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On algebraic characterization of coercive linear partial differential operators with constants coefficients

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Abstract. Let $\mathscr{K}$ denote the Hörmander class of weight functions $k\colon \boldsymbol{R}^n \to \boldsymbol{R}$ and let p be in the interval $[1, \infty)$. Define a norm $\|\cdot\|_{p,k}$ and a scalar product $(\cdot, \cdot)_k$ by

$$\|\varphi\|_{p,k} = \Big((2\pi)^{-n} \int_{\boldsymbol{R}^n} |(\mathscr{F}\varphi)(\xi)\, k(\xi)|^p \, d\xi\Big)^{1/p}$$

and

$$(\varphi, \psi)_k = (2\pi)^{-n} \int_{\boldsymbol{R}^n} (\mathscr{F}\varphi)(\xi)\overline{(\mathscr{F}\psi)(\xi)}\, k^2(\xi)\, d\xi \quad \text{for } \varphi, \psi \in C_0^\infty(G),$$

where $\mathscr{F}$ denotes the Fourier transform and where G is an open set in $\boldsymbol{R}^n$. The paper considers sufficient and necessary algebraic conditions for the validity of the following estimates

$$\|L(D)\varphi\|_{p,k} \geqslant C_1 \|\varphi\|_{p,k\tilde{k}} - C_2 \|\varphi\|_{p,k} \tag{1}$$

and

$$\operatorname{Re}(L(D)\varphi, \varphi)_k \geqslant C_1 \|\varphi\|^2_{k\sqrt{\tilde{k}}} - C_2 \|\varphi\|^2_{p,k}, \tag{2}$$

where $L(D) = \sum_{|\sigma| \leqslant r} a_\sigma D^\sigma$ is a linear partial differential operator with constant coefficients $a_\sigma \in \boldsymbol{C}$. Especially, an algebraic characterization of (1) is verified when G is an open strip or bounded. Also a characterization of (2) is obtained when $\operatorname{Re} L(\xi) \geqslant 0$ and when G is an open strip or bounded.

1. Introduction. Denote by $\mathscr{K}$ the class of positive weight functions such that for each $k \in \mathscr{K}$ one can find constants $C > 0$ and $N \in \boldsymbol{N}$ with which

$$k(\xi+\eta) \leqslant (1 + C|\xi|)^N k(\eta) \quad \text{for all } \xi, \eta \in \boldsymbol{R}^n. \tag{1.1}$$

Furthermore, let p lie in the interval $[1, \infty)$ and let G be an open set in $\boldsymbol{R}^n$. In the space $C_0^\infty(G)$ we define a norm $\|\cdot\|_{p,k}$ and the scalar product $(\cdot, \cdot)_k$ through the relations

$$\|\varphi\|_{p,k} = \Big((2\pi)^{-n} \int_{\boldsymbol{R}^n} |(\mathscr{F}\varphi)(\xi)\, k(\xi)|^p \, d\xi\Big)^{1/p} \tag{1.2}$$

and

$$(1.3) \qquad (\varphi, \psi)_k = (2\pi)^{-n} \int_{\mathbf{R}^n} \overline{(\mathscr{F}\varphi)(\xi)} (\mathscr{F}\psi)(\xi) k^2(\xi) d\xi,$$

where $\mathscr{F}$ is the Fourier transform from the Schwarz space S into itself. The completion of $C_0^\infty(G)$ with respect to the norm $\|\cdot\|_{p,k}$ (with respect to the scalar product $(\cdot, \cdot)_k$) is denoted by $\mathscr{B}_{p,k}(G)$ (and $\mathscr{B}_k(G)$ resp.).

Let p be in $[1, \infty)$ and let k and $k^\sim$ be two weight functions in $\mathscr{K}$. A linear partial differential operator

$$(1.4) \qquad L(D) = \sum_{|\sigma| \leqslant r} a_\sigma D^\sigma \qquad (D = (D_1, \ldots, D_n);\ D_j = -i\frac{\partial}{\partial x_j})$$

of order r ($\in \mathbf{N}$) with constant coefficients $a_\sigma \in \mathbf{C}$ is called *$k^\sim$-coercive* (and *strongly $k^\sim$-coercive*) in the space $\mathscr{B}_{p,k}(G)$ (and in the space $\mathscr{B}_k(G)$ resp.) if there exist constants $C_1 > 0$ and $C_2 \geqslant 0$ such that the inequality

$$(1.5a) \qquad \|L(D)\varphi\|_{p,k} \geqslant C_1 \|\varphi\|_{p,kk^\sim} - C_2 \|\varphi\|_{p,k},$$

resp.

$$(1.5b) \qquad \mathrm{Re}(L(D)\varphi, \varphi)_k \geqslant C_1 \|\varphi\|^2_{k\sqrt{k^\sim}} - C_2 \|\varphi\|_k^2,$$

is valid for all $\varphi \in C_0^\infty(G)$. Here we denoted $\|\varphi\|_k = (\varphi, \varphi)_k^{1/2}$.

In [4] (pp. 5–8), Louhivaara and Simader proved that in the whole space case ($G = \mathbf{R}^n$) an operator $L(D)$ is k_{2t}-coercive in the space $\mathscr{B}_{2,1}(\mathbf{R}^n) = L^2(\mathbf{R}^n)$ if and only if there exists two constants $E > 0$ and $R \geqslant 0$ such that the algebraic condition

$$(1.6) \qquad |L(\xi)| := \Big|\sum_{|\sigma| \leqslant r} a_\sigma \xi^\sigma\Big| \geqslant E|\xi|^{2t}, \qquad t \in \mathbf{N},$$

is valid for all $\xi \in \mathbf{R}^n$ with $|\xi| \geqslant R$. Here $k_{2t} \in \mathscr{K}$ such that $k_{2t}(\xi) = (1 + |\xi|^2)^t$. They also remarked in [5] (p. 340) that in the case of an arbitrary open set G this condition (1.6) is sufficient for the k_{2t}-coercivity in $\mathscr{B}_{2,1}(G) = L^2(G)$.

Combining the method of [5] (pp. 340–342) with the fact that, when G is bounded, for every $\alpha \in \mathbf{N}^n$ there exists a constant $C_\alpha > 0$ such that

$$(1.7) \qquad \|L^{(\alpha)}(D)\varphi\|_{2,1} \leqslant C_\alpha \|L(D)\varphi\|_{2,1}$$

for all $\varphi \in C_0^\infty(G)$ (cf. [1], pp. 265–267, [2], pp. 183–185), one can further prove that $L(D)$ is k_{2t}-coercive in the space $\mathscr{B}_{2,1}(G)$ for a bounded open set G if and only if there exists a constant $\gamma > 0$ with which

$$(1.8) \qquad L^\sim(\xi) := \Big(\sum_\alpha |L^{(\alpha)}(\xi)|^2\Big)^{1/2} \geqslant \gamma|\xi|^{2t} \qquad \Big(L^{(\alpha)}(\xi) := \frac{\partial^{(\alpha)} L(\xi)}{\partial \xi^\alpha}\Big)$$

for all $\xi \in \mathbf{R}^n$.

For strongly k_{2r}-coercive operators one has obtained slightly different conditions (cf. [4], pp. 4–8 and [5], 340–342).

In this paper we shall consider analogous results mentioned above for some classes of unbounded open proper subsets $G \subset \boldsymbol{R}^n$ and for $k^\sim$-coercive and strongly $k^\sim$-coercive operators.

2. On the $k^\sim$-coercivity of $L(D)$.

2.1. For the first instance we consider the case where the open set $G \subset \boldsymbol{R}^n$ has the following property:

PROPERTY A_j. *For a fixed number $j \in \{1, \ldots, n\}$ there exist a constant $a > 0$ and numbers $x_k^0 \in \boldsymbol{R}$ $(k = 1, \ldots, j)$ such that the strip*

$$\mathscr{K}_j^a = \{x = (x_1, \ldots, x_n) \in \boldsymbol{R}^n \mid |x_k - x_k^0| \leqslant a \text{ for } k = 1, \ldots, j\} \tag{2.1}$$

is contained in G.

In the sequel we use for $\alpha = (\alpha_1, \ldots, \alpha_n) \in N_0^n$ the decomposition

$$\alpha = \alpha' + \alpha'',$$

where $\alpha' = (\alpha_1, \ldots, \alpha_j, 0, \ldots, 0)$ and $\alpha'' = (0, \ldots, 0, \alpha_{j+1}, \ldots, \alpha_n)$. Similarly we write for $\xi \in \boldsymbol{R}^n$

$$\xi = \xi' + \xi''.$$

THEOREM 2.1. *Let $G \subset \boldsymbol{R}^n$ be an open set with Property A_j and let $L(D)$ be a $k^\sim$-coercive operator in the space $\mathscr{B}_{p,k}(G)$. Then there exists a constant $\gamma > 0$ such that*

$$\Big(\sum_{|\alpha'| \leqslant r} |L^{(\alpha')}(\xi)|^p\Big)^{1/p} + 1 \geqslant \gamma k^\sim(\xi) \tag{2.2}$$

for all $\xi \in \boldsymbol{R}^n$ (here the summation is taken over such $\alpha' = (\alpha_1, \ldots, \alpha_j, 0, \ldots, 0)$ for which $|\alpha'| \leqslant r$).

Proof. Let $\Phi \in C_0^\infty(\boldsymbol{R}^j)$ such that $\operatorname{supp} \Phi \subset \{x \in \boldsymbol{R}^j \mid |x_k - x_k^0| \leqslant a$ for all $k = 1, \ldots, j\}$ and $\Phi \neq 0$. Furthermore, let $\theta \in C_0^\infty(\boldsymbol{R}^{n-j})$ be chosen so that $\theta(0) = 1$. Define a function $\theta_l \in C_0^\infty(\boldsymbol{R}^n)$ by

$$\theta_l(x) = l^{-(n-j)+(n-j)/p}\, \Phi(x_1, \ldots, x_j)\, \theta(x_{j+1}/l, \ldots, x_n/l). \tag{2.3}$$

One sees that the inclusion $\operatorname{supp} \theta_l \subset K_j^a \subset G$ holds and then θ_l lies in $C_0^\infty(G)$.

For a fixed $\xi \in \boldsymbol{R}^n$ we define a function φ_l from $C_0^\infty(G)$ by

$$\varphi_l(x) = \theta_l(x)\, e^{i(\xi, x)}. \tag{2.4}$$

Then one has

$$(2.5)\qquad (\mathscr{F}\varphi_l)(\eta) = l^{-(n-j)+(n-j)/p} \int_{\mathbf{R}^n} \Phi(x_1, \ldots, x_j)\,\theta(x_{j+1}/l, \ldots, x_n/l) \times$$
$$\times e^{-i(\eta'-\xi',x')} e^{-i(\eta''-\xi'',x'')}\,dx$$
$$= l^{(n-j)/p} \int_{\mathbf{R}^n} \Phi(y_1, \ldots, y_j)\,\theta(y_{j+1}, \ldots, y_n)\, e^{-i(\eta'-\xi',y')} e^{-i(l(\eta''-\xi''),y'')}\,dy$$
$$= l^{(n-j)/p}\, \mathscr{F}(\widetilde{\Phi\theta})(\eta'-\xi'+l(\eta''-\xi'')),$$

where $\widetilde{\Phi\theta} \in C_0^\infty(\mathbf{R}^n)$ is defined by $(\widetilde{\Phi\theta})(y) = \Phi(y_1, \ldots, y_j)\,\theta(y_{j+1}, \ldots, y_n)$.

We now consider the norms $\|\varphi_l\|_{p,k}$ and $\|\varphi_l\|_{p,kk^\sim}$. In virtue of (2.5) and (1.1) one obtains

$$(2.6)\qquad \|\varphi_l\|_{p,k}^p = (2\pi)^{-n} l^{n-j} \int_{\mathbf{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\eta'-\xi'+l(\eta''-\xi''))\,k(\eta)|^p\,d\eta$$
$$= (2\pi)^{-n} \int_{\mathbf{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)\,k(\tau'+\xi'+\tau''/l+\xi'')|^p\,d\tau.$$
$$\leqslant (2\pi)^{-n} k(\xi)^p \int_{\mathbf{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C|\tau'+\tau''/l|)^N|^p\,d\tau$$
$$\leqslant (2\pi)^{-n} k(\xi)^p \int_{\mathbf{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C(|\tau'|+|\tau''|))^N|^p\,d\tau.$$

Hence there is a constant $D_1 > 0$ with which

$$(2.7)\qquad \|\varphi_l\|_{p,k}^p \leqslant D_1^p\, k(\xi)^p.$$

Furthermore one sees that (cf. [3], p. 34)

$$(2.8)\qquad \|\varphi_l\|_{p,kk^\sim}^p = (2\pi)^{-n} \int_{\mathbf{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(kk^\sim)(\tau'+\xi'+\tau''/l+\xi'')|^p\,d\tau$$
$$\geqslant (2\pi)^{-n} (kk^\sim)(\xi)^p \int_{\mathbf{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C_1|\tau'+\tau''/l|)^{-N_1}|^p\,d\tau$$
$$\geqslant (2\pi)^{-n} (kk^\sim)(\xi)^p \int_{\mathbf{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C_1(|\tau'|+|\tau''|))^{-N_1}|^p\,d\tau,$$

where $C_1 > 0$ and $N_1 \in \mathbf{N}$ are chosen so that

$$(kk^\sim)(\xi+\eta) < (1+C_1|\xi|)^{N_1}(kk^\sim)(\eta).$$

Inequality (2.8) shows the existence of a constant $D_2 > 0$ with which

$$(2.9)\qquad \|\varphi_l\|_{p,kk^\sim}^p \geqslant D_2^p (kk^\sim)(\xi)^p.$$

Finally we consider the norm $\|L(D)\varphi_l\|_{p,k}$. For all $\eta \in \mathbf{R}^n$ it holds

$$(2.10)\qquad \mathscr{F}(L(D)\varphi_l)(\eta) = L(\eta)(\mathscr{F}\varphi_l)(\eta)$$

and then we get

$$(2.11)\quad \|L(D)\varphi_l\|_{p,k}^p$$
$$= (2\pi)^{-n} \int_{R^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)\, k(\tau'+\xi'+\tau''/l+\xi'')\, L(\tau'+\xi'+\tau''/l+\xi'')|^p\, d\tau$$
$$\leqslant (2\pi)^{-n} k(\xi)^p \int_{R^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C(|\tau'|+|\tau''|))^N L(\tau'+\xi'+\tau''/l+\xi'')|^p\, d\tau.$$

In view of the Lebesgue Dominated Convergence Theorem the right-hand side of (2.11) is converging to

$$(2\pi)^{-n} k(\xi)^p \int_{R^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C(|\tau'|+|\tau''|))^N L(\tau'+\xi)|^p\, d\tau.$$

Hence by the assumption (1.5a) and by inequalities (2.7), (2.9) and (2.11) we obtain by letting $l\to\infty$ the inequality

$$(2.12)\quad C_1 D_2 (kk^{\sim})(\xi)$$
$$\leqslant C_2 D_1 k(\xi) + k(\xi)\Big((2\pi)^{-n} \int_{R^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C(|\tau'|+|\tau''|))^N L(\tau'+\xi)|^p\, d\tau\Big)^{1/p}.$$

In virtue of the Tayloris formula,

$$(2.13)\qquad L(\tau'+\xi) = \sum_{|\alpha|\leqslant r} \frac{1}{\alpha!} L^{(\alpha)}(\xi)\tau'^{\alpha} = \sum_{|\alpha'|\leqslant r} \frac{1}{\alpha'!} L^{(\alpha')}(\xi)\tau'^{\alpha'}$$

and the validity of (2.2) follows from (2.12). □

2.2. In this subsection we deal with open subsets $G\subset \boldsymbol{R}^n$ satisfying the following condition:

PROPERTY B_j. *For a fixed number $j\in\{1,\ldots,n\}$ there exists a constant $b>0$ such that*

$$(2.14)\qquad G\subset \{x=(x_1,\ldots,x_n)\in \boldsymbol{R}^n \mid |x_k|\leqslant b \text{ for } k=1,\ldots,j\}.$$

Let $\boldsymbol{C}_j$ be a subspace of $\boldsymbol{C}^n$ defined by

$$\boldsymbol{C}_j = \{(z_1,\ldots,z_j,0,\ldots,0)\in \boldsymbol{C}^n \mid z_k\in \boldsymbol{C} \text{ for } 1\leqslant k\leqslant j\}.$$

Denote by $\mathscr{L}\varphi$ the Fourier–Laplace transform of a function $\varphi\in C_0^\infty(G)$. Then we obtain for $p\in[1,\infty)$ and $k\in\mathscr{K}$.

LEMMA 2.2. *Assume that an open set $G\subset \boldsymbol{R}^n$ satisfies Property B_j. Then there exists a constant $K>0$ such that*

$$(2.15)\qquad \int_{R^n} |(\mathscr{L}\varphi)(\tau+z)\, k(\tau)|^p\, d\tau \leqslant K\,\|\varphi\|_{p,k}^p$$

for all $\varphi\in C_0^\infty(G)$ and $z=\xi'+i\eta'\in S_j := \{t\in \boldsymbol{C}_j \mid |t|\leqslant 1\}$.

Proof. Let $\psi' \in C_0^\infty(\boldsymbol{R}^j)$ satisfying $0 \leqslant \psi' \leqslant 1$ and $\psi'(x) \equiv 1$ for $x \in \{y \in \boldsymbol{R}^j \mid |y| \leqslant b\}$. Furthermore choose ψ'' from $C_0^\infty(\boldsymbol{R}^{n-j})$ such that $\psi''(x) \equiv 1$ for $x \in \{y \in \boldsymbol{R}^{n-j} \mid |y| \leqslant 1\}$. Define functions $\Phi_l \in C_0^\infty(\boldsymbol{R}^n)$ through the requirement

$$\Phi_l(x) = \psi'(x_1, \ldots, x_j)\psi''((x_{j+1}, \ldots, x_n)/l). \tag{2.16}$$

Then for every $\varphi \in C_0^\infty(G)$ there exists $l \in N$ such that $\Phi_l \varphi = \varphi$.

For every $l \in N$ and $z = \xi' + i\eta' \in S_j$ one has

$$\begin{aligned}(2.17)\quad & \int_{\boldsymbol{R}^n} |\mathscr{L}(\Phi_l \varphi)(\tau + z) k(\tau)|^p d\tau \\ &= \int_{\boldsymbol{R}^n} |\mathscr{F}(e^{(\eta', x')}\Phi_l \varphi)(\tau + \xi') k(\tau)|^p d\tau \\ &\leqslant M_k(-\xi')^p \int_{\boldsymbol{R}^n} |\mathscr{F}(e^{(\eta', x')}\Phi_l \varphi)(\tau + \xi') k(\tau + \xi')|^p d\tau \\ &= (2\pi)^n M_k(-\xi')^p \|e^{(\eta', x')}\Phi_l \varphi\|_{p,k}^p \\ &\leqslant (2\pi)^n M_k(-\xi')^p \|e^{(\eta', x')}\Phi_l\|_{1,M_k}^p \|\varphi\|_{p,k}^p,\end{aligned}$$

where we used inequality (2.2.3) revealed in [3], p. 39.

We now consider the norm $\|e^{(\eta', x')}\Phi_l\|_{p,k}$. One has for all $\tau \in \boldsymbol{R}^n$.

$$\begin{aligned}(2.18)\quad & \mathscr{F}(e^{(\eta', x')}\Phi_l)(\tau) \\ &= \int_{\boldsymbol{R}^n} e^{(\eta', x')}\psi'(x_1, \ldots, x_j)\psi''((x_{j+1}, \ldots, x_n)/l) e^{-i(\tau, x)} dx \\ &= \int_{\boldsymbol{R}^j} e^{(\eta', x')}\psi'(x_1, \ldots, x_j) e^{-i(\tau', x')} dx_1, \ldots, dx_j \times \\ &\qquad \times \int_{\boldsymbol{R}^{n-j}} \psi''((x_{j+1}, \ldots, x_n)/l) e^{-i(\tau'', x'')} dx_{j+1}, \ldots, dx_n \\ &= l^{n-j} \mathscr{F}_j(e^{(\eta', x')}\psi')(\tau_1, \ldots, \tau_j)(\mathscr{F}_{n-j}\psi'')(l(\tau_{j+1}, \ldots, \tau_n)),\end{aligned}$$

where $\mathscr{F}_j$ (and $\mathscr{F}_{n-j}$) denotes the Fourier transform in $\boldsymbol{R}^j$ (and in $\boldsymbol{R}^{n-j}$ resp.).

Let $C > 0$ and $N_1, N_2 \in N$ be numbers such that

$$\begin{aligned}(2.19)\quad M_k(\tau) &\leqslant C(1 + |\tau'|^2)^{N_1/2}(1 + |\tau''|^2)^{N_2/2} \\ &= C k_{N_1}(\tau') k_{N_2}(\tau'') \quad \text{for all } \tau = \tau' + \tau'' \in \boldsymbol{R}^n\end{aligned}$$

(cf. [3], p. 34). For each $l \in N$ one gets

$$\begin{aligned}(2.20)\quad & l^{n-j} \int_{\boldsymbol{R}^{n-j}} |(\mathscr{F}_{n-j}\psi'')(l(\tau_{j+1}, \ldots, \tau_n)) k_{N_2}(\tau'')| d\tau_{j+1}, \ldots, d\tau_n \\ &= \int_{\boldsymbol{R}^{n-j}} |(\mathscr{F}_{n-j}\psi'')((\tau_{j+1}, \ldots, \tau_n)) k_{N_2}(\tau''/l)| d\tau_{j+1}, \ldots, d\tau_n \\ &\leqslant \int_{\boldsymbol{R}^{n-j}} |(\mathscr{F}_{n-j}\psi'')((\tau_{j+1}, \ldots, \tau_n)) k_{N_2}(\tau'')| d\tau_{j+1}, \ldots, d\tau_n =: C_1.\end{aligned}$$

Furthermore one has

$$(2.21)\qquad \mathscr{F}_j(e^{(\eta',x')}\psi')(\tau_1,\ldots,\tau_j)=(\mathscr{L}_j\psi')((\tau_1,\ldots,\tau_j)+i(\eta_1,\ldots,\eta_j)),$$

where $\mathscr{L}_j$ denotes the Fourier–Laplace transform in C^j. Let A be a number such that $\operatorname{supp}\psi'\in\{y\in R^j|\ |y|\leqslant A\}$. In virtue of the Paley–Wiener Theorem one can find a constant $C'>0$ with which

$$(2.22)\qquad |(\mathscr{L}_j\psi')((\tau_1,\ldots,\tau_j)+i(\eta_1,\ldots,\eta_j))|\leqslant C'(1+|\tau'|^2+|\eta'|^2)^{-(j+N_1+1)/2}e^{A|\eta'|}$$

for all $(\tau_1,\ldots,\tau_j)\in R^j$ and $(\eta_1,\ldots,\eta_j)\in R^j$. Since $|\eta'|\leqslant 1$, the supremum

$$\sup_{|\eta'|\leqslant 1}\int|\mathscr{F}_j(e^{(\eta',x')}\psi')(\tau_1,\ldots,\tau_j)|\,d\tau_1,\ldots,d\tau_j$$

is finite, say C_2. Hence we see by (2.18) and (2.20) that

$$\|e^{(\eta',x')}\Phi_l\|_{1,M_k}\leqslant C_2C_1\qquad\text{for all } |\eta'|\leqslant 1 \text{ and } l\in N$$

and then the validity of (2.15) follows from (2.17). □

The previous lemma yields us the following one.

LEMMA 2.3. *Assume that an open set $G\subset R^n$ satisfies Property B_j. Then for each $\alpha'=(\alpha_1,\ldots,\alpha_j,0,\ldots,0)$ there exists a constant $C_{\alpha'}>0$ such that*

$$(2.23)\qquad \|L^{(\alpha')}(D)\varphi\|_{p,k}\leqslant C_{\alpha'}\|L(D)\varphi\|_{p,k}$$

for all $\varphi\in C_0^\infty(G)$.

Proof. Let $\tau\in R^n$ be arbitrary and let φ lie in $C_0^\infty(G)$. Define functions H, R and χ: $C^j\to C$ through the relations

$$H(z)=(\mathscr{L}\varphi)(\tau+(z_1,\ldots,z_j,0,\ldots,0)),$$
$$R(z)=L(\tau+(z_1,\ldots,z_j,0,\ldots,0))$$

and

$$\chi(z)=\begin{cases}1 & \text{for } |z|\leqslant 1,\\ 0 & \text{for } |z|>1.\end{cases}$$

Let $\bar r\in N_0^j$ such that $\bar r=(r,\ldots,r)$. Then we have

$$(2.24)\qquad |H(0)(D^{\alpha'}R)(0)|\int_{|z|\leqslant 1}|z^{\alpha'}|\chi(z)\,dz\leqslant(\bar r!/(\bar r-\alpha')!)\int_{|z|\leqslant 1}|H(z)R(z)|\chi(z)\,dz$$

(cf. [6], Lemma 3, p. 186). In other words, one has

$$(2.25)\qquad |(\mathscr{F}\varphi)(\tau)L^{(\alpha')}(\tau)|E_{\alpha'}$$
$$\leqslant(\bar r!/(\bar r-\alpha')!)\int_{|z|\leqslant 1}|\mathscr{L}(L(D)\varphi)(\tau+(z_1,\ldots,z_j,0,\ldots,0))|\,dz,$$

where $E_{\alpha'} = \int_{|z|\leqslant 1} |z^{\alpha'}|\,dz > 0$. Hence by the Hölder inequality and by the Fubini Theorem we obtain

$$(2.26)\qquad \|L^{(\alpha')}(D)\,\varphi\|_{p,k}^{p}\,E_{\alpha'}^{p}$$
$$\leqslant (\bar{r}!/(\bar{r}-\alpha')!)^p\, m_j^{p-1} \int_{R^n}\int_{|z|\leqslant 1} \big|\mathscr{L}(L(D)\,\varphi)(\tau+(z_1,\ldots,z_j,0,\ldots,0))\big|^p\,dz\,d\tau$$
$$= (\bar{r}!/(\bar{r}-\alpha')!)^p\, m_j^{p-1} \int_{|z|\leqslant 1}\int_{R^n} \big|\mathscr{L}(L(D)\,\varphi)(\tau+(z_1,\ldots,z_j,0,\ldots,0))\big|^p\,d\tau\,dz,$$

where m_j denotes the integral $\int_{|z|\leqslant 1} 1\,dz$. Combining inequalities (2.15) and (2.26), we get the assertion (2.23).

We are now ready to show

THEOREM 2.4. *Assume that an open set $G \subset \boldsymbol{R}^n$ satisfies Property B_j. Furthermore, assume that there exists a constant $\gamma > 0$ such that the inequality*

$$(2.27)\qquad \Big(\sum_{|\alpha'|\leqslant r} |L^{(\alpha')}(\xi)|^p\Big)^{1/p} + 1 \geqslant \gamma k^{\sim}(\xi)$$

holds for all $\xi \in \boldsymbol{R}^n$. Then $L(D)$ is $k^{\sim}$-coercive in the space $\mathscr{B}_{p,k}(G)$, that is, there exist constants $C_1 > 0$ and $C_2 \geqslant 0$ such that the inequality

$$(2.28)\qquad \|L(D)\,\varphi\|_{p,k} \geqslant C_1\,\|\varphi\|_{p,kk^{\sim}} - C_2\,\|\varphi\|_{p,k}$$

is valid for all $\varphi \in C_0^{\infty}(G)$.

Proof. For every $\varphi \in C_0^{\infty}(G)$ one has by (2.27)

$$(2.29)\qquad \gamma^p |(\mathscr{F}\varphi)(\xi)\,(kk^{\sim})(\xi)|^p \leqslant 2^p\Big(\sum_{|\alpha'|\leqslant r} |L^{(\alpha')}(\xi)\,(\mathscr{F}\varphi)(\xi)\,k(\xi)|^p + |(\mathscr{F}\varphi)(\xi)\,k(\xi)|^p\Big)$$
$$= 2^p\Big(\sum_{|\alpha'|\leqslant r} \big|\mathscr{F}(L^{(\alpha')}(D)\,\varphi)(\xi)\,k(\xi)\big|^p + |(\mathscr{F}\varphi)(\xi)\,k(\xi)|^p\Big).$$

Integrating over $\boldsymbol{R}^n$ and using inequality (2.23), we get the validity of (2.28).

Combining Theorems 2.1 and 2.4, we obtain

COROLLARY 2.5. *Assume that an open set $G \subset \boldsymbol{R}^n$ has Properties A_j and B_j. Then the operator $L(D)$ is $k^{\sim}$-coercive in the space $\mathscr{B}_{p,k}(G)$ if and only if there exist a constant $\gamma > 0$ such that the inequality*

$$(2.30)\qquad \Big(\sum_{|\alpha'|\leqslant r} |L^{(\alpha')}(\xi)|^p\Big)^{1/p} + 1 \geqslant \gamma k^{\sim}(\xi)$$

holds for all $\xi \in \boldsymbol{R}^n$. □

Applying Corollary 2.5 with $j = n$, we see that for a bounded open set G the operator $L(D)$ is $k^{\sim}$-coercive in the space $\mathscr{B}_{p,k}(G)$ if and only if with some constant $\gamma > 0$

(2.31) $$\Big(\sum_{|\alpha|\leqslant r} |L^{(\alpha)}(\xi)|^p\Big)^{1/p}+1 \geqslant \gamma k^{\sim}(\xi) \quad \text{for all } \xi\in \boldsymbol{R}^n.$$

More generally, the open strip

(2.32) $$G = \{y\in \boldsymbol{R}^n \mid |y_k-y_k^0| < a \text{ for } k = 1, \ldots, j\}$$

has Properties A_j and B_j.

The partial differential operator $P(D) = D_1(D_1^2+D_2^2+D_3^2)$ satisfies

$$\Big(\sum_{|(\alpha_1,0,0)|\leqslant 3} |P^{(\alpha')}(\xi)|^p\Big)^{1/p} \geqslant |P^{(1,0,0)}(\xi)| \geqslant |\xi|^2 \quad \text{for all } \xi\in \boldsymbol{R}^3.$$

Hence $P(D)$ is k_2-coercive in the space $\mathscr{B}_{p,k}(G)$, where G is the strip $\{x = (x_1, x_2, x_3)\in \boldsymbol{R}^3 \mid |x_1| < b\}$.

3. On strongly $k^{\sim}$-coercivity of $L(D)$.

3.1. This section deals with strongly $k^{\sim}$-coercive operators. Applying the proof of Theorem 2.1, we get the following necessary algebraic criterion for the strongly $k^{\sim}$-coercive operators $L(D)$.

THEOREM 3.1. *Let $G\subset \boldsymbol{R}^n$ be an open set with Property A_j. Furthermore, let $L(D)$ be a strongly $k^{\sim}$-coercive operator in the space $\mathscr{B}_k(G)$. Then there exists a constant $\gamma>0$ such that the inequality*

(3.1) $$\Big(\sum_{|\alpha'|\leqslant r} |L_{\mathrm{Re}}^{(\alpha')}(\xi)|^2\Big)^{1/2}+1 \geqslant \gamma k^{\sim}(\xi)$$

holds for all $\xi\in \boldsymbol{R}^n$, where $L_{\mathrm{Re}}(\xi) := \operatorname{Re} L(\xi)$.

Proof. In virtue of the strongly $k^{\sim}$-coercivity of the operator $L(D)$ we have for all $\varphi\in C_0^\infty(G)$

(3.2) $$C_1\|\varphi\|_{k,k^{\sim}}^2 \leqslant \operatorname{Re}\big(L(D)\varphi, \varphi\big)_k + C_2\|\varphi\|_k^2 = \big(L_{\mathrm{Re}}(D)\varphi, \varphi\big)_k + C_2\|\varphi\|_k^2.$$

Let φ_l be functions in $C_0^\infty(G)$ defined by (2.4) (with $p=2$). Then by the proof of Theorem 2.1 one sees that there exist constants $D_1>0$ and $D_2>0$ with which

(3.3) $$\|\varphi_l\|_k^2 \leqslant D_1^2 k(\xi)^2$$

and

(3.4) $$\|\varphi_l\|_{k,k^{\sim}}^2 \geqslant D_2^2 k(\xi)^2 k^{\sim}(\xi).$$

Furthermore, we obtain

(3.5) $$\begin{aligned}&\big(L_{\mathrm{Re}}(D)\varphi_l, \varphi_l\big)_k\\ &= (2\pi)^{-n} l^{n-j}\int_{\boldsymbol{R}^n} L_{\mathrm{Re}}(\eta)\,|\mathscr{F}(\widetilde{\Phi\theta})(\eta'-\xi'+l(\eta''-\xi''))k(\eta)|^2\,d\eta\\ &= (2\pi)^{-n}\int_{\boldsymbol{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)|^2\big(k(\tau'+\xi'+\tau''/l+\xi'')\big)^2|L_{\mathrm{Re}}(\tau'+\xi'+\tau''/l+\xi'')|\,d\tau\\ &\leqslant (2\pi)^{-n}k(\xi)^2\int_{\boldsymbol{R}^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C(|\tau'|+|\tau''|))^N|^2\,|L_{\mathrm{Re}}(\xi+\tau'+\tau''/l)|\,d\tau,\end{aligned}$$

where the right-hand side is converging to

$$(2\pi)^{-n} k(\xi)^2 \int_{R^n} |\mathscr{F}(\widetilde{\Phi\theta})(\tau)(1+C(|\tau'|+|\tau''|))^N|^2 |L_{\mathrm{Rc}}(\xi+\tau')|\, d\tau.$$

Hence the assertion follows from (3.2), (3.3), (3.4) and from the Taylor formula (2.13).

3.2. A sufficient condition for the strongly $k^{\sim}$-coercivity follows from Theorem 2.4 by choosing $p = 1$.

THEOREM 3.2. *Assume that an open set $G \subset R^n$ has Property B_j. Furthermore, suppose that there exist constants $\gamma > 0$ and $\mathscr{H} \in R$ such that*

$$L_{\mathrm{Rc}}(\xi) \geqslant \mathscr{H} \quad \textit{for all } \xi \in R^n \tag{3.6}$$

and

$$\Big(\sum_{|\alpha'| \leqslant r} |L_{\mathrm{Rc}}^{(\alpha')}(\xi)|^2\Big)^{1/2} + 1 \geqslant \gamma k^{\sim}(\xi) \quad \textit{for all } \xi \in R^n. \tag{3.7}$$

Then the operator $L(D)$ is strongly $k^{\sim}$-coercive in the space $\mathscr{B}_k(G)$.

Proof. To a function $\varphi \in C_0^{\infty}(G)$ we define $\varphi^{\vee} \in C_0^{\infty}$ by $\varphi^{\vee}(x) = \overline{\varphi(-x)}$. Then one has

$$\operatorname{supp} \varphi^{\vee} \subset -G := \{y \in R^n \mid -y \in G\} \tag{3.8}$$

and

$$\operatorname{supp}(\varphi * \varphi^{\vee}) \subset \operatorname{supp} \varphi + \operatorname{supp} \varphi^{\vee} \subset G + (-G), \tag{3.9}$$

where $*$ denotes the convolution

$$(\varphi * \varphi^{\vee})(x) = \int_{R^n} \varphi(y)\, \varphi^{\vee}(x-y)\, dy.$$

Furthermore, we have for all $\eta \in R^n$.

$$|\mathscr{F}(\varphi * \varphi^{\vee})(\eta)| = |(\mathscr{F}\varphi)(\eta)(\mathscr{F}\varphi^{\vee})(\eta)| = |(\mathscr{F}\varphi)(\eta)|^2. \tag{3.10}$$

The open set $G+(-G)$ satisfies Property B_j. Hence in virtue of Theorem 2.4 there exist constants $C_1 > 0$ and $C_2 \geqslant 0$ such that

$$C_1 \|\varphi * \varphi^{\vee}\|_{1,k^2k^{\sim}} \leqslant \|L_{\mathrm{Rc}}(D)(\varphi * \varphi^{\vee})\|_{1,k^2} + C_2 \|\varphi * \varphi^{\vee}\|_{1,k^2} \tag{3.11}$$

in other words,

$$\begin{aligned} &C_1 (2\pi)^{-n} \int_{R^n} |(\mathscr{F}\varphi)(\eta) k(\eta)|^2 k^{\sim}(\eta)\, d\eta \\ &\leqslant (2\pi)^{-n} \int_{R^n} |L_{\mathrm{Rc}}(\eta)|\, |(\mathscr{F}\varphi)(\eta)|^2 k^2(\eta) + C_2 (2\pi)^{-n} \int_{R^n} |(\mathscr{F}\varphi)(\eta) k(\eta)|^2\, d\eta. \end{aligned} \tag{3.12}$$

Inequality (3.12) yields us for all $\varphi \in C_0^\infty(G)$

$$C_1 \|\varphi\|^2_{k, k^\sim} \leqslant (2\pi)^{-n} \int_{\boldsymbol{R}^n} (L_{\mathrm{Re}}(\eta) - \mathscr{H}) |(\mathscr{F}\varphi)(\eta)|^2 k^2(\eta) + (C_2 + |\mathscr{H}|) \|\varphi\|_k^2 = \operatorname{Re}(L(D)\varphi, \varphi)_k + (C_2 + |\mathscr{H}| - \mathscr{H}) \|\varphi\|_k^2. \tag{3.13}$$

This shows our assertion. □

Combining Theorems 3.1 and 3.2 one sees that when G is the open strip (2.32), the operator $L(D)$, for which $L_{\mathrm{Re}}(\xi) \geqslant 0$, is $k^\sim$-coercive in the space $\mathscr{B}_k(G)$ if and only if there exists a constant $\gamma > 0$ such that

$$\Big(\sum_{|\alpha'| \leqslant r} |L_{\mathrm{Re}}^{(\alpha')}(\xi)|^2\Big)^{1/2} + 1 \geqslant \gamma k^\sim(\xi) \qquad \text{for all } \xi \in \boldsymbol{R}^n.$$

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