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Constants for lower bounds for linear forms in the logarithms of algebraic numbers I
The general case

by

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1. Introduction. In this note we further improve the lower bound for a non-homogeneous linear form in logarithms of non-zero algebraic numbers with algebraic coefficients. All the techniques are to be found in [W] and its companion [PW]; our sole contribution is to observe that the method of [W] which is essentially due to A. Baker [B1] yields better constants and to use the method to explicitly find the constants in [PW] for the general algebraic case. For an excellent account of the history of the subject, see [B2].

Our motivation is completely classical; i.e., to develop the theory to enable one to completely solve specific types of equations. The improvements that we obtain in the constants (by 2^{30} for the logarithms associated with the real quartic extension of the field of rational numbers having least discriminant, for example) ensure that Thue equations of small degree and coefficients over the ring of rational integers can be completely solved therein quite rapidly on a typical university computer (see [BGMS]). We therefore see this paper as providing the necessary results to facilitate practical solutions to number-theoretic problems associated with linear forms in logarithms of algebraic numbers with algebraic coefficients.

Specifically, let A be the field of algebraic numbers and let $\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n \in A$ with $\alpha_1, \ldots, \alpha_n$ non-zero. Let

$$\Lambda = \beta_0 + \beta_1 \log \alpha_1 + \ldots + \beta_n \log \alpha_n$$

and $D = [K:Q]$, where $K = Q(\alpha_1, \ldots, \alpha_n, \beta_0, \ldots, \beta_n)$. Let $h(\alpha)$ be the "ab-

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solute logarithmic height" of α in the sense of Mahler and Weil (see [L]). Define

$$V_1 = \max\{h(\alpha_1),\ 1/D,\ |\log\alpha_1|/D\},$$

$$V_{j+1} = \max\{h(\alpha_{j+1}),\ V_j,\ |\log\alpha_{j+1}|/D\} \qquad (1 \leqslant j \leqslant n-1).$$

Let

$$a_j = DV_j/|\log\alpha_j| \geqslant 1 \quad (1 \leqslant j \leqslant n) \quad \text{and} \quad \frac{1}{a} = \frac{1}{n}\sum_{j=1}^{n}\frac{1}{a_j}.$$

Let

$$V_j' = jV_j,\ V_0^+ = \bar{V}_0 = 1,\ V_j^+ = \max\{V_j, 1\},\ \bar{V}_j = \max\{V_j', 1\} \qquad (1 \leqslant j \leqslant n).$$

Let

$$W \geqslant h(\beta_j) \qquad (0 \leqslant j \leqslant n).$$

Let

$$\frac{1}{a_j} = \frac{1}{j}\sum_{i=1}^{j}\frac{1}{a_i} \quad (1 \leqslant j \leqslant n); \quad \text{so } a_j \geqslant 1 \quad (1 \leqslant j \leqslant n).$$

Let q be a prime number.

Let $E_2 = \min\{e^{qDV_1}, 2qa\}$. Note that $E_2 \geqslant 2q$. Let

$$M = 2(2^6 q^2 nDV_{n-1}^+ E_2)^n.$$

Assume that

$$W \geqslant \max\{n\log(nq^2 2^7 DV_n^+),\ \log E_2,\ (q/nD)\log E_2\}.$$

We first prove:

PROPOSITION. *If* $[K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K] = q^n$, *then* $\Lambda = 0$ *or*

$$|\Lambda| > \exp(-C(n)\, q^{3n}(q-1)\, D^{n+2}\, V_1 \ldots V_n\, W \log M/(\log E_2)^{n+1})$$

where $C(n) = (n^{2n+1}/n!)(3e^2)^n 2^{21}$.

Observe that $E_2 \geqslant E_1$ of [W] so the term to the nth power has been shrunk still further.

Let $\bar{E}_2 = \min\{e^{qDV_1}, 2qa\}$ and $\bar{M} = 2(2^6 q^2 nD\bar{V}_{n-1}\bar{E}_2)^n$. Let

$$x_n^* = \begin{cases} \log(2^{13}\bar{V}_1) & \text{if } n = D = 1, \\ n^2(n+1)\log(6n/\log D) + n(n+1)\log(n!) + \log n & \text{if } D \geqslant 2, \\ n^2(n+1)\log(9n) + n(n+1)\log(n!) + \log n & \text{if } D = 1 < n. \end{cases}$$

Let

$$W^* = \max\{nW + n^2(n+1)\log(D^3\bar{V}_n) + x_n^*,\ \log E_2,\ (q/nD)\log\bar{E}_2\}$$

and assume that $W^* \geqslant \log(2q)$. As noted on page 282 of [W], the worst bound for Λ occurs when $\{\log\alpha_1, \ldots, \log\alpha_n\}$ is linearly independent (over the rational integers) but not necessarily strongly independent with respect to q (i.e., $[K(\alpha_1^{1/q}, \ldots, \alpha_n^{1/q}) : K] < q^n$). As on page 281 of [W], using [W, Lemma 2.6], we may replace $\alpha_1, \ldots, \alpha_n$ and $\beta_0, \ldots, \beta_n$ by $\alpha_1', \ldots, \alpha_n', \beta_0', \ldots, \beta_n'$ to get a strongly independent set provided we use W^* in place of W and V_j' in place of V_j $(1 \leqslant j \leqslant n)$. Thus we obtain

THEOREM A. *If* $\Lambda \neq 0$, *then*

$$|\Lambda| > \exp\left\{-\hat{C}_1(n)\, D^{n+2}\frac{V_1 \ldots V_n}{(\log\bar{E}_2)^{n+1}}(\log\bar{M})(W + \hat{C}_2(n))\right\}$$

where

$$\hat{C}_1(n) = \begin{cases} n^{2n+1}(24e^2)^n 2^{20} & \text{if } n \geqslant 3, \\ n^{2n+1}(24e^2)^n 2^{21} & \text{if } n = 1, 2 \end{cases} \quad \text{and} \quad \hat{C}_2(n) = n(n+1)\log(D^3\bar{V}_n) + x_n^*/n.$$

Observe that $\bar{E}_2 \geqslant E$ of [W] so the term to the nth power has been shrunk still further. An alternative form of the Proposition and Theorem A will be given at the end of Section 2; although it is somewhat more complicated to state, it often leads to smaller constants in practical cases.

Let

$$E^* = \max\{2^{5n+4} q^{n+1} n^{2n} D\bar{E}_2^n,\ \bar{E}_2^{n^2}\}$$

and if $V_1 \geqslant n/D$, let

$$\hat{E}_2 = \min\{e^{W^*}, e^{2DV_1}, 4a\}.$$

THEOREM B. (i) *If* $\Lambda \neq 0$ *and* $n < 29$, *then*

$$|\Lambda| > \exp\left\{-\frac{\hat{C}_1(n)}{n} D^{n+2}\frac{V_1 \ldots V_n}{(\log\bar{E}_2)^{n+1}}(\log E^*)(W + \hat{C}_2(n))\right\}$$

where $\hat{C}_1(n)$ *and* $\hat{C}_2(n)$ *are as in Theorem* A.

(ii) *If* $V_1 \geqslant n/D$ *and* $\Lambda \neq 0$, *then*

$$|\Lambda| > \exp\left\{-\hat{C}_1(n)\, D^{n+2}\frac{V_1 \ldots V_n}{(\log\bar{E}_2)^{n+1}}(\log\hat{E}_2)(W + \hat{C}_2(n))\right\}$$

where $\hat{C}_1(n)$ *and* $\hat{C}_2(n)$ *are as above.*

Note that the assumptions in Theorem B(ii) could increase the value of $V_1 \ldots V_n$ by n^n, which would make the result far worse. However, if V_1 turns out to exceed n/D in a particular problem, Theorem B(ii) would apply whatever the value of n. From Theorems A and B(i) we obtain

COROLLARY. *If* $\Lambda \neq 0$ *and* $n < 29$, *then*

$$|\Lambda| > \exp\left\{-\hat{C}_1(n)\, D^{n+2}\frac{V_1 \ldots V_n}{(\log\bar{E}_2)^{n+1}}(W + \hat{C}_2(n))\min\{n\log(2^{6+1/n} nq^2 D\bar{V}_{n-1}\bar{E}_2),\ \max\{n\log\bar{E}_2,\ \log 2^{5+4/n} q^{1+1/n} n^2 D^{1/n}\bar{E}_2\}\}\right\}$$

where $\hat{C}_1(n)$ *and* $\hat{C}_2(n)$ *are given in Theorem* A.

2. The proof of the Proposition. Our proof follows that of [W]. We will number our equations (and use ′) throughout this section to mirror those of [W]; so, for example, (3.7) will refer to [W] but (3.7′) to an equation in this section. We will assume throughout that

$$c_0 \geq 2, \quad c_0' = (c_0+2^{-9})e^{1/256}, \quad c_0'' = (c_0+2^{-8})e^{1/256},$$
$$c_0'' c_4 c_3 \leq 2^{22}, \quad c_0'' c_4 \leq 2^{14}, \quad c_4 \geq 2^4,$$
$$2^3 \leq c_3 \leq 2^{12}, \quad 2, 2^4/n \leq c_2 \leq 2^7/e,$$
$$c_0 c_1 c_2^n c_3 c_4 \geq 2^{23+\upsilon}/(nq)^n, \tag{3.2′}$$

where

$$\upsilon = \begin{cases} 2 & \text{if } n = D = 1 \text{ and } q \in \{2, 3\}, \\ 1 & \text{if } n = 1 \text{ and } qD \in \{4, 5, 6\}, \\ 0 & \text{otherwise;} \end{cases}$$

$$c_1 = 2.5 \tag{3.3}$$

(e.g., $c_0 = 2^2$, $c_2 = 3e^2$, $c_3 = 2^9$, $c_4 = 2^9$ if $n \geq 3$).

The notation used is the same as in [W] unless otherwise stated. Indeed, rather than merely repeat the part of our argument that is the same as [W], we will only include here the parts where the arguments differ. The reader should therefore have a copy of [W] in front of him to refer to. The excrutiating yet straightforward verifications of both [W] and this section can be found in [BGMMS1], although the interested reader should have no trouble supplying these routine omitted details.

Proof of Proposition. Let

$$U_1 = 2^{22+\upsilon} n^2 q^{n+1} D^2 \max\{W, V_n^+, WV_n^+/\log E_2\}$$

and

$$U = c_0 c_1 c_2^n c_3 c_4 \frac{n^{2n+1}}{n!} q^{3n}(q-1) D^{n+2} \frac{V_1 \ldots V_n W \log M}{(\log E_2)^{n+1}}.$$

As in [W, page 264], we let

$$S = q[c_3 nDW/\log E_2], \qquad T = [U/c_1 c_3 q^n DW],$$
$$L_{-1} = [W/(\log E_2)^{n+1}], \qquad L_0 = [U/c_1 c_4 q^n D(L_{-1}+1)\log M],$$
$$L_j = [U/c_1 c_2 nq^{n+1} DSV_j] \qquad (1 \leq j \leq n).$$

Note that our conditions imply $L_1 \geq \ldots \geq L_n \geq 1$. We also assume that $\beta_n = -1$.

It is easy to see that

$$S \sum_{j=1}^{n} L_j V_j \leq \frac{U}{c_1 c_2 q^{n+1} D}, \tag{3.4}$$

$$E_2 S \sum_{j=1}^{n} L_j |\log \alpha_j| \leq \frac{2U}{c_1 c_2 q^n} \tag{3.5}$$

and

$$c_0\left(1-\frac{1}{q}\right) S \binom{T+n}{n} \leq \prod_{j=-1}^{n} (L_j+1). \tag{3.6}$$

We now improve [W (3.7)] with $M = 2(2^6 nq^2 DV_{n-1}^+ E_2)^n$ in place of $(2^{13} nq^2 DV_{n-1}^+ E_2)^n$. First assume that $n, D \geq 2$. Since $(\log x)/x \leq 1/e$ if $x \geq 1$,

$$\begin{aligned} \log M &= \log 2 + n(6\log 2 + \log n + 2\log q + \log D + \log V_{n-1}^+) + n\log E_2 \\ &\leq \log 2 + n(6\log 2 + (1/e)(n + 2q + D + V_{n-1}^+)) + n\log E_2 \\ &\leq \log 2 + n(6\log 2 + (nqDV_{n-1}^+/e)(\tfrac{1}{4}+\tfrac{1}{2}+\tfrac{1}{4}+\tfrac{1}{8})) + n\log E_2 \\ &\leq \log 2 + n(6\log 2 + (nqDV_{n-1}^+)(9/8e)) + n\log E_2 \\ &\leq 0.98n^2 qDV_{n-1}^+ + n\log E_2. \end{aligned}$$

Since $\log E_2 \geq \log 4$,

$$\log M \leq 0.98n^2 qDV_{n-1}^+ + n\log E_2 \leq 0.84n^2 qDV_{n-1}^+ \log E_2.$$

If $n = 1$ or $D = 1$, then it can be deleted from the expression for M. Hence we obtain

$$\log M \leq 1.78n^2 qV_{n-1}^+ + n\log E_2 \leq 1.54n^2 qV_{n-1}^+ \log E_2 \quad \text{if } n > 1 = D;$$
$$\log M \leq 1.78qD + \log E_2 \leq 1.54qD \log E_2 \quad \text{if } D > 1 = n;$$
$$\log M \leq 3.17q + \log E_2 \leq 2.79q \log E_2 \quad \text{if } n = D = 1.$$

Thus we get

$$\log M \leq Bn^2 qDV_{n-1}^+ + n\log E_1 \leq An^2 qDV_{n-1}^+ \log E_2 \tag{3.7′}$$

where

$$A = \begin{cases} 0.84 & \text{if } n, D \geq 2, \\ 2.79 & \text{if } n = 1 = D, \\ 1.54 & \text{otherwise;} \end{cases} \qquad B = \begin{cases} 0.98 & \text{if } n, D \geq 2, \\ 3.17 & \text{if } n = 1 = D, \\ 1.78 & \text{otherwise.} \end{cases}$$

It is now straightforward to verify from the definitions that

$$T \leq e^{(1+2/n)W}, \tag{3.9}$$

$$4E_2 qL_n \leq M^{1+1/n}, \tag{3.10′}$$

$$(3.12') \qquad (L_{-1}+1)\log M \leqslant \frac{U}{2^{22}q^n D}\left(\frac{1}{(\log 4)^n}+1\right)\left(B+\frac{1}{nqD}\right),$$

$$(3.13') \qquad L_1+L_n \leqslant 2L_1 \leqslant e^{(1+1/n)W},$$

$$(3.11') \qquad (L_{-1}+1)(L_0+1)\dots(L_n+1)D^2 U \leqslant \exp(U/2^{22}q^n D)$$

and

$$(3.14') \qquad 12q^n L_n E_2 S \leqslant (L_{-1}+1)M^{1+2/n}$$

using (3.7′) and (3.2′) and $\log E_2 \leqslant E_2/e$.

We also correct [W (3.15)]:

$$(3.15') \qquad T\log(L_1+L_n e^{DW}) \leqslant (1+D+1/n)U/c_1 c_3 q^n D.$$

We next determine f_1–f_7 appropriately. To ensure that (3.17) holds, using (3.9) and (3.14′) we see that

$$\log|\Delta(z+\lambda_{-1}; L_{-1}+1; \lambda_0+1; t)|$$
$$\leqslant TW\left(1+\frac{2}{n}\right)+\left(\frac{U}{c_1 c_4 q^n D\log M}+(L_{-1}+1)\right)\log 4e\left(\frac{M^{1+2/n}}{12}+1\right)$$

for $0 \leqslant \lambda_{-1} \leqslant L_{-1}$, $0 \leqslant \lambda_0 \leqslant L_0$, $0 \leqslant t \leqslant T$ and $|z| \leqslant 2q^n L_n E_2 S$.

But $M^{1+2/n} \geqslant 2^{33} \geqslant (12)2^{29}$. Hence (3.17) is true if

$$f_1 = \frac{1+\frac{2}{n}}{c_1 c_3}+\frac{1+\frac{2}{n}}{c_1 c_4}+\frac{\left(1+\frac{2}{n}\right)\left(\frac{1}{(\log 4)^n}+1\right)\left(B+\frac{1}{nqD}\right)}{2^{22}}.$$

Using (3.12′), $E_2 \geqslant 2q$ and $W \geqslant 9n\log 2$ we obtain (3.18) of [W] by the same method provided

$$f_2 = \left(\frac{\log 3}{(\log 4)^{n+1}}+\frac{\log 3}{9n\log 2}\right)\frac{1}{c_1 c_3}+\frac{2\left(1+\frac{1}{n}\right)}{c_1 c_4}+\frac{2\left(\frac{1}{(\log 4)^n}+1\right)\left(B+\frac{1}{nqD}\right)\left(1+\frac{1}{n}\right)}{2^{22}}.$$

Arguing as in [W, Lemma 3.2] we obtain

$$f_3 = \frac{1}{c_0-1}\left\{f_1+f_2+\frac{1}{qc_1 c_2}+\frac{2+1/n}{c_1 c_3}+\frac{1+2/nq}{2^{22}}\right\}+\frac{1}{2^{24}n}$$

and, as in [W, page 268], we obtain (3.20) provided that

$$f_3' = f_3+\frac{2D}{2^{22}n^2 q}.$$

The calculation in [W, Lemma 3.3] shows that we need only take

$$\frac{1}{qc_1 c_2}+q^{-n}\left(\frac{f_1}{D}+\frac{f_3'}{D}+\frac{1+1/n+D}{c_1 c_3 D}+\frac{1}{2^{22}D}\right) = f_4.$$

That $f_4 \leqslant 1/2$ follows easily once (3.1′a) is established (see below).

The calculation in [W, Lemma 3.4] shows that we can take

$$f_5 = f_1+f_2+f_3+\frac{1}{qc_1 c_2}+\frac{2+\frac{1}{n}}{c_1 c_3}+\frac{1+\frac{1}{n^2 q}}{2^{22}}.$$

However, the calculation there for f_6 can be improved. In the calculation of the upper bound on $-\log|d_{J+1,\tau,s/q}\varphi_{J,\tau}(s/q)|$, the term $U/c_1 c_2$ comes from bounding above

$$Dq^n \sum_{j=1}^{n} L_j Sqh(\alpha_j^{1/q}).$$

Since

$$h(\alpha_j^{1/q}) = \frac{1}{q}h(\alpha_j) \leqslant \frac{1}{q}V_j,$$

this term is bounded above by

$$Dq^n S\sum_{j=1}^{n} L_j V_j \leqslant \frac{U}{c_1 c_2 q} \quad \text{by (3.4).}$$

Hence the term $U/c_1 c_2$ can be replaced by the improvement $U/c_1 c_2 q$ in f_6. Thus we obtain

$$f_6 = f_1+f_2+f_3+\frac{1}{c_1 c_2 q}+\frac{2+\frac{1}{n}}{c_1 c_3}+\frac{1+\frac{1}{n^2 q}}{2^{22}} = f_5.$$

To calculate f_7 we first observe that it is trivial to improve [CW, Lemma 2] to:

$$|f|_{2r} < 2|f|_R\left(\frac{4r}{R}\right)^{kt}+\frac{10}{3}\left(\frac{18r}{\delta k}\right)^{kt}\max\left\{\frac{|f^{(\tau)}(x)|}{\tau!}: x\in E; 0\leqslant\tau\leqslant t-1\right\}$$

if $r < R/4$ since if $|z| = 2r$ and $\zeta\in\Gamma_x$, then $|\zeta-z| \geqslant 3r/4$ and $(k-1)! \geqslant (k/3)^k$ by induction.

Hence, as in the proof of [W, Lemma 3.5] we obtain

$$f_7 = \left(\frac{q-1}{2q}\right)\left(\frac{1}{c_1}-\frac{1}{c_1 c_3 q}-\frac{c_3}{2^{22}qn^2 D}\right)$$
$$-\left\{\frac{f_1}{Dq}+\frac{f_3}{Dq}+\frac{1+\frac{1}{D}+\frac{1}{nD}}{c_1 c_3 q}+\frac{4}{c_1 c_2}+\frac{1}{2^{22}Dq}\left(2+\frac{2D}{n^2}\right)\right\}.$$

(Note that there is a minor error in [W];

$$lt\log\left(\frac{18r}{\delta l}\right) \leqslant \left(\frac{q-1}{q}\right)q^k\left(\frac{ST}{2n}+SqL_n\right)\log\left(\frac{9q^2}{q-1}\right) < (0.3)U-\log 5.)$$

To obtain the desired contradiction, as in [W] we require $f_7 \geqslant f_6+1/2^{24}n$. This is implied by (and is equivalent to if $q=2$):

$$\begin{aligned}(3.1')\qquad 1 \geqslant\ & \frac{18}{c_2}+\frac{1}{c_3}\left(14.5+\frac{4}{D}+\frac{12}{n}+\frac{6}{nD}+\frac{4\log 3}{9n\log 2}+\frac{4\log 3}{(\log 4)^{n+1}}\right)\\ & +\frac{1}{c_4}\left(12+\frac{16}{n}+\frac{2}{D}+\frac{4}{nD}\right)+\frac{c_1}{2^{22}}\left\{\left(\frac{1}{(\log 4)^n}+1\right)\left(B+\frac{1}{2nD}\right)\left(12+\frac{16}{n}+\frac{2}{D}\left(1+\frac{2}{n}\right)\right)\right.\\ & \left.+4+\frac{2}{n}+\frac{8}{n^2}+\frac{1}{D}\left(4+\frac{1}{2n}+\frac{c_3}{n^2}\right)\right\}+\frac{\left(4+\frac{2}{D}\right)}{c_0-1}\left\{\frac{1}{2c_2}+\frac{3+\frac{3}{n}+\frac{\log 3}{9n\log 2}+\frac{\log 3}{(\log 4)^{n+1}}}{c_3}\right.\\ & \left.+\frac{3+\frac{4}{n}}{c_4}+\frac{c_1}{2^{22}}\left(1+\frac{1}{n}+\left(3+\frac{4}{n}\right)\left(B+\frac{1}{2nD}\right)\left(\frac{1}{(\log 4)^n}+1\right)\right)\right\}.\end{aligned}$$

Since $D \geqslant 1$, the above inequality holds for all D if

$$\begin{aligned}(3.1'\mathrm{a})\qquad 1 \geqslant\ & \frac{18}{c_2}+\frac{1}{c_3}\left(18.5+\frac{18}{n}+\frac{4\log 3}{9n\log 2}+\frac{4\log 3}{(\log 4)^{n+1}}\right)+\frac{1}{c_4}\left(14+\frac{20}{n}\right)\\ & +\frac{c_1}{2^{22}}\left(4+\frac{2}{n}+\frac{8}{n^2}+\frac{c_3}{n^2}+\left(B+\frac{1}{2n}\right)\left(14+\frac{20}{n}\right)\left(1+\frac{1}{(\log 4)^n}\right)\right)\\ & +\frac{1}{c_0-1}\left\{\frac{3}{c_2}+\left(18+\frac{18}{n}+\frac{6\log 3}{9n\log 2}+\frac{6\log 3}{(\log 4)^{n+1}}\right)\frac{1}{c_3}\right.\\ & \left.+\left(18+\frac{24}{n}\right)\frac{1}{c_4}+\frac{6c_1}{2^{22}}\left(1+\frac{1}{n}+\left(3+\frac{4}{n}\right)\left(1+\frac{1}{(\log 4)^n}\right)\left(B+\frac{1}{2n}\right)\right)\right\}.\end{aligned}$$

From this last inequality it is easy to see that $f_4 \leqslant 1/2$. Indeed (3.1′a) is almost the same as [W (3.1)].

To complete the proof of the Proposition, we proceed as in the proof of [W, Proposition 3.8] using

$$c_0=\begin{cases}5 & \text{if } n\geqslant 3,\\ 4 & \text{if } n=2,\\ 3 & \text{if } n=1,\end{cases}\qquad c_2=\begin{cases}3e^2 & \text{if } n\geqslant 4,\\ 2^5 & \text{if } n=1,2,3,\end{cases}\qquad c_4=\begin{cases}2^7 & \text{if } n=2,3,\\ 2^8 & \text{otherwise}\end{cases}$$

and

$$c_3=\begin{cases}533 & \text{if } n\geqslant 4,\\ 161 & \text{if } n=3,\\ 295 & \text{if } n=2,\\ 342 & \text{if } n=1;\end{cases}$$

but the maximum is clearly attained by $W\max\{V_n^+,\log E_2\}/\log E_2$. This establishes the Proposition. ∎

To prove Theorem A, we need only note that as in [W, §4], we must replace V_j by jV_j, W by W^*, E_2 by $\bar{E}_2$ and M by $\bar{M}$. However, for $n \geqslant 3$, we can show

$$(3.9')\qquad T \leqslant e^{W^*/(n+1)}$$

and

$$(3.13')\qquad 2L_1 \leqslant e^{W^*/(n+1)}.$$

Hence we obtain

$$\begin{aligned}(3.1^+)\qquad 1 \geqslant\ & \frac{18}{c_2}+\frac{1}{c_3}\left(6.5+\frac{8}{n+1}+\frac{4}{(n+1)D}+4\log 3\left(\frac{1}{(\log 4)^{n+1}}+\frac{1}{5n(n+1)}\right)\right)\\ & +\frac{1}{c_4}\left(12+\frac{16}{n}+\frac{2}{D}+\frac{4}{nD}\right)\\ & +\frac{c_1}{2^{22}}\left(4+\frac{2}{n}+\frac{4}{D}+\frac{1}{2nD}+\frac{8}{n^2}+\frac{c_3}{n^2D}\right.\\ & \left.+\left(12+\frac{16}{n}+\frac{2}{D}+\frac{4}{nD}\right)\left(1+\frac{1}{(\log 4)^n}\right)\left(B+\frac{1}{2nD}\right)\right)\\ & +\frac{\left(4+\frac{2}{D}\right)}{c_0-1}\left\{\frac{1}{2c_2}+\frac{1}{c_3}\left(1+\frac{2}{n+1}+\log 3\left(\frac{1}{(\log 4)^{n+1}}+\frac{1}{5n(n+1)}\right)\right)\right.\\ & \left.+\frac{3+\frac{4}{n}}{c_4}+\frac{c_1}{2^{22}}\left(\left(3+\frac{4}{n}\right)\left(B+\frac{1}{2n}\right)\left(1+\frac{1}{(\log 4)^n}\right)+1+\frac{1}{n}\right)\right\}.\end{aligned}$$

If $n=3$, this is satisfied by $c_0=5$, $c_2=2^5$, $c_3=67$ and $c_4=2^7$; if $n \geqslant 4$, this is satisfied by $c_0=5$, $c_2=3e^2$, $c_3=218$ and $c_4=2^8$. This establishes Theorem A. ∎

It should be noted that if $n \geqslant 28$ and we let $c_0=4$, $c_3=2^{10}$ and $c_4=2^{11}$, then by (3.1′) c_2 can be taken to be 19.8, assuming $q=2$ (and $c_2 \to 19.16$ approximately as $n \to \infty$). Thus we can replace $2^{8n+20}n^{2n+1}/e^n$ by $(19.8)^n 2^{25} n^{2n+1}$ if $n \geqslant 28$. Let $E_3=2q$. Then (3.5) becomes

$$E_3 S \sum_{j=1}^{n} L_j |\log \alpha_j| \leqslant 2U/c_1 c_2 q^n a.$$

The proofs of the Proposition and Theorem A go through with the modifications that

(i) $E_2^{\cdot}(\bar{E}_2)$ is replaced throughout by $2q$ in the Proposition (Theorem A), and

(ii) the first summand on the right-hand side of (3.1′) and (3.1⁺) is replaced by $\dfrac{2+(16/a)}{c_2}$ for the Proposition and Theorem A respectively.

If, for example, $a = 16$ and $n \geqslant 4$, $\hat{C}_1(n)$ in the Proposition can be replaced by $\dfrac{n^{2n+1}}{n!} 5^n 2^{21}$.

The same considerations hold equally for Theorem B.

Tables of computed values for $c = c_0'' c_1 c_2^n c_3 c_4$ where $q = 2$, $n \leqslant 10$ and $a = 1, 2, 4, 8, 16, 64$

So if $n = 1$, $D = 1$ and $a = 1$, $c = 1.7 \times 10^7 = 1.7\ \ 7$

$D = 1$

n	$a=1$	$a=2$	$a=4$	$a=8$	$a=16$	$a=64$
1	1.7 7	1.7 7	1.7 7	1.7 7	1.7 7	
2	3.1 8	1.1 8	4.3 7	2.2 7	1.4 7	9.6 6
3	3.4 9	6.5 8	1.7 8	5.7 7	2.8 7	1.5 7
4	8.4 10	9.3 9	1.5 9	3.5 8	1.3 8	4.8 7
5	2.1 12	1.3 11	1.3 10	2.0 9	5.7 8	1.6 8
6	4.8 13	1.8 12	1.1 11	1.2 10	2.4 9	5.1 8
7	1.1 15	2.3 13	8.1 11	6.0 10	9.9 9	1.7 9
8	2.5 16	2.9 14	6.3 12	3.2 11	3.9 10	4.8 9
9	5.5 17	3.6 15	4.8 13	1.7 12	1.5 11	1.4 10
10	1.3 19	4.5 16	3.7 14	8.3 12	5.7 11	4.0 10

$D = 2, 3$

n	$a=1$	$a=2$	$a=4$	$a=8$	$a=16$	$a=64$
1	1.1 7	8.4 6	8.4 6	8.4 6	8.4 6	
2	2.4 8	7.9 7	3.2 7	1.6 7	9.7 6	7.1 6
3	2.7 9	5.2 8	1.3 8	4.4 7	2.1 7	1.2 7
4	6.8 10	7.4 9	1.2 9	2.7 8	9.8 7	3.7 7
5	1.7 12	1.1 11	9.7 9	1.6 9	4.4 8	1.3 8
6	3.9 13	1.4 12	7.9 10	8.6 9	1.9 9	4.0 8
7	8.8 14	1.8 13	6.3 11	4.7 10	7.6 9	1.3 9
8	2.0 16	2.3 14	4.9 12	2.5 11	3.0 10	3.7 9
9	4.4 17	2.9 15	3.8 13	1.3 12	1.2 11	1.1 10
10	9.7 18	3.5 16	2.9 14	6.5 12	4.5 11	3.1 10

$4 \leqslant D \leqslant 9$

n	$a=1$	$a=2$	$a=4$	$a=8$	$a=16$	$a=64$
1	8.6 6	4.8 6	4.2 6	4.2 6	4.2 6	
2	2.0 8	6.7 7	2.7 7	1.4 7	8.2 6	6.0 6
3	2.4 9	4.6 8	1.2 8	3.9 7	1.9 7	9.7 6
4	6.0 10	6.5 9	9.9 8	2.4 8	8.5 7	3.2 7
5	1.5 12	9.0 10	8.4 9	1.4 9	3.8 8	1.1 8
6	3.4 13	1.2 12	6.9 10	7.5 9	1.6 9	3.4 8
7	7.8 14	1.6 13	5.5 11	4.1 10	6.6 9	1.2 9
8	1.8 16	2.0 14	4.3 12	2.2 11	2.6 10	3.2 9
9	3.9 17	2.5 15	3.3 13	1.2 12	9.9 10	9.2 9
10	8.6 18	3.1 16	2.5 14	5.6 12	3.9 11	2.7 10

$D \geqslant 10$

n	$a=1$	$a=2$	$a=4$	$a=8$	$a=16$	$a=64$
1	7.8 6	4.4 6	4.2 6	4.2 6	4.2 6	
2	1.9 8	6.0 7	2.4 7	1.2 7	7.4 6	5.4 6
3	2.2 9	4.2 8	1.1 8	3.5 7	1.7 7	8.9 6
4	5.5 10	6.0 9	9.1 8	2.2 8	7.8 7	2.9 7
5	1.4 12	8.2 10	7.7 9	1.3 9	3.5 8	9.6 7
6	3.2 13	1.1 12	6.4 10	6.8 9	1.5 9	3.1 8
7	7.2 14	1.5 13	5.1 11	3.7 10	6.1 9	1.1 9
8	1.7 16	1.9 14	3.9 12	2.0 11	2.4 10	2.9 9
9	3.6 17	2.3 15	3.0 13	1.1 12	9.0 10	8.4 9
10	7.9 18	2.9 16	2.3 14	5.1 12	3.5 11	2.5 10

3. The proof of Theorem B. As in Section 1, the result follows from the strongly independent case. So let W and E_2 be as defined there. We assume that (3.2′) and (3.3) hold but make no assumptions that $V_1 \geqslant n/D$ first of all. Then if we replace M by $E^* = \max\{2^{5n+4} q^{n+1} n^{2n} D\bar{E}_2^n, \bar{E}_2^{n^2}\}$, then a similar computation to (3.7′) yields

$$\log E^* \leqslant Bn^2 qD + n\log\bar{E}_2 \leqslant An^2 qD \log\bar{E}_2 \tag{3.7″}$$

where A and B are as in the previous section if $n, D \geqslant 2$; $A = 3.28$ and $B = 3.86$ if $n = D = 1$; and $A = 1.78$ and $B = 2.12$ otherwise. From this it follows easily that if $N_0 = 2^{5n+10} q^2 n^{2n+2}$, then

$$qN_0 DE_2^{n+1} \leqslant E^{*^{1+1/n}}, \tag{3.10″}$$

$$(L_{-1}+1)\log E^* \leqslant \left(B + \frac{1}{2nD}\right)\left(\frac{1}{(\log 4)^n} + 1\right)\frac{U}{2^{22} q^n D} \tag{3.12″}$$

and

$$8N_0 q^n SE_2 \leqslant (L_{-1}+1)E^{*^{1+2/n}}. \tag{3.14″}$$

Since $E^{*^{1+2/n}} \geqslant 2^{39}, f_1$–$f_7$ are defined as in the previous section. Thus we obtain (3.1″) and (3.1″a) which are identical to (3.1′) and (3.1′a) respectively, and (3.1⁺) if $n \geqslant 3$. Moreover, the values and tables of the previous section hold even with the new A and B.

If we assume $V_1 \geqslant n/D$ and replace $\bar{E}_2$ by $\hat{E}_2$ where $\hat{E}_2 = \min\{e^{W^*}, e^{qDV_1/n}, 2qa\}$, then we obtain Theorem B as in [PW, §4c]. If, on the other hand, we only assume that $n \geqslant 2$, $c_2 \geqslant 2^4$ and $c_3, c_4 \geqslant 2^7$ (but not $V_1 \geqslant n/D$), we can modify the argument in [PW, §4c] provided that $n < 29$. We follow the notation of [PW, §4c] except that we take $\bar{E}_2$ as above,

$$J_0 = [\log N_0/\log q] \quad \text{and} \quad J_1 = J_0 - [\log(20n^2)/\log q].$$

So

$$N_0 \geqslant q^{J_0} \geqslant 20n^2 q^{J_1} \geqslant 2^{5n+10} n^{2n+2}.$$

It easily follows that $T/\tilde{L}_j > 31.935$ and $TS/\tilde{L}_0 \geqslant 27.476$. Since $L_j^{(J_1)} \geqslant 20n^2$,

$$\prod_{j=1}^{n} (L_j^{(J_1)}+1) \leqslant \left(1+\frac{1}{20n^2}\right)^n \prod_{j=1}^{n} L_j^{(J_1)}.$$

For $i \in \{0, 1\}$

$$S_1\binom{T_1+\sigma_i}{\sigma_i} \geqslant \frac{ST^{\sigma_i}}{\sigma_i!(n+1)^{\sigma_i+1}} q^{(1-\sigma_i)J_1}\left(1-\frac{5}{21n^{2n}2^{5n+11}}\right)\left(1-\frac{n+2}{629n^4}\right)^{\sigma_i}$$

and

$$\binom{T_1+\sigma_i}{\sigma_i} \geqslant \frac{T^{\sigma_i} q^{-J_1\sigma_i}}{\sigma_i!(n+1)^{\sigma_i}}\left(1-\frac{n+2}{629n^4}\right)^{\sigma_i}.$$

Using (4.10) and (4.11) of [PW] together with the above and the equation

$$\frac{H(\mathscr{L}, L^{(J_1)})}{\prod_{j=1}^{n} L_j^{(J_1)}} \geqslant q^{J_1 r}\frac{H(\mathscr{L}, \tilde{L})}{\prod_{j=0}^{n} \tilde{L}_j}$$

of [PW, §4c] we get the desired contradiction to [PW, Lemma 3.6] provided that

$$\left(1-\frac{n+2}{629n^4}\right)^{r+1}\left(1-\frac{5}{21n^{2n}2^{5n+11}}\right)(31.935n^2)^r(27.476)n^3$$
$$> \left(1+\frac{1}{20n^2}\right)^n (n+1)^{r+2}\frac{(n+1)!}{(n-r)!}(r+1)!$$

for $0 \leqslant r \leqslant n$ and

$$\left(1-\frac{n+2}{629n^4}\right)^r (31.935n^2)^r > \left(1+\frac{1}{20n^2}\right)^n (n+1)^{r+1}\frac{(n+1)!}{(n-r)!}r!$$

for $1 \leqslant r \leqslant n$.

These hold for $n < 29$ as was verified by computer but fail if $n = 29$. This completes the proof of Theorem B. ■

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Constants for lower bounds for linear forms in the logarithms of algebraic numbers II
The homogeneous rational case

by

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1. Introduction. In this note we compute the constant for the lower bound of a homogeneous linear form in logarithms of non-zero algebraic numbers with rational coefficients. The constant we obtain improves that in [Wa]. Actually, we will derive the result from the special case when the rational coefficients are integers and a certain strong independence holds. In this paper, unlike the previous one, we only address the strongly independent case; although reduction to the strongly independent case can be done as in the previous paper (see Corollary 2 below), there may be cases when a reduction to strong independence is possible without increasing the bounds quite so much (see, e.g., [BS]). We will again follow [Wa] but with the modification given in [LMPW]; the reader will need to consult both papers since we will only give those steps in the proof which are different from those of [Wa] and [LMPW] (for more details, see [BGMMS1]). We will not bother to determine the constants of [PW, §5] since they are far greater ($c_5 \geqslant 2^n(n+1)^{n+2}n!$; since $c_0 \geqslant 1$,

$$c_0 c_1 c_2^n c_3 c_4 n^n/n! \geqslant 2^{n^2+2n}(n+1)^{n^2+5n+4}(n!)^{n+1}.$$

Let $\alpha_1, \ldots, \alpha_n$ be non-zero algebraic numbers, $K = \boldsymbol{Q}(\alpha_1, \ldots, \alpha_n)$ and $D = [K:\boldsymbol{Q}]$. Let

$$V_1 = \max\{h(\alpha_1), |\log\alpha_1|/D, 1/D\}$$

and

$$V_{j+1} = \max\{h(\alpha_{j+1}), |\log\alpha_{j+1}|/D, V_j\} \quad (1 \leqslant j \leqslant n-1),$$

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