APPROXIMATION AND FUNCTION SPACES
BANACH CENTER PUBLICATIONS, VOLUME 22
PWN-POLISH SCIENTIFIC PUBLISHERS
WARSAW 1989

A NEW CLASSIFICATION OF PERIODIC FUNCTIONS AND THEIR APPROXIMATION

A. I. STEPANETS

Institute of Mathematics, Academy of Sciences of the Ukrainian S.S.R.
Kiev, U.S.S.R.

In 1983 the author proposed a classification of periodic functions based on transformations of Fourier series by using multipliers and translations of the argument. The basic idea is the following.

Let $f \in L(0, 2\pi)$ and let $a_k = a_k(f)$, $b_k = b_k(f)$, $k = 0, 1, \ldots$, be the Fourier coefficients of f. Furthermore, let $\psi(k)$ be an arbitrary function on the positive integers and $\beta \in \mathbf{R}$ a fixed number. Suppose that the series

$$(1) \qquad \sum_{k=1}^{\infty} \frac{1}{\psi(k)} (a_k \cos(kx + \beta\pi/2) + b_k \sin(kx + \beta\pi/2))$$

is the Fourier series of some function from $L(0, 2\pi)$. We denote this function by $f_\beta^\psi(\cdot)$, and call it the (ψ, β)-*derivative* of f; the set of all functions f with the above property will be denoted by L_β^ψ. Let, moreover, $\mathfrak{N}$ be a subset of $L(0, 2\pi)$. Then if $f \in L_\beta^\psi$ and $f_\beta^\psi \in \mathfrak{N}$, we say that f is of class $L_\beta^\psi \mathfrak{N}$. Denote by C_β^ψ the set of all continuous functions from L_β^ψ. For $\psi(k) = k^{-r}$, $r > 0$, the class $C_\beta^\psi \mathfrak{N}$ turns into the well-known Weyl–Nagy class $W_\beta^r \mathfrak{N}$, which for r an integer and $\beta = r$ is the class of 2π-periodic functions whose rth derivative is in $\mathfrak{N}$. In the case where $\sum_{k=1}^{\infty} \psi(k) \cos(kt + \beta\pi/2)$ is the Fourier series of an integrable function $\Psi_\beta(t)$, the class $L_\beta^\psi \mathfrak{N}$ coincides with the set of all f which can be represented as a convolution $f = a_0/2 + \varphi * \Psi_\beta$ with $\varphi \in \mathfrak{N}$; such classes were studied by many authors (see e.g. [3], [4], [6], [7], [10], [12], [13], [15], [18]).

Our approach permits a wide range of periodic functions to be classified, including the ones with divergent Fourier series, the C^∞ functions and in particular analytic and entire functions.

Up to the present, the author and his students have considered for the new classes all principal problems of approximation theory which were earlier formulated for classes of differentiable functions. The results obtained are final to the same extent as the corresponding ones for classes of functions defined by means of Weyl's fractional derivatives.

The results are formulated in terms of the parameters which determine a given class. They embrace the known assertions for classes of differentiable functions, but also, as was to be expected, exhibit some new phenomena not observed earlier.

The basic method of study in C and L spaces is to obtain and investigate the integral representations for deviations of polynomials on function classes. When studying problems in integral norms, multiplier theory is also used.

The approximation properties of $L^{\psi}_{\beta}\mathfrak{N}$ essentially depend on the function $\psi(k)$. When considering problems in C and L spaces, we assume that $\psi(k)$ is a convex sequence tending to 0 as $k \to \infty$ whose terms are the values of a convex function $\psi(v)$ of the continuous argument $v \geqslant 1$. Denote by $\mathfrak{M}$ the set of all such functions. If $\psi \in \mathfrak{M}$ and $\int_1^{\infty} v^{-1}\psi(v+1)\,dv < \infty$, then we write $\psi \in F$. Further, with every $\psi \in \mathfrak{M}$ we associate a pair of functions $\eta(t) = \eta(\psi; t)$, $\mu(t) = \mu(\psi; t)$ given by

$$\eta(t) = \psi^{-1}\left(\tfrac{1}{2}\psi(t)\right), \qquad \mu(t) = t\left(\eta(t) - t\right)^{-1},$$

and we put

$$\mathfrak{M}_C = \{\psi \in \mathfrak{M}\colon\ 0 \leqslant K_1 \leqslant \mu(\psi; t) \leqslant K_2\},$$

$$\mathfrak{M}_0 = \{\psi \in \mathfrak{M}\colon\ 0 < \mu(\psi; t) \leqslant K_3\}.$$

Here and in the sequel, K and K_i, $i = 1, 2, \ldots$, are absolute constants. Denote also by $\mathfrak{M}_{\infty}$ the subset of those $\psi \in \mathfrak{M}$ for which $\mu(\psi; t)$ is increasing and unbounded from above: $\mathfrak{M}_{\infty} = \{\psi \in \mathfrak{M}\colon\ \mu(\psi; t) \uparrow \infty\}$. It can be shown that if $\psi \in \mathfrak{M}_{\infty}$, then the classes C^{ψ}_{β} consist of C^{∞} functions. If, moreover, $\eta(\psi; t) - t \leqslant K$, then $C^{\psi}_{\beta}\mathfrak{N}$ is a class of analytic functions.

The classes $\mathfrak{M}_{\infty}$, $\mathfrak{M}_C$ and $\mathfrak{M}_0$ have natural representatives: $\psi_r(v) = \exp(-\delta v^r)$, $r > 0$, $\delta > 1$, for $\mathfrak{M}_{\infty}$; $\varphi_r(v) = v^{-r}$, $r > 0$, for $\mathfrak{M}_C$; and $\chi_r(v) = \ln^{-r}(v+e)$, $r > 0$, for $\mathfrak{M}_0$.

We will take for $\mathfrak{N}$ the unit balls S_p in L_p spaces: $S_p = \{\varphi\colon\ \|\varphi\|_p \leqslant 1\}$, where $\|\varphi\|_p = \left(\int_{-\pi}^{\pi} |\varphi(t)|^p\,dt\right)^{1/p}$ for $1 \leqslant p < \infty$ and $\|\varphi\|_{\infty} = \operatorname{ess\,sup}|\varphi(t)|$; we then write $L^{\psi}_{\beta}S_p = L^{\psi}_{\beta,p}$, $1 \leqslant p < \infty$, and $C^{\psi}_{\beta}S_{\infty} = C^{\psi}_{\beta,\infty}$. We will also take for $\mathfrak{N}$ the function classes H_{ω} and H_{ω_p} defined by

$$H_{\omega} = \{\varphi\colon\ \|\varphi(x+t) - \varphi(x)\|_C \leqslant \omega(t)\},$$

$$H_{\omega_p} = \{\varphi\colon\ \|\varphi(x+t) - \varphi(x)\|_{L_p} \leqslant \omega(t)\},$$

where $\omega = \omega(t)$ is a modulus of continuity.

We will use the following approximation characteristics:

$\varrho_n(f; x) = f(x) - S_{n-1}(f; x)$ — the deviation of the partial Fourier sum $S_{n-1}(f; x)$ from the function $f(x)$;

$\mathcal{E}_n(A)_X = \sup\{\|\varrho_n(f;x)\|_X: f \in A\}$, where X is C or L_p, $1 \leqslant p < \infty$, and A is a function class;

$E_n(f)_X$ – the best approximation of the function f by trigonometric polynomials of order $n-1$;

$E_n(A)_X = \sup\{E_n(f)_X: f \in A\}$.

We now formulate some of the results obtained.

1. A representation of deviations for linear means of Fourier series

Let $\Lambda = \{\lambda_k^{(n)}\}_{n=0}^{\infty}$ be a triangular matrix of real numbers such that $\lambda_k^{(n)} = 0$ for $k \geqslant n$ and $\lambda_0^{(n)} = 1$. To every function $f \in L(0, 2\pi)$ with the Fourier series

$$S[f] = a_0/2 + \sum_{k=1}^{\infty} (a_k \cos kx + b_k \sin kx) \equiv \sum_{k=0}^{\infty} A_k(f; x)$$

we associate the sequence of polynomials

$$U_n(f; x; \Lambda) = \sum_{k=0}^{n-1} \lambda_k^{(n)} A_k(f; x). \tag{2}$$

Further, let $\{\lambda_n(v)\}_{n=1}^{\infty}$ be a sequence of functions defined on [0, 1] such that $\lambda_n(k/n) = \lambda_k^{(n)}$, $k = 0, 1, \ldots$

Put

$$\tau_n(v) = \tau_n(v; \Lambda; \psi) = \begin{cases} (1-\lambda_n(v))\psi(1), & 0 \leqslant v \leqslant 1/n, \\ (1-\lambda_n(v))\psi(nv), & 1/n \leqslant v \leqslant 1, \\ \psi(nv), & v \geqslant 1, \end{cases} \tag{3}$$

so that

$$\tau_n(k/n) = \begin{cases} (1-\lambda_k^{(n)})\psi(k), & 1 \leqslant k \leqslant n-1, \\ \psi(k), & k \geqslant n, \end{cases} \tag{4}$$

and let

$$\hat{\tau}_n(t) = \pi^{-1} \int_0^{\infty} \tau_n(v) \cos(vt + \beta\pi/2)\, dt.$$

Denote by M the set of all essentially bounded functions $\varphi \in L(0, 2\pi)$:

$$M = \{\varphi \in L(0, 2\pi): \operatorname{ess\,sup}|\varphi| < \infty\} \overset{\mathrm{df}}{=} L_\infty.$$

With the above notation we have the following assertions.

THEOREM 1. *If the function $\tau_n(v) = \tau_n(v; \Lambda; \psi)$ defined by (3) is continuous and its transform $\hat{\tau}_n(t)$ is integrable on the whole real line, then for all $f \in C_\beta^\psi M$, at each x,*

$$f(x) - U_n(f; x; \Lambda) = \int_{-\infty}^{\infty} f_\beta^\psi(x + t/n)\hat{\tau}_n(t)\, dt, \quad n \in N. \tag{5}$$

THEOREM 1′. *If $\tau_n(v) = \tau_n(v; \Lambda; \psi)$ is continuous, if its transform $\hat{\tau}_n(t)$ is integrable on* $\mathbf{R}$, *and if the function*

$$\hat{\tau}_n^{(0)}(t) = \begin{cases} \sup\limits_{x \geq t} |\tau_n(x)|, & t > 0, \\ \sup\limits_{x \leq t} |\tau_n(x)|, & t < 0, \end{cases}$$

is integrable on the set $|t| > A$ for some positive A, then for all $f \in L_\beta^\psi$ equality (5) *is valid at almost every point of any interval of length 2π.*

Let us point out that to obtain the representation (5) for a given polynomial $U_n(f; x; \Lambda)$ it is only necessary to choose a function $\tau_n(v)$ of a continuous argument in such a way that $\tau_n(v)$ is continuous, its transform is integrable and (4) is satisfied. This fact may be used to choose $\tau_n(v)$ so as to facilitate the investigation of the integral on the right of (5). In particular, to obtain an integral representation of the deviation $\varrho_n(f; x)$ of the partial Fourier sum $S_{n-1}(f; x)$ from the function $f \in C_\beta^\psi M$ (or $f \in L_\beta^\psi$), it suffices to take for $\tau_n(v)$ any continuous function $\tau_n^*(v)$, $v \geq 0$, satisfying

$$\tau_n^*(k/n) = \begin{cases} 0, & k \leq n-1, \\ \psi(k), & k \geq n. \end{cases}$$

The following function turns out to be convenient:

$$\tau_n^*(v) = \begin{cases} 0, & 0 \leq v \leq 1-1/n, \\ 1+n(v-1)\psi(n), & 1-1/n \leq v \leq 1, \\ \psi(nv), & v \geq 1. \end{cases}$$

It meets all the necessary requirements and is used to obtain, with the aid of Theorem 1,

THEOREM 2. *If $\psi \in F$, then for all $f \in C_\beta^\psi M$, at each x,*

$$\varrho_n(f; x) = \int_{-\infty}^{\infty} f_\beta^\psi(x+t/n) I_2(t)\, dt + \tfrac{1}{2} A_n(f; x), \tag{6}$$

where

$$I_2(t) = I_2^{(\beta)}(\psi; t) = \pi^{-1} \int_1^{\infty} \psi(nv) \cos(vt + \beta\pi/2)\, dt. \tag{7}$$

If $\psi \in F$ and $f \in L_\beta^\psi$, then (6) *holds almost everywhere.*

For $\beta = 0$, (6) *holds for all $\psi \in \mathfrak{M}$ at each x if $f \in C_0^\psi M$, and almost everywhere on the period if $f \in L_0^\psi$.*

Note that the improper integrals in (5) and (6) are understood in the principal value sense, i.e. as limits of integrals over symmetric increasing intervals.

For $\psi(k) = k^{-r}$, $r > 0$, and β an integer, formula (5) was obtained by Nagy [6]. It was then extended to all real β by S. A. Telyakovskii [18]. The proofs of Theorems 1 and 2 are given in the papers [12] and [13] by the author.

2. Approximation by Fourier sums in C and L_1

Proceeding from equality (6), we obtain the following statement concerning $\varrho_n(f; x)$:

THEOREM 3. *Let $\psi \in F$ and let $a = a(n)$, $n \in N$, be any numerical sequence with $a(n) \geqslant a_0 > 0$. Then we have the following asymptotic formulas as $n \to \infty$:*

$$\mathscr{E}_n(C^{\psi}_{\beta,\infty})_C = \frac{4}{\pi^2}\psi(n)\ln^+\frac{n\pi}{a(n)} + b^{\psi}_n(a), \tag{8}$$

where

$$b^{\psi}_n(a) = O(1)\Big[\psi(n) + \int_{1/a(n)}^{\infty} t^{-1}\psi(nt+t)\,dt + \int_{a(n)}^{\infty} t^{-1}\big(\psi(n) - \psi(n+n/t)\big)\,dt\Big],$$

and

$$\mathscr{E}_n(C^{\psi}_{\beta} H_{\omega}) = \frac{2}{\pi^2}\psi(n)s_n(\omega)\ln^+\frac{n\pi}{a(n)} + d^{\psi}_n(a, \omega), \tag{8'}$$

where

$$d^{\psi}_n(a, \omega) = O(1)\,b^{\psi}_n(a)\,\omega(1/n),$$

$$s_n(\omega) = \Theta_{\omega}\int_0^{\pi/2}\omega(2t/n)\sin t\,dt, \quad 2/3 \leqslant \Theta_{\omega} \leqslant 1, \tag{9}$$

and $\Theta_{\omega} = 1$ if $\omega(t)$ is a concave modulus of continuity. Here $\ln^+ t = \max\{\ln t, 0\}$ *and $O(1)$ denotes the quantities uniformly bounded with respect to n and β.*

An analogous statement is valid for the space L_1. More precisely, under the assumptions of Theorem 3, on the left-hand sides of (8) and (8′) one can write $\mathscr{E}_n(L^{\psi}_{\beta,1})$ and $\mathscr{E}_n(L^{\psi}_{\beta} H_{\omega_1})_{L_1}$ respectively provided that the quantity Θ_{ω} in (9) is subject to the condition $\Theta_{\omega} \in [1/2, 1]$.

Using the freedom of the choice of the sequence $a(n)$ and putting $a(n) = \mu(n) = \mu(\psi; n)$ in (8), (8′) and in their analogues for L_1, we obtain

THEOREM 4. *Let* $\psi \in \mathfrak{M}_{C,\infty} = \mathfrak{M}_C \cup \mathfrak{M}_\infty$. *Then for all* $\beta \in \boldsymbol{R}$ *we have as* $n \to \infty$

$$\mathcal{E}_n(C^{\psi}_{\beta,\infty})_C = \mathcal{E}_n(L^{\psi}_{\beta,1})_1 + O(1)\psi(n) = \frac{4}{\pi^2}\ln^+ \pi(\eta(n)-n) + O(1)\psi(n), \tag{10}$$

$$\mathcal{E}_n(C^{\psi}_{\beta} H_\omega)_C = \frac{2}{\pi^2}\psi(n) s_n(\omega)\ln^+ \pi(\eta(n)-n) + O(1)\psi(n)\omega(1/n). \tag{11}$$

On the other hand, if $\psi \in \mathfrak{M}_0$, *then*

$$\mathcal{E}_n(C^{\psi}_{0,\infty})_C = \mathcal{E}_n(L^{\psi}_{0,1})_1 + O(1)\psi(n) = \frac{4}{\pi^2}\psi(n)\ln n + O(1)\psi(n), \tag{12}$$

$$\mathcal{E}_n(C^{\psi}_{0} H_\omega)_C = \frac{2}{\pi^2}\psi(n) s_n(\omega)\ln n + O(1)\psi(n)\omega(1/n). \tag{13}$$

Here the $O(1)$ *are quantities uniformly bounded in* n *and* $\beta \in \boldsymbol{R}$, *and* $\eta(n) = \eta(\psi; n) = \psi^{-1}(\frac{1}{2}\psi(n))$. *In* (11) *and* (13), *the left-hand sides can be replaced by* $\mathcal{E}(L^{\psi}_{\beta} H_{\omega_1})_1$ *and* $\mathcal{E}(L^{\psi}_{0} H_{\omega_1})_1$ *provided that* $\Theta_\omega \in [1/2, 1]$.

The function $\psi_1(t) = t^{-r}$, $r > 0$, $t > 0$, belongs to $\mathfrak{M}_C$ and has $\eta(n) - n = (2^{1/r} - 1)n$. Hence Theorem 4 implies

THEOREM 4′. *Let* $W^r_\beta = C^{\psi_1}_{\beta,\infty}$ *and* $W^r_\beta H_\omega = C^{\psi_1}_{\beta} H_\omega$. *Then for all* $\beta \in \boldsymbol{R}$ *as* $n \to \infty$

$$\mathcal{E}_n(W^r_\beta)_C = \frac{4}{\pi^2 n^r}\ln n + O(1) n^{-r}, \tag{14}$$

$$\mathcal{E}_n(W^r_\beta H_\omega)_C = \frac{2 s_n(\omega)}{\pi^2 n^r}\ln n + O(1) n^{-r}\omega(1/n). \tag{15}$$

Equality (14) for r being a positive integer and $\beta = r$ was obtained by A. N. Kolmogorov in his celebrated paper [5] which initiated a new trend in approximation theory and in the theory of Fourier series — looking for asymptotic formulas for l.u.b. of deviations of linear means of Fourier series on fixed function classes.

For all $r > 0$ and $\beta = r$, equality (14) was obtained by V. T. Pinkevich [9]. Under the same assumptions, equality (15) is due to S. M. Nikol'skii [7]; for all $\beta \in \boldsymbol{R}$, it was proved by A. V. Efimov [4].

The function $\psi_r(t) = \exp(-\delta t^r)$ belongs to $\mathfrak{M}$ for all $\delta > 0$ and $r > 0$, and satisfies $\eta(\psi_r; n) - n = n^{1-r}(\ln 2/(r\delta) + O(1))$. Hence Theorem 4 yields

THEOREM 4″. *Let $\delta > 1$, $r > 0$, $\beta \in \mathbf{R}$ and $\psi_r(t) = \exp(-\delta t^r)$. Then, as $n \to \infty$,*

$$\mathscr{E}_n(C^{\psi_r}_{\beta,\infty})_C = \frac{4}{\pi^2}\exp(-\delta n^r)\ln^+ n^{1-r} + O(1)\exp(-\delta n^r),$$

$$\mathscr{E}_n(C^{\psi_r}_{\beta} H_\omega)_C = \frac{2}{\pi^2}\exp(-\delta n^r) s_n(\omega)\ln^+ n^{1-r} + O(1)\exp(-\delta n^r)\,\omega(1/n).$$

The proof of Theorem 4″ can be found in [14]; in the general case in the space C, Theorem 4 was proved in [12] and [13].

An analogue of Theorem 4′ in the space L_1 is also known. The fundamental results here are due to S. M. Nikol'skii [7] (see also [1], [2], [11]). The assertions of Theorem 4 in the L_1 norm were proved in [17].

3. Best approximation by trigonometric polynomials in C and L_1

For l.u.b. of best approximations in C and L_1 we have the following assertion.

THEOREM 5. *Let $\psi \in \mathfrak{M}_{C,\infty}$ and $\beta \in \mathbf{R}$. Then there are absolute positive constants K_1 and K_2 such that*

$$K_1\psi(n) \leqslant \{E_n(C^{\psi}_{\beta,\infty})_C,\ E_n(L^{\psi}_{\beta,1})_1\} \leqslant K_2\psi(n), \tag{16}$$

$$K_1\psi(n)\,\omega(1/n) \leqslant \{E_n(C^{\psi}_{\beta} H_\omega)_C,\ E_n(L^{\psi}_{\beta} H_\omega)_1\} \leqslant K_2\psi(n)\,\omega(1/n), \tag{17}$$

where $\omega = \omega(t)$ is an arbitrary modulus of continuity. The estimates (16) *and* (17) *are also valid for $\psi \in \mathfrak{M}_0$ and $\beta = 0$.*

For $\psi(k) = k^{-r}$, $\beta = r$, the estimates (16), (17) are the well-known Jackson inequalities. The proof of Theorem 5 can be found in [16]. We point out that the required order of best approximations is realized by polynomials generated by linear transforms of Fourier series, i.e. by polynomials of the form (2) with

$$\lambda_n(v) = \begin{cases} 0, & 0 \leqslant v \leqslant c_n, \\ 1 - \dfrac{\psi(n)(v - c_n)}{(1 - c_n)\psi(nv)}, & c_n \leqslant v \leqslant 1, \\ 0, & v \geqslant 1, \end{cases}$$

where

$$c_n = 1 - \big(\psi^{-1}(\tfrac{1}{2}\psi(n)) - n\big)/n.$$

4. Approximation by Fourier sums and best approximations in L_s spaces

As is well known, for $s \in (1, \infty)$ the orders of the best approximations $E_n(f)_s$ and of the deviations $\|\varrho_n(f; x)\|_s$ coincide. The following assertions are true:

THEOREM 6. *Let $\psi \in \mathfrak{M}_\infty$ and suppose $\eta(\psi; t) - t \leqslant K$ for all $t \geqslant 1$. If $1 \leqslant p, s \leqslant \infty$ and $f \in L^\psi_\beta L_p$, then for all $n \in N$*

$$E_n(f)_s \leqslant \|\varrho_n(f; x)\|_s \leqslant K_1 \psi(n) E_n(f^\psi_\beta)_p, \tag{18}$$

$$K_1 \psi(n) \leqslant E_n(L^\psi_{\beta,p})_s \leqslant K_2 \psi(n), \tag{19}$$

where K_1 and K_2 are absolute constants.

On the other hand, if either $\psi \in \mathfrak{M}_\infty$ and $\eta(\psi; t) - t \geqslant K > 0$, or $\psi \in \mathfrak{M}_C$ and $1 < p, s < \infty$, then for all $f \in L^\psi_\beta L_p$ and $n \in N$

$$E_n(f)_s \leqslant \|\varrho_n(f; x)\|_s \leqslant C_{p,s} \psi(n) (\eta(n) - n)^\alpha E_n(f^\psi_\beta)_p, \tag{20}$$

$$C_{p,s} \psi(n) (\eta(n) - n)^\alpha \leqslant E_n(L^\psi_{\beta,p})_s \leqslant C^{(1)}_{p,s} \psi(n) (\eta(n) - n)^\alpha, \tag{21}$$

where $\eta(n) = \psi^{-1}(\frac{1}{2}\psi(n))$, $\alpha = \max(0, p^{-1} - s^{-1})$, and $C_{p,s}$, $C^{(1)}_{p,s}$ are positive constants which may depend on p and s only.

It has been observed above that the functions $\psi_1(v) = v^{-r}$, $r > 0$, belong to $\mathfrak{M}_C$. They satisfy $\eta(n) - n = (2^{1/r} - 1)n$, and then (16)–(21) turn into well-known classical relations established earlier by D. Jackson, A. N. Kolmogorov, S. M. Nikol'skii, B. Nagy, J. Favard, and also V. K. Dzyadyk, A. V. Efimov, S. B. Stechkin, S. A. Telyakovskii and others. Relations (18), (19) are proved in [16].

If ψ satisfies

$$\lim_{k \to \infty} \ln |\psi(k)|^{1/k} = \infty, \tag{22}$$

then all continuous functions from L^ψ_β have regular extensions to the whole complex plane, i.e. the functions from C^ψ_β are restrictions of entire functions to the real axis. This means that in this case $C^\psi_\beta \mathfrak{N}$ and $L^\psi_\beta \mathfrak{N}$ are classes of entire 2π-periodic functions and of equivalence classes of such functions respectively. For these, the following assertions are valid.

THEOREM 7. *Let $\psi(k)$ be such that $|\psi(k)|$ is decreasing and satisfies* (22). *Then for all $f \in L^\psi_\beta L_p$, $1 \leqslant p < \infty$, we have a.e.*

$$\varrho_n(f; x) = \frac{\psi(n)}{\pi} \int_{-\pi}^{\pi} f^\psi_\beta(x+t) \cos(nt + \beta\pi/2)\, dt + \varrho_{n+1}(f; x), \qquad n \in N. \tag{23}$$

Moreover, for all $n \geqslant n_0 = \min\{k\colon |\psi(k)|^{1/k} < 1\}$

$$\|\varrho_{n+1}(f; x)\|_s \leqslant \frac{2\pi|\psi(n)|^{1+1/n}}{1-|\psi(n)|^{1/n}} E_n(f_\beta^\psi)_p, \qquad 1 \leqslant p, s \leqslant \infty,$$

where $E_n(\varphi)_p$ *is the best approximation in* L_p *of the function* φ *by trigonometric polynomials of order* $n-1$.

If $f \in C_\beta^\psi M = C_\beta^\psi L_\infty$, *then* (23) *holds at each point.*

If B is a function class contained in $L(0, 2\pi)$, we put

$$A_n(B)_s = \sup_{\varphi\in B} \Big\|\pi^{-1} \int_{-\pi}^{\pi} \varphi(x+t)\cos(nt+\beta\pi/2)\,dt\Big\|_s.$$

THEOREM 8. *Let* $\psi(k)$ *and* n_0 *satisfy the assumptions of Theorem* 7. *Then for all* $n \geqslant n_0$

$$\mathcal{E}_n(L_{\beta,p}^\psi)_s = \psi(n)\,[A_n(S_p)_s + \gamma_p(\psi; n)], \qquad 1 \leqslant p, s < \infty,$$

$$|\gamma_p(\psi; n)| \leqslant \frac{2\pi|\psi(n)|^{1/n}}{1-|\psi(n)|^{1/n}} \stackrel{\text{def}}{=} \gamma_n,$$

$$\mathcal{E}_n(L_\beta^\psi H_{\omega_p})_s = \psi(n)\,[A_n(H_{\omega_p})_s + \gamma_p(\psi; n; \omega)],$$

$$|\gamma_p(\psi; n; \omega)| \leqslant C_p \gamma_n \omega(1/n),$$

where C_p *is a quantity which depends on* p *only. In particular,*

$$\mathcal{E}_n(L_{\beta,1}^\psi)_1 = \psi(n)\big(1/\pi + \gamma_1(\psi; n)\big),$$

$$\mathcal{E}_n(L_\beta^\psi H_{\omega_1})_1 = \psi(n)\left(\frac{2\Theta_\omega}{\pi}\int_0^{\pi/2} \omega(2t/n)\sin t\,dt + \gamma_1(\psi; n; \omega)\right),$$

where $\Theta_\omega \in [1/2, 1]$ *and* $\Theta_\omega = 1$ *if* $\omega(t)$ *is a concave modulus of continuity,*

$$\mathcal{E}_n(C_{\beta,\infty}^\psi)_C = \psi(n)\big(4/\pi + \gamma_\infty(\psi; n)\big), \qquad |\gamma_\infty(\psi; n)| \leqslant \gamma_n,$$

$$\mathcal{E}_n(C_\beta^\psi H_\omega)_C = \psi(n)\left(\frac{2\Theta_\omega}{\pi}\int_0^{\pi/2} \omega(2t/n)\sin t\,dt + \gamma_\infty(\psi; n; \omega)\right),$$

$$|\gamma_\infty(\psi; n; \omega)| \leqslant K\gamma_n \omega(1/n),$$

where K *is an absolute constant.*

For all p, $s \geqslant 1$

$$A_n(S_p)_s = \|\cos x\|_s \sup_{\varphi\in S_p} \sqrt{a_n^2(\varphi) + b_n^2(\varphi)},$$

and if $p = 2$ then

$$\sup_{\varphi\in S_2} \sqrt{a_n^2(\varphi) + b_n^2(\varphi)} = \pi^{-1/2}.$$

This yields

COROLLARY. *If ψ and n_0 satisfy the assumptions of Theorem 7, then for all $s \geqslant 1$*

$$\mathscr{E}_n(L^{\psi}_{\beta,2})_s = \psi(n)\big(\|\cos x\|_s \pi^{-1/2} + \gamma_2(\psi; n)\big), \qquad |\gamma_2(\psi; n)| \leqslant \gamma_n.$$

In particular, for $s = 2$

$$\mathscr{E}_n(L^{\psi}_{\beta,2})_2 = \psi(n)\big(1 + \gamma_2(\psi; n)\big).$$

The results are most final in L_2 (see [16]).

THEOREM 9. *Suppose $|\psi(k)| \leqslant M = \text{const}$ and let $\nu(n) = \sup_{k \geqslant n} |\psi(k)|$. Then for all $n \in \boldsymbol{N}$*

$$\mathscr{E}_n(L^{\psi}_{\beta,2})_2 = E_n(L^{\psi}_{\beta,2})_2 = \nu(n).$$

THEOREM 10. *Let $f \in L^{\psi}_{\beta} L_2$. Then $f \in L_2$ if and only if the series*

$$\sum_{k=2}^{\infty} \big(\psi^2(k) - \psi^2(k-1)\big) E_k^2(f^{\psi}_{\beta})_2$$

is convergent. In the convergence case, for all $n \in \boldsymbol{N}$

$$E_n^2(f)_2 = \psi^2(n) E_n^2(f^{\psi}_{\beta})_2 + \sum_{k=n+1}^{\infty} \big(\psi^2(k) - \psi^2(k-1)\big) E_k^2(f^{\psi}_{\beta})_2. \tag{24}$$

On the other hand, if $f \in L_2$ then $f \in L^{\psi}_{\beta} L_2$ for all (or, equivalently, for some) $\beta \in \boldsymbol{R}$ if and only if the series

$$\sum_{k=2}^{\infty} \big(\psi^{-2}(k) - \psi^{-2}(k-1)\big) E_k^2(f)_2 \tag{25}$$

is convergent. In that case, for all $n \in \boldsymbol{N}$

$$E_n^2(f^{\psi}_{\beta})_2 = \psi^{-2}(n) E_n^2(f)_2 + \sum_{k=n+1}^{\infty} \big(\psi^{-2}(k) - \psi^{-2}(k-1)\big) E_k^2(f)_2.$$

Observe that the first part of the theorem enables us to draw conclusions about the rate of convergence of $E_n(f)_2$ to zero from the information about the (ψ, β)-derivative of f. Such assertions are usually called direct theorems of approximation theory. The second part is an inverse theorem — from the properties of $E_n(f)_2$ we derive some properties of the function itself and of its derivatives. In particular, (24) shows that $f \in L_2$ has (ψ, β)-derivative with finite L_2-norm if and only if the series (25) is convergent. It follows, in particular, that for all $f \in L_2$ the (ψ, β)-derivative for $\psi_1(k) = E_k(f)_2$ is in L_2 for no $\beta \in \boldsymbol{R}$.

References

[1] V. I. Berdyshev, *Approximation of functions in the mean by trigonometric polynomials*, Izv. Akad. Nauk SSSR Ser. Mat. 10 (1946), 207–256 (in Russian).

[2] A. G. Demchenko, *Approximation in the mean of functions from the classes* $W^r H_{[\omega]_L}$, Ukrain. Mat. Zh. 25 (1973), 262–271 (in Russian).

[3] V. K. Dzyadyk, *On the best approximation on the classes of periodic functions defined by integrals of linear combinations of absolutely monotonous kernels*, Mat. Zametki 16 (5) (1974), 691–701 (in Russian).

[4] A. V. Efimov, *Approximation of continuous periodic functions by Fourier sums*, Izv. Akad. Nauk SSSR Ser. Mat. 131 (2) (1960), 243–296 (in Russian).

[5] A. Kolmogoroff, *Zur Grössenordnung des Restgliedes Fourierschen Reihen differenzierbarer Funktionen*, Ann. of Math. (2) 36 (2) (1935), 521–526.

[6] B. Nagy, *Sur une classe générale de procédés de sommation pour les séries de Fourier*, Acta Math. Hungar. 1 (3) (1948), 14–62.

[7] S. M. Nikol'skii, *Approximation of periodic functions by trigonometric polynomials*, Trudy Mat. Inst. Steklov. 15 (1945), 1–76 (in Russian).

[8] —, *Approximation of functions in the mean by trigonometric polynomials*, Izv. Akad. Nauk SSSR Ser. Mat. 10 (1946), 207–256 (in Russian).

[9] V. T. Pinkevich, *On the order of the remainder term in the Fourier series of functions differentiable in Weyl's sense*, ibid. 4 (5) (1940), 521–528 (in Russian).

[10] S. B. Stechkin, *An estimate for the remainder of Fourier series of differentiable functions*, Trudy Mat. Inst. Steklov. 145 (1980), 126–151 (in Russian).

[11] S. B. Stechkin and S. A. Telyakovskii, *On approximation of differentiable functions by trigonometric polynomials in the L-norm*, ibid. 86 (1967), 20–29.

[12] A. I. Stepanets, *Classification of periodic functions and their approximation by Fourier sums*, preprint 83.69, Inst. Math., Acad. Sci. Ukrain. SSR, Kiev 1983, 57 pp. (in Russian).

[13] —, *Classes of periodic functions and approximation of their elements by Fourier sums*, Dokl. Akad. Nauk SSR 277 (1984), 1074–1077 (in Russian).

[14] —, *Deviations of Fourier sums on classes of infinitely differentiable functions*, Ukrain. Mat. Zh. 36 (6) (1984), 750–758 (in Russian).

[15] —, *Classification of periodic functions and rate of convergence of their Fourier series*, Izv. Akad. Nauk SSSR Ser. Mat. 50 (1) (1986), 101–136 (in Russian).

[16] A. I. Stepanets and A. K. Kushpel', *Best approximations and widths of classes of periodic functions*, preprint 84.15, Inst. Math., Acad. Sci. Ukrain. SSR, Kiev 1984, 44 pp. (in Russian).

[17] A. I. Stepanets and A. K. Novikova, *Approximation in the mean of periodic functions by Fourier sums*, Ukrain. Mat. Zh. 38 (2) (1986), 204–210 (in Russian).

[18] S. A. Telyakovskii, *On the norms of trigonometric polynomials and approximation of differentiable functions by linear means of their Fourier series*, Trudy Mat. Inst. Steklov. 62 (1961), 61–97 (in Russian).

Translated from the Russian
by Jerzy Trzeciak

Presented to the Semester
Approximation and Function Spaces
February 27–May 27, 1986