d_1(x) \ldots d_n(x) = 0$ for all $x \in R$ and, by Lemma 3.5, we have $d_i = 0$ for some i.

The corollary below is a generalization of the Posner theorem [7] for commutative rings.

COROLLARY 3.7. *Let R be a commutative $n!$-torsion free prime ring and let h be a singular polynomial in $R[D]$ of degree $n \geqslant 2$ such that $h = at_1 \ldots t_n + g$, where $0 \neq a \in R$, $\deg g < n$, and $t_1, \ldots, t_n$ are the indeterminates corresponding to the derivations $d_1, \ldots, d_n \in D$, respectively. Moreover, let $\delta\colon R \to R$ be the mapping defined by $\delta(r) = h * r$. If δ is a derivation of R, then $d_i = 0$ for some i.*

Proof. Apply Theorem 3.6 to the polynomial $f = h - t$ of the ring $R[D']$, where $D' = D \cup \{\delta\}$ and t is the indeterminate corresponding to δ.

Remark. If R is a prime non-commutative ring, then Corollary 3.7 is not true in general. For example, let $R = M_2(Q)$ be the ring of (2×2)-matrices over the field Q of rational numbers, and let $d\colon R \to R$ be the inner derivation

$$d(X) = AX - XA, \quad \text{where } A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}.$$

Then $d^3 = 0$, $d^2 \neq 0$ and $d \neq 0$.

4. The case of one derivation. In this section we get some additional information about $\operatorname{Ann}_S M$ if $|D| = 1$. Throughout this section $D = \{d\}$, where d is a fixed derivation of R. The terms D-modules and D-ideals are replaced by the terms d-modules and d-ideals, respectively. Now the ring $R[D]$ of differential polynomials is equal to the well-known Ore extension $R[t, d]$ of R (see [6] and [3]). We denote $R[t, d]$ by S.

If M is a d-module, then

$$\operatorname{Ann}_S M = \{r_n t^n + \ldots + r_1 t + r_0 \in S\colon r_n \bar{d}^n(m) + \ldots + r_1 \bar{d}(m) + r_0 m = 0, \quad \text{for any } m \in M\}.$$

Some of the properties of $\operatorname{Ann}_S M$ in the case $M = R$ are given in [5]. Using methods similar to those in [5] we can prove the following two theorems:

THEOREM 4.1. *Let M be a d-module such that* $\operatorname{Ann}_R M = 0$, $\operatorname{Ann}_S M \neq 0$, *and let f be a non-zero polynomial in* $\operatorname{Ann}_S M$ *of minimal degree. If f is monic, then* $\operatorname{Ann}_S M = Sf$.

THEOREM 4.2. *Let A be a d-ideal such that* $\operatorname{Ann}_R A = 0$, $\operatorname{Ann}_S A \neq 0$, *and let f be a non-zero polynomial in* $\operatorname{Ann}_S M$ *of minimal degree. Assume that*

$$f = r_1 t^{n_1} + \ldots + r_k t^{n_k},$$

where $n_1 > \ldots > n_k$, $r_i \neq 0$ for $i = 1, \ldots, k$. Then there exists a prime number p such that

$$n_1 = p^{u_1}, \ldots, n_k = p^{u_k} \quad \text{and} \quad u_1, \ldots, u_k \geqslant 0.$$

Moreover, if $\operatorname{char} R = n \geqslant 0$, *then $p | n$.*

By Theorems 4.1 and 4.2 and by the theorems of Sections 2 and 3 we obtain the following corollaries.

COROLLARY 4.3. *Let A be a d-ideal such that* $\operatorname{Ann}_R A = 0$. *If* $\operatorname{Ann}_S A \neq 0$, *then there exist a non-zero element $b \in R$ and a prime number p such that $p^k b d(R) = 0$ for some $k \geqslant 0$. Moreover, if* $\operatorname{char} R = n \geqslant 0$, *then $p | n$.*

Proof. Let $f = bt^m + r_{m-1} t^{m-1} + \ldots + r_0$, where $b \neq 0$, be a non-zero polynomial of $\operatorname{Ann}_S A$ of minimal degree. Theorem 4.2 implies that $m = p^k$ for some prime number p. If $r \in R$, then

$$fr - rf \in \operatorname{Ann}_S A \quad \text{and} \quad fr - rf = p^k b d(r) t^{m-1} + g,$$

where $\deg g \leqslant m-2$. Therefore, by the minimality of f, we get $p^k b d(R) = 0$.

We say that a ring R is *reduced* if it has no nilpotent elements except 0.

COROLLARY 4.4. *Let R be an $n!$-torsion free reduced ring and let A be a d-ideal of R such that* $\operatorname{Ann}_R A = 0$. *If there exists a polynomial* $f \in \operatorname{Ann}_S A$ *of the form $f = t^n + g$, where* $\deg g < n$, *then $d = 0$.*

Proof. By Theorem 2.1 we have $n!(d(r))^n = 0$ for any $r \in R$, so the proof is completed.

We end this paper with the following

COROLLARY 4.5. *Let R be an $n!$-torsion free reduced ring and let A be a d-ideal of R such that* $\operatorname{Ann}_R A = 0$. *If $d^i(d^j(A)^k) = 0$ for $i+j = n$ and some k, then $d = 0$.*

Proof. Let $a_1, \ldots, a_k \in A$ and $a = d^j(a_1)d^j(a_2)\ldots d^j(a_{k-1})$. Since

$$0 = d^i\big(ad^j(a_k)\big) = ad^n(a_k) + id(a)\,d^{n-1}(a_k) + \ldots,$$

the polynomial $at^n + id(a)\,t^{n-1} + \ldots$ belongs to $\mathrm{Ann}_S A$. Hence, by Theorem 2.1, we have $n!\,ad(r_k)^n = 0$ for any $r_k \in R$. Let

$$b = n!\,d^j(a_1)\ldots d^j(a_{k-2})\,d(r_k)^n.$$

Then $bd^j(a_{k-1}) = 0$ and, by Theorem 2.1, we have $j!\,bd(r_{k-1})^j = 0$ for any $r_{k-1} \in R$. Therefore

$$n!\,j!\,d^j(a_1)\ldots d^j(a_{k-2})\,d(r_{k-1})^j\,d(r_k)^n = 0$$

for any $a_1, \ldots, a_{k-2} \in A$, r_{k-1}, $r_k \in R$, i.e.,

$$cd^j(a_{k-2}) = 0 \quad \text{for any } a_{k-2} \in A,$$

where

$$c = n!\,j!\,d^j(a_1)\ldots d^j(a_{k-3})\,d(r_{k-1})^j\,d(r_k)^n.$$

Continuing this process we get the following equality:

$$n!\,(j!)^k\,d(r_1)^j\ldots d(r_{k-1})^j\,d(r_k)^n = 0 \quad \text{for any } r_1, \ldots, r_k \in R.$$

In particular, for $r_1 = \ldots = r_k = x$ we have

$$n!\,(j!)^k\,d(x)^s = 0, \quad \text{where } s = (k-1)j + n,$$

which completes the proof.

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