

## TWO THEOREMS ON CONVERGENCE OF FOURIER INTEGRALS AND FOURIER SERIES

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### Introduction

We shall first consider Fourier integrals in  $\mathbf{R}^n$  where  $n \geq 2$ .

Let  $\Phi \in C^1(\mathbf{R}^n \setminus \{0\})$  and assume that  $\Phi$  is positive and homogeneous of degree 1. Also let  $\Phi(0) = 0$ . We set

$$D = \{\xi \in \mathbf{R}^n; \Phi(\xi) < 1\}.$$

Let  $\hat{f}$  denote the Fourier transform of a function  $f$  in  $\mathbf{R}^n$ , defined by

$$\hat{f}(\xi) = \int_{\mathbf{R}^n} e^{-i\xi \cdot x} f(x) dx, \quad \xi \in \mathbf{R}^n.$$

For  $f \in L^2(\mathbf{R}^n)$  we also set

$$S_R f(x) = (2\pi)^{-n} \int_{RD} e^{i\xi \cdot x} \hat{f}(\xi) d\xi, \quad x \in \mathbf{R}^n, R > 0.$$

We shall prove the following theorem.

**THEOREM 1.** *Assume that  $f \in L^2(\mathbf{R}^n)$  and that  $\text{supp } f$  is a bounded set (where the support is taken in the sense of distribution theory). Then*

$$\lim_{R \rightarrow \infty} S_R f(x) = 0$$

*for almost every  $x$  in  $\mathbf{R}^n \setminus \text{supp } f$ .*

In the case when  $\Phi(\xi) = |\xi|$  Theorem 1 was essentially proved in P. Sjölin [4] and also in A. I. Bastis [1].

We shall also study the partial sums of Fourier series in one variable.

Set  $T = [0, 2\pi]$  and let  $S_n f$  denote the  $n$ th partial sum of the trigonometric Fourier series of a function  $f \in L^1(T)$ . Then set

$$Mf(x) = \sup_{n \geq 0} |S_n f(x)|, \quad x \in T.$$

The following results are then well known from R. A. Hunt [2] and P. Sjölin [3].

**THEOREM A.** *If  $f \in L \log L \log \log L(T)$  then*

- (1)  $S_n f(x)$  converges a.e.,
- (2)  $\|Mf\|_1 \leq C \int_T |f| (\log^+ |f|)^2 dx + C,$
- (3)  $\|Mf\|_p \leq C_p \|f\|_p, \quad 1 < p < \infty.$

We shall here prove an extension of the estimate (2) to the case when  $f \in L(\log L)^{1+\varepsilon}$ ,  $0 < \varepsilon < 1$ . Set  $\log^{(+)} y = 1$ ,  $0 \leq y \leq e$ , and  $\log^{(+)} y = \log y$ ,  $y > e$ . We then have the following inequality.

**THEOREM 2.** *If  $0 < \varepsilon < 1$  then*

- (4)  $\int_T Mf (\log^{(+)} Mf)^{\varepsilon-1} dx \leq C \int_T |f| (\log^+ |f|)^{1+\varepsilon} dx + C.$

We remark that the proof which gives Theorem 2 also yields the analogous result for Walsh–Fourier series.

### 1. Proof of Theorem 1

We shall need the following notation.

First set

$$K(x) = (2\pi)^{-n} \int_D e^{i\xi \cdot x} d\xi, \quad x \in \mathbb{R}^n,$$

and

$$K_R(x) = R^n K(Rx) \quad \text{for } R > 0.$$

We also let  $\chi$  denote the characteristic function of  $D$  and set  $\chi_R(\xi) = \chi(\xi/R)$ ,  $R > 0$ .

It is then clear that  $\hat{K} = \chi$ ,  $\hat{K}_R = \chi_R$  and that  $S_R f = K_R * f$  if  $f$  belongs to the Schwartz class  $\mathcal{S}$ . We now choose a function  $\varphi \in \mathcal{S}$  with the property that  $\varphi(0) = 0$  and set

$$T_R f = (\varphi K_R) * f$$

for  $f \in L^2(\mathbb{R}^n)$  and  $R > 0$ .

We shall also consider the maximal operator

$$T^* f = \sup_{R \geq 1} |T_R f|$$

and first prove the following lemma.

LEMMA 1. *If  $\varphi$  has the above properties then  $T^*$  is a bounded operator on  $L^2(\mathbb{R}^n)$ .*

*Proof.* We first assume that  $f \in \mathcal{S}$ . From the equality

$$(5) \quad (\widehat{T_R f}) = (\widehat{\varphi K_R}) \widehat{f} = (2\pi)^{-n} (\widehat{\varphi} * \chi_R) \widehat{f}$$

it follows that

$$(6) \quad T_R f(x) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \widehat{\varphi} * \chi_R(\xi) \widehat{f}(\xi) e^{i\xi \cdot x} d\xi$$

and

$$(7) \quad \frac{d}{dR} T_R f(x) = (2\pi)^{-2n} \int_{\mathbb{R}^n} \frac{d}{dR} (\widehat{\varphi} * \chi_R(\xi)) \widehat{f}(\xi) e^{i\xi \cdot x} d\xi, \quad R > 0.$$

We have

$$|T_R f|^2 \leq |(T_R f)^2 - (T_1 f)^2| + |T_1 f|^2$$

and

$$(T_R f)^2 - (T_1 f)^2 = \int_1^R \frac{d}{dt} (T_t f)^2 dt = \int_1^R 2T_t f \frac{d}{dt} T_t f dt.$$

Thus

$$|(T_R f)^2 - (T_1 f)^2| \leq 2 \int_1^R |T_t f| \left| \frac{d}{dt} T_t f \right| dt, \quad R \geq 1,$$

and hence

$$(8) \quad |T^* f|^2 \leq 2 \int_1^\infty |T_R f| \left| \frac{d}{dR} T_R f \right| dR + |T_1 f|^2 \leq 2(Gf)(Hf) + |T_1 f|^2,$$

where we have set

$$Gf(x) = \left( \int_1^\infty |T_R f(x)|^2 dR \right)^{1/2}, \quad x \in \mathbb{R}^n,$$

and

$$Hf(x) = \left( \int_1^\infty \left| \frac{d}{dR} T_R f(x) \right|^2 dR \right)^{1/2}, \quad x \in \mathbb{R}^n.$$

Applying the Cauchy-Schwarz inequality to the estimate (8) we obtain

$$\int_{\mathbb{R}^n} |T^* f|^2 dx \leq 2 \|Gf\|_2 \|Hf\|_2 + \int_{\mathbb{R}^n} |T_1 f|^2 dx.$$

$T_1$  is bounded on  $L^2$  since  $\varphi K_1 \in L^1$  and hence

$$\|T^* f\|_2 \leq 2 \|Gf\|_2^{1/2} \|Hf\|_2^{1/2} + C \|f\|_2.$$

To obtain the estimate

$$\|T^*f\|_2 \leq C\|f\|_2$$

it is therefore sufficient to prove that

$$(9) \quad \|Gf\|_2 \leq C\|f\|_2$$

and

$$(10) \quad \|Hf\|_2 \leq C\|f\|_2.$$

To estimate  $Gf$  and  $Hf$  we shall use the inequalities

$$(11) \quad |\hat{\varphi} * \chi_R(\xi)| \leq C \frac{1}{1 + |\Phi(\xi) - R|^N}, \quad \xi \neq 0, R \geq 1,$$

and

$$(12) \quad \left| \frac{d}{dR} (\hat{\varphi} * \chi_R)(\xi) \right| \leq C \frac{1}{1 + |\Phi(\xi) - R|^N}, \quad \xi \neq 0, R \geq 1,$$

where  $N$  denotes a large integer. We postpone the proofs of (11) and (12).

From (11) and (12) we conclude that

$$(13) \quad \int_1^\infty |\hat{\varphi} * \chi_R(\xi)|^2 dR \leq C, \quad \xi \neq 0,$$

Invoking the Plancherel theorem and the estimate (13) we then obtain

$$\begin{aligned} \int_{\mathbb{R}^n} |Gf(x)|^2 dx &= \int_{\mathbb{R}^n} \left( \int_1^\infty |T_R f(x)|^2 dR \right) dx = \int_1^\infty \left( \int_{\mathbb{R}^n} |T_R f(x)|^2 dx \right) dR \\ &= (2\pi)^{-n} \int_1^\infty \left( \int_{\mathbb{R}^n} |(T_R f)^\wedge(\xi)|^2 d\xi \right) dR \\ &= (2\pi)^{-3n} \int_1^\infty \left( \int_{\mathbb{R}^n} |(\hat{\varphi} * \chi_R)(\xi) \hat{f}(\xi)|^2 d\xi \right) dR \\ &= (2\pi)^{-3n} \int_{\mathbb{R}^n} \left( \int_1^\infty |\hat{\varphi} * \chi_R(\xi)|^2 dR \right) |\hat{f}(\xi)|^2 d\xi \\ &\leq C \int_{\mathbb{R}^n} |\hat{f}(\xi)|^2 d\xi = C \int_{\mathbb{R}^n} |f(x)|^2 dx \end{aligned}$$

and hence (9) follows.

In the same way (10) can be proved if we use the inequality (14).

It remains to prove (11) and (12) and we shall first prove (11). Since  $\varphi(0) = 0$  we can write

$$(15) \quad \hat{\varphi} * \chi_R(\xi) = \int \hat{\varphi}(\xi - \eta) \chi_R(\eta) d\eta = \int \hat{\varphi}(\xi - \eta) (\chi_R(\eta) - 1) d\eta.$$

If  $\xi \in \partial(RD)$  then  $\Phi(\xi) = R$  and

$$|\hat{\varphi} * \chi_R(\xi)| \leq \int |\hat{\varphi}(\eta)| d\eta = C,$$

which proves (11) in this case.

If  $\xi \in RD$  we conclude from (15) that

$$(16) \quad |\hat{\varphi} * \chi_R(\xi)| \leq \int_{\mathbb{R}^n \setminus RD} |\hat{\varphi}(\xi - \eta)| d\eta.$$

If  $\eta \in \partial(RD)$  then  $\Phi(\eta) = R$  and it follows from the mean value theorem that

$$|\Phi(\xi) - R| = |\Phi(\xi) - \Phi(\eta)| \leq C|\xi - \eta|,$$

since the derivatives  $D_i \Phi = \partial \Phi / \partial x_i$  are homogeneous of degree 0 and bounded in  $\mathbb{R}^n \setminus \{0\}$ . Hence

$$d(\xi, \partial(RD)) \geq c|\Phi(\xi) - R|,$$

where  $c$  is a positive constant and  $d(x, A)$  denotes the distance between a point  $x$  and a set  $A$ .

It follows that also

$$d(\xi, \mathbb{R}^n \setminus RD) \geq c|\Phi(\xi) - R|$$

and using (16) we then get

$$|\hat{\varphi} * \chi_R(\xi)| \leq \int_{|\xi - \eta| \geq c|\Phi(\xi) - R|} |\hat{\varphi}(\xi - \eta)| d\eta = \int_{|\eta| \geq c|\Phi(\xi) - R|} |\hat{\varphi}(\eta)| d\eta$$

and (11) follows in the case  $\xi \in RD$  since  $\hat{\varphi} \in \mathcal{S}$ .

We shall then study the case when  $\xi \in \mathbb{R}^n \setminus \overline{RD}$ . We use the estimate

$$|\hat{\varphi} * \chi_R(\xi)| \leq \int_{RD} |\hat{\varphi}(\xi - \eta)| d\eta.$$

In the same way as above one finds that

$$d(\xi, RD) \geq c|\Phi(\xi) - R|$$

and hence

$$|\hat{\varphi} * \chi_R(\xi)| \leq \int_{|\xi - \eta| \geq c|\Phi(\xi) - R|} |\hat{\varphi}(\xi - \eta)| d\eta = \int_{|\eta| \geq c|\Phi(\xi) - R|} |\hat{\varphi}(\eta)| d\eta,$$

and (11) follows.

We shall then prove (12). Performing a change of variable we find that

$$\hat{\varphi} * \chi_R(\xi) = \int \hat{\varphi}(\xi - \eta) \chi(\eta/R) d\eta = \int \hat{\varphi}(\xi - Ru) \chi(u) du R^n$$

and it follows that

$$\begin{aligned} \frac{d}{dR}(\hat{\varphi} * \chi_R)(\xi) &= \int \hat{\varphi}(\xi - Ru) \chi(u) du nR^{n-1} + \int \frac{d}{dR}(\hat{\varphi}(\xi - Ru)) \chi(u) du R^n \\ &= \int \hat{\varphi}(\xi - \eta) \chi_R(\eta) d\eta \frac{n}{R} - \sum_{j=1}^n \int D_j \hat{\varphi}(\xi - Ru) u_j \chi(u) du R^n. \end{aligned}$$

The desired estimate for the first term on the right-hand side follows from (11) and to estimate the second term it suffices to prove that

$$(17) \quad |B_j| \leq C \frac{1}{1 + |\Phi(\xi) - R|^N}, \quad \xi \neq 0, R \geq 1, j = 1, 2, \dots, n,$$

where

$$(18) \quad B_j = \int D_j \hat{\varphi}(\xi - Ru) u_j \chi(u) du R^n = \int D_j \hat{\varphi}(\xi - \eta) \frac{\eta_j}{R} \chi_R(\eta) d\eta.$$

We have  $\int D_j \hat{\varphi}(\eta) d\eta = 0$  and  $\int D_j \hat{\varphi}(\eta) \eta_j d\eta = 0$  since  $\varphi(0) = 0$  and we can therefore write

$$(19) \quad B_j = \int D_j \hat{\varphi}(\xi - \eta) \frac{\xi_j}{R} \chi_R(\eta) d\eta + \int D_j \hat{\varphi}(\xi - \eta) \frac{\eta_j - \xi_j}{R} \chi_R(\eta) d\eta \\ = \int D_j \hat{\varphi}(\xi - \eta) \frac{\xi_j}{R} (\chi_R(\eta) - 1) d\eta + \int D_j \hat{\varphi}(\xi - \eta) \frac{\eta_j - \xi_j}{R} (\chi_R(\eta) - 1) d\eta.$$

If  $\xi \in \partial(RD)$  then (17) is a consequence of (18) since  $|\eta_j/R| \leq C$  if  $\eta \in RD$ .

We shall then consider the case  $\xi \in RD$ . Using (19) one finds that

$$|B_j| \leq C \int_{\mathbb{R}^n \setminus RD} |D_j \hat{\varphi}(\xi - \eta)| d\eta + \int_{\mathbb{R}^n \setminus RD} |D_j \hat{\varphi}(\xi - \eta)| |\xi - \eta| d\eta \\ \leq C \int_{|\xi - \eta| \geq c|\Phi(\xi) - R|} |D_j \hat{\varphi}(\xi - \eta)| d\eta + \int_{|\xi - \eta| \geq c|\Phi(\xi) - R|} |D_j \hat{\varphi}(\xi - \eta)| |\xi - \eta| d\eta \\ \leq C \int_{|\eta| \geq c|\Phi(\xi) - R|} (|D_j \hat{\varphi}(\eta)| + |\eta| |D_j \hat{\varphi}(\eta)|) d\eta$$

and (17) follows since  $D_j \hat{\varphi}$  and  $\eta_k D_j \hat{\varphi}$  belong to the class  $\mathcal{S}$ .

It remains to prove (17) in the case  $\xi \in \mathbb{R}^n \setminus \overline{RD}$ . We use (18) and obtain

$$|B_j| \leq C \int_{RD} |D_j \hat{\varphi}(\xi - \eta)| d\eta \leq C \int_{|\xi - \eta| \geq c|\Phi(\xi) - R|} |D_j \hat{\varphi}(\xi - \eta)| d\eta \\ = C \int_{|\eta| \geq c|\Phi(\xi) - R|} |D_j \hat{\varphi}(\eta)| d\eta$$

and (17) follows also in this case.

We have now proved the inequality

$$\|T^*f\|_2 \leq C \|f\|_2$$

for  $f \in \mathcal{S}$ . The same inequality for  $f \in L^2$  then follows from approximation of  $f$  with functions in  $\mathcal{S}$  and an application of Fatou's lemma.

The proof of Lemma 1 is complete.

For  $f \in L^2(\mathbb{R}^n)$  we set  $S^*f = \sup_{R \geq 1} |S_R f|$ . Also let

$$B(x; r) = \{y \in \mathbb{R}^n; |y - x| < r\}.$$

The following lemma is then a consequence of Lemma 1.

LEMMA 2. Assume that  $0 < b < a < M$  and set  $\Omega = \{x \in \mathbb{R}^n; a < |x| < M\}$ . Then

$$(20) \quad \int_{B(0; b)} |S^*f(x)|^2 dx \leq C \int_{\mathbb{R}^n} |f(x)|^2 dx$$

for all  $f \in L^2(\mathbb{R}^n)$  with  $\text{supp } f \subset \bar{\Omega}$ .

Also  $\lim_{R \rightarrow \infty} S_R f(x) = 0$  for a.e.  $x \in B(0; a)$  if  $f$  is of the above type.

*Proof.* Assume that  $f \in L^2(\mathbb{R}^n)$  and  $\text{supp } f \subset \bar{\Omega}$ , i.e.  $f = 0$  a.e. outside  $\Omega$ . Choose  $\varphi \in C_0^\infty(\mathbb{R}^n)$  so that  $\varphi(x) = 1$  for  $a - b \leq |x| \leq M + b$  and  $\varphi(x) = 0$  if  $|x|$  is small. If  $|x| < b$  it then follows that

$$\begin{aligned} S_R f(x) &= K_R * f(x) = \int K_R(x - y) f(y) dy = \int \varphi(x - y) K_R(x - y) f(y) dy \\ &= (\varphi K_R) * f(x) = T_R f(x) \end{aligned}$$

(since  $a \leq |y| \leq M$  implies  $a - b \leq |x - y| \leq M + b$ ).

We therefore conclude from Lemma 1 that (20) holds.

To prove the remaining part of the lemma we assume that there exist a function  $f \in L^2(\mathbb{R}^n)$  with  $\text{supp } f \subset \bar{\Omega}$  and a number  $b$  with  $0 < b < a$  such that it is not true that

$$(21) \quad \lim_{R \rightarrow \infty} S_R f(x) = 0 \quad \text{for a.e. } x \in B(0; b).$$

Then there exist a positive number  $\delta$  and a set  $F \subset B(0; b)$  with Lebesgue measure  $mF = \delta$  such that

$$\overline{\lim}_{R \rightarrow \infty} |S_R f(x)| \geq \delta$$

for  $x \in F$ . We choose  $g \in C_0^\infty(\Omega)$  so that  $\|f - g\|_2$  is small and observe that  $x \in F$  implies

$$\delta \leq \overline{\lim}_{R \rightarrow \infty} |S_R f(x)| = \overline{\lim}_{R \rightarrow \infty} |S_R(f - g)(x)| \leq S^*(f - g)(x).$$

Using (20) one therefore finds that

$$\delta^3 = \int_F \delta^2 dx \leq \int_{B(0; b)} |S^*(f - g)|^2 dx \leq C \|f - g\|_2^2$$

and we obtain a contradiction if  $g$  is chosen suitably. Thus we have proved that if  $f \in L^2$ ,  $\text{supp } f \subset \bar{\Omega}$  and  $0 < b < a$  then (21) holds and the last statement in the lemma follows since  $b$  is arbitrary.

Theorem 1 is an easy consequence of Lemma 2.

*Proof of Theorem 1.* Let  $\mathcal{A}$  denote the set of all balls  $B(r; 1/k)$  in  $\mathbb{R}^n$  such that  $r$  has rational coordinates,  $k$  is a positive integer and  $f = 0$  a.e. in  $B(r; 1/k)$ . Hence  $\mathcal{A}$  is countable. If  $B \in \mathcal{A}$  it follows from Lemma 2 that  $\lim_{R \rightarrow \infty} S_R f(x) = 0$  for a.e.  $x \in B$ . If we set  $E = \bigcup_{B \in \mathcal{A}} B$  we can therefore conclude that  $\lim_{R \rightarrow \infty} S_R f(x) = 0$  for a.e.  $x$  in  $E$ .

The theorem then follows from the observation that  $E = \mathbb{R}^n \setminus \text{supp } f$ .

## 2. Proof of Theorem 2

We shall need the following notation. We say that  $f$  is a *special function* on  $T$  if  $f = g\chi_F$ , where  $g$  is a measurable function satisfying  $1/2 < g(x) \leq 1$  and  $\chi_F$  is the characteristic function of a measurable subset  $F$  of  $T$ . The following lemma is essentially proved in R. A. Hunt [2].

LEMMA 3. *If  $f$  is a special function,  $f = g\chi_F$ , then*

$$m\{x; Mf(x) > y\} \leq B_p^p y^{-p} mF, \quad 1 < p < \infty, y > 0,$$

where  $B_p = Cp^2/(p-1)$ .

*Proof.* Set

$$S_n^* f(x) = \int_0^{2\pi} \frac{e^{-in\theta} f(t)}{x-t} dt, \quad x \in T, n \in \mathbb{Z}, f \in L^1(T),$$

and

$$M^* f(x) = \sup_n |S_n^* f(x)|.$$

It was proved by R. A. Hunt [2] that

$$(22) \quad m\{x; M^* \chi_F(x) > y\} \leq B_p^p y^{-p} mF, \quad y > 0, 1 < p < \infty,$$

with  $B_p = Cp^2/(p-1)$ .

The proof in [2] shows that one may replace  $\chi_F$  in the above estimate with a special function  $f = g\chi_F$ . We have

$$Mf \leq CM^* f + C\|f\|_1$$

and hence if  $f = g\chi_F$  is a special function

$$(23) \quad Mf \leq CM^* f + CmF.$$

The lemma now follows from a combination of (22) with  $\chi_F$  replaced by  $f = g\chi_F$  and (23).



Arguing as in P. Sjölin [3], p. 563, we obtain from Lemma 3 the estimate

$$(24) \quad m\{x; Mf(x) > y\} \leq C \frac{1}{y} \log \frac{1}{y} mF, \quad 0 < y < 1/2,$$

if  $f$  is a special function.

We also obviously have the inequalities

$$(25) \quad m\{x; Mf(x) > y\} \leq 2\pi, \quad y > 0,$$

and

$$(26) \quad m\{x; Mf(x) > y\} \leq Cy^{-2} mF, \quad y > 0,$$

if  $f = g\chi_F$  is a special function. These three estimates will be used in the proof of Theorem 2.

*Proof of Theorem 2.* We shall first construct an auxiliary function  $\Phi$  on the interval  $[0, \infty[$ . Let  $\beta$  satisfy the inequality

$$(27) \quad \frac{1}{1+\varepsilon} = 1 - \frac{\varepsilon}{1+\varepsilon} < \beta < 1.$$

We want  $\Phi$  to have the following properties:

$$(28) \quad \Phi(x) = \begin{cases} x^\beta, & 0 \leq x \leq a, \\ x(\log x)^{\varepsilon-1}, & x \geq e^2, \end{cases}$$

where  $0 < a < e^2$ ,

$$(29) \quad \Phi \in C^1(]0, \infty[),$$

$$(30) \quad \Phi'(x) > 0, \quad x > 0,$$

and

$$(31) \quad \Phi' \text{ is decreasing on } ]0, \infty[.$$

First define  $\Phi$  on  $[0, a]$  and  $[e^2, \infty[$  by (28). For  $x > e^2$  we have

$$\Phi'(x) = (\varepsilon - 1)(\log x)^{\varepsilon-2} + (\log x)^{\varepsilon-1}$$

and

$$\Phi''(x) = (\varepsilon - 1)(\log x)^{\varepsilon-3} x^{-1} (\varepsilon - 2 + \log x).$$

It follows that  $\Phi''(x) < 0$  for  $x > e^2$  and hence  $\Phi'$  is decreasing on  $[e^2, \infty[$ .

We also have  $\Phi(e^2) = e^2 2^{\varepsilon-1}$  and  $\Phi'_+(e^2) = (1+\varepsilon)2^{\varepsilon-2}$ , where  $\Phi'_+$  denotes the right-hand derivative. Since  $0 < \varepsilon < 1$  it is easy to see that  $e^2 \Phi'_+(e^2) < \Phi(e^2)$ . Because of this we can choose  $a$  so small that

$$(32) \quad (e^2 - a) \Phi'_+(e^2) < \Phi(e^2) - \Phi(a) < (e^2 - a) \Phi'_-(a).$$

(since  $\Phi(x) \rightarrow 0$  and  $\Phi'(x) \rightarrow \infty$  as  $x \rightarrow 0$  from the right). We then choose a function  $\varphi$  continuous and decreasing on the interval  $[a, e^2]$  such that  $\varphi(a) = \Phi'(a)$  and  $\varphi(e^2) = \Phi'_+(e^2)$ . Because of (32) we may also assume that

$$\int_a^{e^2} \varphi(t) dt = \Phi(e^2) - \Phi(a).$$

Then set

$$\Phi(x) = \Phi(a) + \int_a^x \varphi(t) dt, \quad a < x < e^2.$$

$\Phi$  will then have the required properties and since  $\Phi$  is concave we also have

$$(33) \quad \Phi(x+y) \leq \Phi(x) + \Phi(y), \quad x, y \geq 0.$$

To prove (4) it is sufficient to prove the estimate

$$(34) \quad \int_T \Phi(Mf) dx \leq C \int_T |f| (\log^+ |f|)^{1+\varepsilon} dx + C$$

for  $f \in L(\log L)^{1+\varepsilon}(T)$ .

Without loss of generality we may assume that  $f \geq 0$  in the proof of (34). We set

$$F_k = \{x; 2^{k-1} < f(x) \leq 2^k\}$$

and

$$f_k(x) = \begin{cases} f(x), & x \in F_k, \\ 0, & x \notin F_k \end{cases}$$

for  $k \geq 5$ . Also set  $f_0(x) = f(x)$ ,  $x \in T \setminus \bigcup_{k=5}^{\infty} F_k$ , and  $f_0(x) = 0$  otherwise. It is then clear that  $f_0(x) \leq 16$  and  $f = f_0 + \sum_{k=5}^{\infty} f_k$ . We observe that

$$\int_T \Phi(Mf_0) dx \leq C + \int_T |Mf_0|^2 dx \leq C,$$

since  $M$  is bounded on  $L^2$ . We have

$$M\left(\sum_5^{\infty} f_k\right) < \sum_5^{\infty} Mf_k$$

and invoking (33) we obtain

$$\Phi\left(M\left(\sum_5^{\infty} f_k\right)\right) \leq \Phi\left(\sum_5^{\infty} Mf_k\right) \leq \sum_5^{\infty} \Phi(Mf_k).$$

We conclude that

$$(35) \quad \int_T \Phi(Mf) dx \leq \int_T \Phi(Mf_0 + M(\sum_5^\infty f_k)) dx \\ \leq \int_T \Phi(Mf_0) dx + \int_T \Phi(M(\sum_5^\infty f_k)) dx \leq C + \sum_5^\infty \int_T \Phi(Mf_k) dx.$$

We shall now estimate  $\int_T \Phi(Mf_k) dx$  for  $k \geq 5$ . We set

$$\lambda_k(y) = m\{x; Mf_k(x) > y\}, \quad y \geq 0,$$

and then have

$$(36) \quad \int_T \Phi(Mf_k) dx = - \int_0^\infty \Phi(y) d\lambda_k(y) = \int_0^\infty \Phi'(y) \lambda_k(y) dy.$$

Since  $2^{-k}f_k$  is a special function we conclude from the estimates (24)–(26) that

$$\lambda_k(y) = m\{x; M(2^{-k}f_k)(x) > 2^{-k}y\}$$

$$(37) \quad \begin{cases} 2\pi, & y \geq 0, \\ C \frac{2^k}{y} \log \frac{2^k}{y} mF_k, & 0 < y < 2^{k-1}, \\ C \frac{2^{2k}}{y^2} mF_k, & y > 0. \end{cases}$$

(38)

(39)

Using the properties of  $\Phi'$  we obtain

$$\int_T \Phi(Mf_k) dx \leq C \int_0^{e^2} y^{\theta-1} \lambda_k(y) dy + \int_{e^2}^\infty (\log y)^{\epsilon-1} \lambda_k(y) dy.$$

We may assume that  $mF_k > 0$  and set

$$\alpha_k = 2^k mF_k \log(4\pi/mF_k).$$

We first consider the case  $\alpha_k < e^2$  and invoke (37), (38) and (39) to obtain

$$\int_T \Phi(Mf_k) dx \leq C \int_0^{\alpha_k} y^{\theta-1} dy + C \int_{\alpha_k}^{e^2} y^{\theta-1} \frac{2^k}{y} \log \frac{2^k}{y} dy mF_k \\ + C \int_{e^2}^{2^{k-1}} (\log y)^{\epsilon-1} \frac{2^k}{y} \log \frac{2^k}{y} dy mF_k + C \int_{2^{k-1}}^\infty (\log y)^{\epsilon-1} \frac{2^{2k}}{y^2} dy mF_k \\ = a_k + b_k + c_k + d_k.$$

If  $\alpha_k \geq e^2$  we have with the same notation

$$\int_T \Phi(Mf_k) dx \leq a_k + c_k + d_k.$$

One finds that

$$a_k \leq C \cdot \alpha_k^\beta = C \cdot 2^{k\beta} (mF_k)^\beta \left( \log \frac{4\pi}{mF_k} \right)^\beta.$$

In estimating  $b_k$  we may assume that  $\alpha_k < e^2$  and performing a change of variable  $t = 2^{-k}y$  we get

$$\begin{aligned} b_k &= C \int_{mF_k \log(4\pi/mF_k)}^{2^{-k}e^2} (2^k t)^{\beta-1} \frac{1}{t} \log \frac{1}{t} dt \, 2^k mF_k \\ &= C \cdot 2^{k\beta} \int_{mF_k \log(4\pi/mF_k)}^{2^{-k}e^2} t^{\beta-2} \log \frac{1}{t} dt \, mF_k. \end{aligned}$$

Since

$$\frac{d}{dt} \left( -t^{\beta-1} \log \frac{1}{t} \right) = (1-\beta) t^{\beta-2} \log \frac{1}{t} + t^{\beta-2}$$

we conclude that

$$\begin{aligned} b_k &\leq C \cdot 2^{k\beta} \left( t^{\beta-1} \log \frac{1}{t} \right) \Big|_{t=mF_k \log(4\pi/mF_k)}^{mF_k} \\ &\leq C \cdot 2^{k\beta} \left( mF_k \log \frac{4\pi}{mF_k} \right)^{\beta-1} \log \frac{1}{mF_k \log(4\pi/mF_k)} mF_k \\ &\leq C \cdot 2^{k\beta} (mF_k)^\beta \left( \log \frac{4\pi}{mF_k} \right)^\beta. \end{aligned}$$

It remains to estimate  $c_k$  and  $d_k$ . Using the inequality

$$\log(2^k/y) \leq \log(2^k/e^2)$$

we obtain

$$\begin{aligned} c_k &\leq C \int_{e^2}^{2^{k-1}} (\log y)^{\varepsilon-1} y^{-1} dy \, 2^k k mF_k \\ &\leq C \cdot 2^k k [(\log y)^\varepsilon]_{e^2}^{2^{k-1}} mF_k \leq C \cdot 2^k k^{1+\varepsilon} mF_k. \end{aligned}$$

We finally have

$$\begin{aligned} d_k &\leq C \cdot 2^{2k} \int_{2^{k-1}}^{\infty} (\log y)^{\varepsilon-1} y^{-2} dy \, mF_k \leq C \cdot 2^{2k} \left[ -\frac{(\log y)^{\varepsilon-1}}{y} \right]_{2^{k-1}}^{\infty} mF_k \\ &\leq C \cdot 2^{2k} k^{\varepsilon-1} 2^{-k} mF_k = C \cdot 2^k k^{\varepsilon-1} mF_k. \end{aligned}$$

Hence

$$\int_I \Phi(Mf_k) dx \leq p_k + q_k,$$

where

$$p_k = C \cdot 2^{k\beta} (mF_k)^\beta \left( \log \frac{4\pi}{mF_k} \right)^\beta$$

and

$$q_k = C \cdot 2^k k^{1+\varepsilon} mF_k.$$

It follows that

$$\int_T \Phi(Mf) dx \leq C + \sum_5^\infty p_k + \sum_5^\infty q_k.$$

It is clear that

$$(40) \quad \sum_5^\infty q_k \leq C \int_T |f| (\log^+ |f|)^{1+\varepsilon} dx$$

and it remains to estimate  $\sum_5^\infty p_k$ .

We set  $\gamma = \varepsilon\beta$  and observe that since  $\beta > 1 - \varepsilon/(1 + \varepsilon)$  we have

$$1 - \beta < \varepsilon/(1 + \varepsilon)$$

and

$$\frac{\gamma}{1 - \beta} > \frac{\varepsilon(1/(1 + \varepsilon))}{\varepsilon/(1 + \varepsilon)} = 1.$$

Applying Hölder's inequality we obtain

$$\begin{aligned} \sum_5^\infty p_k &= C \sum_5^\infty 2^{k\beta} (mF_k)^\beta \left( \log \frac{4\pi}{mF_k} \right)^\beta k^\gamma k^{-\gamma} \\ &\leq C \left( \sum_5^\infty 2^k mF_k \log \frac{4\pi}{mF_k} k^{\gamma/\beta} \right)^\beta \left( \sum_5^\infty k^{-\gamma/(1-\beta)} \right)^{1-\beta} \\ &\leq C \left( \sum_5^\infty 2^k k^\varepsilon \log \frac{4\pi}{mF_k} mF_k \right)^\beta. \end{aligned}$$

If  $mF_k < 2^{-2k}$  then

$$\left( \log \frac{4\pi}{mF_k} \right) mF_k \leq (mF_k)^{2/3} < 2^{-(4/3)k}$$

and if  $mF_k \geq 2^{-2k}$  then

$$\frac{4\pi}{mF_k} \leq 4\pi 2^{2k} \quad \text{and} \quad \log \frac{4\pi}{mF_k} \leq Ck.$$

Hence

$$\sum_5^\infty p_k \leq C \left( C + \sum_5^\infty 2^k k^{1+\varepsilon} mF_k \right)^\beta \leq C + CI^\beta \leq C + CI,$$

where

$$I = \int_T |f| (\log^+ |f|)^{1+\varepsilon} dx.$$

A combination of this estimate and (40) now yields (34) and the proof of the theorem is complete.

### References

- [1] A. I. Bastis, *A generalized localization principle for the N-fold Fourier integral*, Soviet Math. Dokl. 30 (1984), 447-448.
- [2] R. A. Hunt, *On the convergence of Fourier series. Orthogonal expansions and their continuous analogues*, Proc. of the Conference held at Southern Illinois University, Edwardsville, 1967. SIU Press, Carbondale, Ill. 1968.
- [3] P. Sjölin, *An inequality of Paley and convergence a.e. of Walsh-Fourier series*, Ark. Mat. 7 (1968), 551-570.
- [4] —, *Regularity and integrability of spherical means*, Monatsh. Math. 96 (1983), 277-291.

*Presented to the Semester  
Approximation and Function Spaces  
February 27-May 27, 1986*

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