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On Kronecker's limit formula for Dirichlet series with periodic coefficients

by

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1. Throughout the paper, we use $\boldsymbol{Q}$ for the field of all rational numbers. Let f be an arithmetical function with period N. The *Dirichlet series associated to* f is defined by

$$L(s, f) = \sum_{n=1}^{\infty} f(n)/n^s \quad (\mathrm{Re}(s) > 1).$$

The main aim of the paper is to show a limit formula for $L(s, f)$ at $s = 1$ in Theorem 5 of Section 3, and to give some of its applications after Section 4. We shall study periodic arithmetical functions, in particular, Dirichlet characters in Sections 2 and 4. Theorem 10 in Section 4 is a generalization of Hasse [10], Addendum due to M. Newman. In Theorem 11 in Section 5, we shall give four distinct representations of the values at $s = 1$ of Dirichlet L functions for every odd character modulo N, not necessarily primitive. When N is an odd prime, it is obvious that the numbers $\sin(2\pi r/N)$ (resp. $\cos(2\pi r/N)$) with $r = 1, \ldots, (N-1)/2$ are linearly independent over $\boldsymbol{Q}$. When N is a prime congruent to 3 modulo 4, Chowla [7] proved that so are $\cot(\pi r/N)$ if and only if all Dirichlet L functions for odd characters modulo N do not vanish at $s = 1$. By Hasse [10], Chowla's theorem can be rewritten in terms of "tan". Generalizations of these assertions will be established in Sections 7 and 8. When N is any odd number, we know that the numbers $\sin(2\pi r/N)$ (resp. $\cos(2\pi r/N)$) with $r = 1, \ldots, (N-1)/2$ and $(r, N) = 1$ are linearly independent over $\boldsymbol{Q}$ if and only if N is square free. The proof, using Gaussian sums, is elementary. On the other hand, the numbers $\cot(\pi r/N)$ (resp. $\tan(\pi r/N)$, $\sec(2\pi r/N)$) are always linearly independent over $\boldsymbol{Q}$. The proofs, which are given in Theorem 20 of Section 7 and Theorem 24 of Section 8, are analytical. Elementary proofs are known only in special cases. See Ayoub [3], Fujisaki [9], Hasse [10], Okada [18], and Wang [20]. Further, the numbers $\mathrm{cosec}(\pi r/N)$ are linearly independent over $\boldsymbol{Q}$ if and only if the multiplicative order of 2 mod N is even. This is a straightforward generalization of Jager and Lenstra [12] and will be proved in Theorem 21 of Section 7. In the case of odd prime powers N, this has been proved by Bundschuh [6].

2. The *Fourier transform* of an arithmetical function f with period N is defined by

$$f^{\wedge}(x)=\frac{1}{\sqrt{N}}\sum_{n=1}^{N} f(n)e(-nx/N),$$

where $e(x)=\exp(2\pi ix)$.

LEMMA 1. *Let f be any arithmetical function with period N. Then $f^{\wedge\wedge}(x)=f(-x)$ holds for any x.*

This lemma is called the *inversion formula*. If f has the two properties: (i) f induces a character $(\mathbf{Z}/N\mathbf{Z})^* \to \mathbf{C}^*$; (ii) $f(x)=0$ if and only if x is not prime to N, then f is said to be a *Dirichlet character modulo* N. The *principal character* χ_0 is defined by the formula $\chi_0(x)=1$ or 0 according as $(x, N)=1$ or not. A nonvanishing arithmetical function f is *completely multiplicative* if $f(xy)=f(x)f(y)$ for all x, y. We can easily show

LEMMA 2. *Let f be any arithmetical function with period N. Then f is completely multiplicative if and only if f is a Dirichlet character whose conductor is a divisor m of N prime to N/m.*

LEMMA 3. *Let f be any arithmetical function with period N. Then the primitive period of $f^{\wedge}$ is N if $f(a)\neq 0$ for some a prime to N.*

Proof. Let m be any period of $f^{\wedge}$. We set

$$g(x)=f(x)(e(xm/N)-1).$$

By assumption, we get $g^{\wedge}=0$, which is equivalent to $g=0$ by Lemma 1. Thus we have $e(am/N)=1$ for $(a, N)=1$ with $f(a)\neq 0$. Hence, N divides m.

It is well known that any primitive Dirichlet character modulo N satisfies $\chi^{\wedge}=\chi^{\wedge}(1)\bar{\chi}$, where $\bar{\chi}$ is the complex-conjugate function of χ. Apostol proved in an elementary way that the converse is also true. Joris [13] gave another proof, using the functional equation of the Dirichlet L function.

LEMMA 4. *Condition $\chi^{\wedge}=\chi^{\wedge}(1)\bar{\chi}$ characterizes primitive characters with conductor N in the class of all completely multiplicative functions with period N.*

Proof. We see $\chi^{\wedge}(1)\neq 0$ since $\chi^{\wedge}$ does not vanish identically. By Lemma 3, the primitive period of $\chi^{\wedge\wedge}$ is N. Lemma 2 combined with Lemma 1 asserts that χ is a Dirichlet character modulo N. The rest is completed by Apostol ([2], Theorem 1).

We can easily rewrite properties of $\chi^{\wedge}$ as properties of χ only in the primitive case. On the other hand, all non-principal characters modulo N are primitive if and only if N is prime. Therefore the composite cases are not parallel with the prime cases. For details, see Section 4 and later.

3. The *Hurwitz zeta function* for r/N is defined by

$$\zeta(s, r/N)=\sum_{n=0}^{\infty}(n+r/N)^{-s} \quad (\operatorname{Re}(s)>1).$$

Then $L(s,f)$ can be represented as

$$L(s,f)=\frac{1}{N^s}\sum_{r=1}^{N} f(r)\zeta(s, r/N).$$

Since $\zeta(s, r/N)$ can be continued analytically to the whole complex plane and is holomorphic except at $s=1$, the same holds for $L(s,f)$. We easily see that $L(s,f)$ is holomorphic at $s=1$ if and only if $f(1)+f(2)+\dots+f(N)=0$. Assuming that $L(s,f)$ is holomorphic at $s=1$, Lehmer [15] showed

$$L(1,f)=-\frac{1}{\sqrt{N}}\sum_{r=1}^{N-1} f^{\wedge}(r)\log(1-e(r/N)),$$

where the logarithms have their principal values between $-\pi/2$ and $\pi/2$. Livingston [16] gave another formula. We show the so-called Kronecker limit formula for $L(s,f)$ when this function is not necessarily holomorphic at $s=1$.

THEOREM 5.

$$\begin{aligned}\lim_{s\to 1}&\left(L(s,f)-\frac{f^{\wedge}(N)}{\sqrt{N}(s-1)}\right)\\ &=-\frac{1}{\sqrt{N}}\sum_{r=1}^{N-1} f^{\wedge}(r)\log(\sin(\pi r/N))+\frac{\pi}{2N}\sum_{r=1}^{N-1} f(r)\cot(\pi r/N)\\ &\qquad+\frac{f^{\wedge}(N)\gamma}{\sqrt{N}}+\left(\frac{f^{\wedge}(N)}{\sqrt{N}}-f(N)\right)\log 2,\end{aligned}$$

where γ is Euler's constant.

Proof. Our proof is based on Kronecker's limit formula for the Hurwitz zeta function, that is,

$$\lim_{s\to 1}\left(\zeta(s, a)-\frac{1}{s-1}\right)=-\psi(a),$$

where $\psi(a)=\Gamma'(a)/\Gamma(a)$ is called the *digamma function*.

Now we have

$$L(s,f)-\frac{f^{\wedge}(N)}{\sqrt{N}(s-1)}=\frac{1}{N^s}\sum_{r=1}^{N}\left\{\zeta(s, r/N)-\frac{1}{s-1}\right\}f(r)+\frac{N^{-s}-N^{-1}}{s-1}\sum_{r=1}^{N} f(r).$$

Therefore, as $s\to 1$, the right-hand side tends to

$$(1)\qquad -\frac{1}{N}\sum_{r=1}^{N} f(r)\psi(r/N)-\frac{f^{\wedge}(N)}{\sqrt{N}}\log N.$$

By Legendre's formula, the digamma function can be expressed in the form

$$\psi(a) = -\gamma + \int_0^1 \frac{t^{a-1}-1}{t-1}\,dt, \qquad \mathrm{Re}(a) > 0.$$

See Whittaker-Watson ([22], p. 260). Thus the first term of (1) reduces to

$$-\frac{1}{N}\int_0^1 \sum_{r=1}^{N} \frac{f(r)(t^{r/N-1}-1)}{t-1}\,dt + \frac{\gamma}{N}\sum_{r=1}^{N} f(r).$$

On writing t^N for t, this is equal to

$$-\int_0^1 \frac{F(t,f)}{t(t^N-1)}\,dt + \frac{\gamma f^{\wedge}(N)}{\sqrt{N}}, \tag{2}$$

where

$$F(t,f) = \sum_{r=1}^{N} f(r)(t^r - t^N).$$

Since $F(t,f)$ is a polynomial in t of degree at most N such that $F(0,f) = F(1,f) = 0$, we can use the idea of Dirichlet-Dedekind ([8], Section 185) for the Dirichlet L functions. So we see that

$$-\int_0^1 \frac{F(t,f)}{t(t^N-1)}\,dt = -\frac{1}{N}\sum_{m=1}^{N-1} F(e(-m/N),f)\log(1-e(m/N))$$

$$= -\frac{1}{\sqrt{N}}\sum_{m=1}^{N-1} f^{\wedge}(m)\log(1-e(m/N)) + \frac{1}{N}\sum_{r=1}^{N} f(r)\sum_{m=1}^{N-1}\log(1-e(m/N)).$$

Here

$$\sum_{r=1}^{N-1}\log(1-e(r/N)) = \sum_{r=1}^{N-1}\log(2\sin(\pi r/N)) = \log N$$

since $\prod_{r=1}^{N-1}\sin(\pi r/N) = N/2^{N-1}$.

We denote by g and h the even and odd parts of f respectively. Namely, $g(N-x) = g(x)$ and $h(N-x) = -h(x)$ for $x = 1, 2, \ldots, N$, and f is represented as $g+h$ uniquely. Then

$$\sum_{r=1}^{N-1} g^{\wedge}(r)\log(1-e(r/N)) = \sum_{r=1}^{N-1} g^{\wedge}(r)\log(\sin(\pi r/N)) + \sum_{r=1}^{N-1} g^{\wedge}(r)\log 2.$$

Here

$$\sum_{r=1}^{N-1} g^{\wedge}(r) = g(N)\sqrt{N} - g^{\wedge}(N).$$

Now we have

$$\sum_{r=1}^{N-1} h^{\wedge}(r)\log(1-e(r/N)) = \frac{\pi i}{N}\sum_{r=1}^{N-1} h^{\wedge}(r)r$$

$$= \frac{\pi i}{N\sqrt{N}}\sum_{t=1}^{N-1} h(t)\sum_{r=1}^{N-1} e(-rt/N)r$$

$$= -\frac{\pi}{2\sqrt{N}}\sum_{r=1}^{N-1} h(r)\cot(\pi r/N).$$

Further we have

$$\sum_{r=1}^{N-1} f^{\wedge}(r)\log(\sin(\pi r/N)) = \sum_{r=1}^{N-1} g^{\wedge}(r)\log(\sin(\pi r/N)),$$

$$\sum_{r=1}^{N-1} f(r)\cot(\pi r/N) = \sum_{r=1}^{N-1} h(r)\cot(\pi r/N),$$

and $f^{\wedge}(N) = g^{\wedge}(N)$, $f(N) = g(N)$. Summing up, we establish the theorem.

Theorem 5 immediately implies

COROLLARY 6. *Let χ be any non-principal Dirichlet character modulo N. Then*

$$(1-\chi(-1))L(1,\chi) = \frac{\pi}{N}\sum_{r=1}^{N}\chi(r)\cot(\pi r/N),$$

$$(1+\chi(-1))L(1,\chi^{\wedge}) = -\frac{2}{\sqrt{N}}\sum_{r=1}^{N}\chi(r)\log(\sin(\pi r/N)).$$

4. In order to give applications of Theorem 5, we need to prepare some equalities involving trigonometrical functions.

LEMMA 7.

$$\sum_{r=1}^{N-1} f^{\wedge}(r)(2r-N) = i\sqrt{N}\sum_{r=1}^{N-1} f(r)\cot(r\pi/N).$$

Proof. The left-hand side can be written as

$$\frac{1}{\sqrt{N}}\sum_{m=1}^{N} f(m)\sum_{r=1}^{N-1} e(-rm/N)(2r-N),$$

which is equal to the right-hand side.

LEMMA 8. *Whenever N is even, assume that $f(N/2) = 0$ and the term with $N/2$ is excepted from the right sum below. Then*

$$\sum_{r=1}^{N-1} f^{\wedge}(r)\{(-1)^r r + (-1)^{r-N}(r-N)\} = i\sqrt{N}\sum_{r=1}^{N-1} f(r)\tan(r\pi/N).$$

Proof. It is similar to Lemma 7.

LEMMA 9. *Let χ be any odd Dirichlet character modulo N. Then the following identities hold:*

$$A_\chi \sum_{r=1}^{N-1} \chi(r)\cot(r\pi/N) = B_\chi \sum_{r=1}^{N-1} \chi(r)\tan(r\pi/N),$$

$$\bar{A}_\chi \sum_{r=1}^{N-1} \chi^\wedge(r)\cot(r\pi/N) = \bar{B}_\chi \sum_{r=1}^{N-1} \chi^\wedge(r)\tan(r\pi/N),$$

where A_χ and B_χ are as follows:

	$N \equiv 1 \bmod 2$	$N \equiv 0 \bmod 4$	$N \equiv 2 \bmod 4$
A_χ:	$2\bar{\chi}(2)-1$	-1	-1
B_χ:	-1	$\chi(1+N/2)$	$2\bar{\chi}(2+N/2)-1.$

Proof. The first case follows from $2\cot 2x = \cot x - \tan x$. The second case follows from $\cot(\pi/2 - x) = \tan x$. The third case reduces to the first case, since there exists a Dirichlet character ξ modulo $N/2$ such that $\xi(r) = \chi(r)$ for all odd r.

This lemma leads to a generalization of the equality in Hasse ([10], Addendum due to M. Newman).

THEOREM 10. *Let N be any odd number and let χ be any Dirichlet character modulo N, not necessarily primitive. Then*

$$(1-2\chi(2)) \sum_{r=1}^{N-1} \chi(r) r = \chi(2) N \sum_{r=1}^{(N-1)/2} \chi(r).$$

Proof. If χ is even then both sides are equal to 0. Assuming that χ is odd, we easily see that

$$\sum_{r=1}^{N-1} \chi(r)(-1)^r = 2\chi(2) \sum_{r=1}^{(N-1)/2} \chi(r).$$

Therefore the theorem follows from the first case of Lemma 9 and Lemmas 7 and 8.

When N is even, the two sums in Theorem 10 vanish for a primitive character χ. For, we can assume $N \equiv 0 \bmod 4$, so that it is obvious that the right sum vanishes, and the left sum is equal to

$$(1+\chi(1+N/2)) \sum_{r=1}^{N/2} \chi(r) r = 0,$$

since the primitiveness of χ implies $\chi(1+N/2) = -1$.

5. We give some known and some not-so-known representations of the values of Dirichlet L functions at $s = 1$ for odd Dirichlet characters.

THEOREM 11. *Let χ be any odd Dirichlet character modulo N, not necessarily primitive. Then*

$$\begin{aligned} L(1, \chi) &= \frac{\pi}{N} \sum_{r=1}^{[N/2]} \chi(r)\cot(\pi r/N) \\ &= \frac{\pi}{iN\sqrt{N}} \sum_{r=1}^{N-1} \chi^\wedge(r) r \\ &= \frac{\pi B_\chi}{A_\chi N} \sum_{r=1}^{[N/2]} \chi(r)\tan(\pi r/N) \\ &= \frac{\pi B_\chi}{2iA_\chi N\sqrt{N}} \sum_{r=1}^{N-1} \chi^\wedge(r)\{(-1)^r r + (-1)^{r-N}(r-N)\}. \end{aligned}$$

Proof. The first formula is just Corollary 6. The second formula follows from Lemma 7. These formulas are well known. The third and fourth formulas are consequences of Lemma 9.

COROLLARY 12. *For any odd character χ,*

$$\frac{A_\chi + B_\chi}{B_\chi} L(1, \chi) = \frac{2\pi}{N} \sum_{r=1}^{[N/2]} \chi(r)\operatorname{cosec}(2\pi r/N).$$

Proof. Since $2\operatorname{cosec} 2x = \tan x + \cot x$, we just add the first and third formulas in Theorem 11 to obtain the corollary.

6. Okada ([19], Lemma) showed the following lemma on the Frobenius determinant.

LEMMA 13. *Let G be a finite abelian group and let H be a subgroup of G. Let λ be a character of H and let Δ be the set of all characters of G whose restriction to H is equal to λ. Then for any complex-valued function f on G with $f(ah) = \lambda(h) f(a)$ $(a \in G,\ h \in H)$, we have*

$$\det_{a,b \in T} f(a^{-1} b) = \prod_{\chi \in \Delta} \Big(\sum_{a \in T} \bar{\chi}(a) f(a)\Big),$$

where T is a complete representative system of G by H.

We denote by χ^* the primitive character corresponding to a Dirichlet character χ. Let p be any prime and let N' be a positive integer such that $N = N' p^a$ $(a \geqslant 0)$ and $(N', p) = 1$. We define $M(p)$ and $L(p)$ as follows: if $N' \neq 1$ then

$$M(p) = \min\{m > 0 \mid p^m \equiv 1 \bmod N'\} \quad \text{and} \quad L(p) = \varphi(N')/M(p);$$

if $N' = 1$ then both are 0. We note that if $N' \neq 1$ then $M(p)$ is called the *multiplicative order of p modulo N'*. This coincides with the residue class degree of p as an ideal in the Nth cyclotomic field K. The number $L(p)$ coincides with the number of prime ideals lying above p in K.

LEMMA 14.

$$\prod_{\chi \text{ odd}} (1-\chi^*(p)X) = \begin{cases} (1+X^{M(p)/2})^{L(p)} & \text{if } M(p) \text{ is even,} \\ (1-X^{M(p)})^{L(p)/2} & \text{if } M(p) \text{ is odd.} \end{cases}$$

Proof. The lemma follows from

$$\prod_{\chi \text{ odd}} (1-\chi^*(p)X) \prod_{\chi \text{ even}} (1-\chi^*(p)X) = (1-X^{M(p)})^{L(p)}.$$

For the product over even characters is $(1-X^{M(p)/2})^{L(p)}$ or $(1-X^{M(p)})^{L(p)/2}$ according as $M(p)$ is even or odd.

Using Lemma 14, we explicitly calculate the determinants of trigonometric values partially shown by Ayoub [3], Okada [18] and others. From now on, we use φ for the Euler totient function.

LEMMA 15. *Set*

$$D_N = \prod_{\chi \text{ odd}} L(1, \chi), \qquad n = \varphi(N)/2.$$

Then we have

$$\det(\cot(ab\pi/N)) = \pm(N/\pi)^n D_N,$$

$$\det(\tan(ab\pi/N)) = \pm(N/\pi)^n D_N E_N,$$

$$\det(\operatorname{cosec}(2ab\pi/N)) = \pm(N/\pi)^n D_N F_N,$$

where a as row and b as column run over all positive integers prime to N and less than N/2. The constants E_N and F_N are given as follows:

E_N	F_N	
1	2^n	*if* $N \equiv 0 \bmod 4$
$(2^{M/2}+1)^L$	2^{L+n}	*if* $N \equiv 1 \bmod 2,\ M \equiv 0 \bmod 2$
$(2^M-1)^{L/2}$	0	*if* $N \equiv 1 \bmod 2,\ M \equiv 1 \bmod 2$
$(2^{M/2}+1)^{-L}$	$(2^{M/2+1}/(1+2^{M/2}))^L$	*if* $N \equiv 2 \bmod 4,\ M \equiv 0 \bmod 2$
$(2^M-1)^{-L/2}$	0	*if* $N \equiv 2 \bmod 4,\ M \equiv 1 \bmod 2,$

where $M = M(2)$ and $L = L(2)$.

Proof. It follows from Theorem 11, Corollary 12 and Lemma 13 that

$$E_N = \pm \prod_{\chi \text{ odd}} A_\chi/B_\chi, \qquad F_N = \pm \prod_{\chi \text{ odd}} (A_\chi + B_\chi)/B_\chi.$$

Since $A_\chi = A_{\chi^*}$ and $B_\chi = B_{\chi^*}$ in any case, the number E_N is immediately determined from Lemma 14. Taking a prime p such that $p \equiv 1 + N/2 \bmod N$, we easily see $M(p) = 2$ and $L(p) = \varphi(N)$ if $N \equiv 0 \bmod 4$, so that we also determine F_N.

7. Let χ^* be the primitive character with conductor f_χ corresponding to a Dirichlet character modulo N. Then from Hasse [11], Kimura [14] and Washington [21] for example, we know the following formula:

$$L(s, \chi) = L(s, \chi^*) \prod_{p|N} (1-\chi^*(p)p^{-s}).$$

Let K^+ be the maximal real subfield of the Nth cyclotomic field K. Let h^- be the quotient of their class numbers, $h(K)/h(K^+)$. The discriminant $d(K)$ of K is

$$\pm N^{\varphi(N)} / \prod_{p|N} p^{\varphi(N)/(p-1)}.$$

By the conductor-discriminant formula, we get

$$\prod_{\chi \text{ odd}} f_\chi = \begin{cases} \sqrt{d_K} & \text{if } N \text{ is not a prime power,} \\ \sqrt{d_K/p} & \text{if } N \text{ is an odd prime power,} \\ \sqrt{d_K/4} & \text{if } N \text{ is an even prime power,} \end{cases}$$

which is denoted by d^-. By the class number formula, we have

$$\prod_{\chi \text{ odd}} L(1, \chi^*) = \frac{(2\pi)^{\varphi(N)/2} h^-}{Qw\sqrt{d^-}},$$

where $Q = 1$ or 2 according as N is a prime power or not and where $w = N$ or $2N$ according as N is even or odd.

LEMMA 16. *Let χ be any odd character modulo N. Then*

$$\sum_{m=1}^{[N/2]} \chi(m)(2m-N) = \frac{c(\chi)N}{i\pi} \prod_{p|N} (1-\chi^*(p)) L(1, \bar{\chi}^*),$$

where

$$c(\chi) = \sum_{r=1}^{f_\chi} \chi^*(r) e(-r/f_\chi).$$

Proof. Let $p, q, \ldots$ be the primes dividing N but not dividing f_χ. The left-hand side is equal to

$$\sum_{m=1}^{N} \chi(m)m = \sum_{\substack{m=1 \\ (m,N)=1}}^{N} \chi^*(m)m$$

$$= \sum_{m=1}^{N} \chi^*(m)m - \sum_{p} \chi^*(p)p \sum_{m=1}^{N/p} \chi^*(m)m$$

$$+ \sum_{p \neq q} \chi^*(pq)pq \sum_{m=1}^{N/pq} \chi^*(m)m - \ldots$$

$$= \frac{N}{f_\chi} \sum_{m=1}^{f_\chi} \chi^*(m)m - \sum_p \chi^*(p) p \frac{N}{pf_\chi} \sum_{m=1}^{f_\chi} \chi^*(m)m$$

$$+ \sum_{p \neq q} \chi^*(pq) pq \frac{N}{pqf_\chi} \sum_{m=1}^{f_\chi} \chi^*(m)m - \dots$$

$$= \frac{N}{f_\chi} \prod_p (1-\chi^*(p)) \sum_{m=1}^{f_\chi} \chi^*(m)m.$$

By the second formula of Theorem 11, we get

$$\sum_{m=1}^{f_\chi} \chi^*(m)m = \frac{c(\chi) f_\chi}{i\pi} L(1, \bar{\chi}^*).$$

This completes the proof.

LEMMA 17. *Let χ be an odd Dirichlet character modulo N. If N is odd, then*

$$\sum_{m=1}^{(N-1)/2} \chi(m)(-1)^m = \frac{c(\chi) A_\chi}{i\pi B_\chi} \prod_{p|N} (1-\chi^*(p)) L(1, \bar{\chi}^*),$$

where $c(\chi)$ is as in Lemma 16.

Proof. It is similar to Lemma 16.

LEMMA 18. *For positive integers a, b prime to N and less than $N/2$, we define an integer $c(a, b)$ by $a \equiv c(a, b) b \bmod N$ and $1 \leqslant c(a, b) \leqslant N-1$. Set*

$$\delta(N) = \prod_{p|N} (1+(-1)^{M(p)})^{L(p)}.$$

Denote $\varphi(N)/2$ by n. Then we have

$$\det(2c(a, b)-N) = \pm\delta(N)(2\pi)^n h^-/Qw,$$

where a as row and b as column run over all positive integers prime to N and less than $N/2$. If N is odd, then we, further, have

$$\det((-1)^{c(a,b)}) = \pm\delta(N) 2^n h^- E_N/Qw.$$

Proof. This follows from Lemmas 16 and 17.

Under the assumptions and the notations of Lemma 13, we can show

LEMMA 19. *Assume that for every $b \in T$ there exists an automorphism σ of the field of all algebraic numbers over $\boldsymbol{Q}$ such that $\sigma(f(a)) = f(ab)$ for all $a \in T$. Then the determinant given in Lemma 13 does not vanish if and only if the values $f(a)$ $(a \in T)$ are linearly independent over $\boldsymbol{Q}$.*

Here we give a generalization of Fujisaki [9].

THEOREM 20-I. *For positive integers N, the following assertions are equivalent:*

(I-0) *$L(1, \chi) \neq 0$ for all odd Dirichlet characters.*

(I-1) *The numbers $\cot(r\pi/N)$ with $r = 1, \dots, [N/2]$ and $(r, N) = 1$ are linearly independent over $\boldsymbol{Q}$.*

(I-2) *The numbers $\tan(r\pi/N)$ with $r = 1, \dots, [N/2]$ and $(r, N) = 1$ are linearly independent over $\boldsymbol{Q}$.*

(I-3) *For every divisor d of N, the number $\sin(2\pi/d)$ can be represented as a linear combination with rational coefficients of the numbers $\cot(r\pi/N)$ with $r = 1, \dots, [N/2]$ and $(r, N) = 1$.*

(I-4) *For every divisor d of N, the number $\sin(2\pi/d)$ can be represented as a linear combination with rational coefficients of the numbers $\tan(r\pi/N)$ with $r = 1, \dots, [N/2]$ and $(r, N) = 1$.*

Proof. The equivalence of (I-0), (I-1), and (I-2) follows from Theorem 11 and Lemmas 13, 15, and 19.

Let K be the Nth cyclotomic field. Put $K^\pm = \{x \in K; \bar{x} = \pm x\}$. We note that K^+ is the maximal real subfield but K^- is not a field. $K^+ + K^- = K$ as vector spaces over $\boldsymbol{Q}$ and the dimensions of K^+ and K^- are $\varphi(N)/2$. If (I-1) holds then the numbers

$$i \cot\frac{r\pi}{N} = \frac{1+e(r/N)}{1-e(r/N)} \quad \text{for } 1 \leqslant r \leqslant [N/2], \quad (r, N) = 1,$$

are a basis of K^-. Since $i\sin(2\pi/d)$ belongs to K^-, (I-3) holds. For every positive integer t prime to N and less than $N/2$, there exists an automorphism σ of K such that $\sigma(e(r/N)) = e(tr/N)$ for all r. Conversely, if (I-3) holds then for all r, the numbers $\sin(2\pi r/N)$ can be represented as a linear combination of the numbers $\cot(r\pi/N)$. Since the numbers $i\sin(2\pi r/N)$ span K^-, (I-1) holds. Hence (I-3) is equivalent to (I-1). Similarly, (I-4) is equivalent to (I-2). Now the proof is complete.

THEOREM 20-II. *Assume that one of the assertions in Theorem 20-I holds. Then the following assertions are equivalent:*

(II-1) *All $M(p)$ with $p|N$ are even, where*

$$M(p) = \begin{cases} 0 & \text{if } N \text{ is a power of } p, \\ \min\{m > 0 |\ p^m \equiv 1 \bmod N'\} & \text{otherwise,} \end{cases}$$

for $N' = N/p^a$ with $p^a \| N$.

(II-2) *For any odd Dirichlet character χ modulo N,*

$$\sum_{a=1}^{N-1} \chi(a) a \neq 0.$$

(II-3) *The square matrix $(2c(a, b)-N)$ with $\varphi(N)/2$ rows is regular, where notations are the same as in Lemma 18.*

Assume further that N is odd. Then the above assertions are also equivalent to either of the following assertions:

(II-4) *For any odd Dirichlet character* χ *modulo* N,

$$\sum_{a=1}^{N-1} \chi(a)(-1)^a \neq 0.$$

(II-5) *The square matrix* $((-1)^{c(a,b)})$ *with* $\varphi(N)/2$ *rows is regular.*

Proof. This follows from Theorem 20-I and Lemmas 13–19.

THEOREM 20-III. *Assume that* (II-1) *in Theorem* 20-II *holds. Then the remaining claims* (II-2)–(II-5) *are equivalent to the assertions in Theorem* 20-I.

Proof. This follows from Theorem 20-II and Lemma 18.

We remark that if N is a prime power then (II-1) is true and that if N has two distinct prime divisors p and q with $q \equiv 1 \bmod p$ then (II-1) is false. We have

$$\sqrt{N}\chi^{\wedge}(1) = \begin{cases} \displaystyle\sum_{r=1}^{N-1} \chi(r)\cos\frac{2r\pi}{N} & \text{if } \chi \text{ is even,} \\ \displaystyle i\sum_{r=1}^{N-1} \chi(r)\sin\frac{2r\pi}{N} & \text{if } \chi \text{ is odd.} \end{cases}$$

All Dirichlet characters χ modulo N satisfy $\chi^{\wedge}(1) \neq 0$ if and only if N is square free. Therefore the numbers $\sin(2r\pi/N)$ (resp. $\cos(2r\pi/N)$) with $r = 1, \ldots, [N/2]$ and $(r, N) = 1$ are linearly independent over $\boldsymbol{Q}$ if and only if N is square-free. Therefore we cannot replace "For every divisor d of N, the number $\sin(2\pi/d)$" by "the number $\sin(2\pi/N)$" in (I-3) and (I-4) of Theorem 20-I. As a generalization of Jager and Lenstra [12], we state

THEOREM 21. *Assume that one of the assertions in Theorem* 20-I *holds. In order that the numbers* $\operatorname{cosec}(2r\pi/N)$ *with* $r = 1, \ldots, [N/2]$ *and* $(r, N) = 1$ *are linearly independent over* $\boldsymbol{Q}$, *it is necessary and sufficient that one of the following conditions holds*: (i) $N \equiv 0 \bmod 4$; (ii) $N \equiv 2 \bmod 4$ *and the multiplicative order of* 2 *modulo* $N/2$ *is even*; (iii) $N \equiv 1, 3 \bmod 4$ *and the multiplicative order of* 2 *modulo* N *is even.*

Proof. This follows from Corollary 12 and Lemma 15.

8. Finally, we shall consider the linear independence of the values of "sec".

LEMMA 22. *Let* N *be odd and let* χ *be any even Dirichlet character modulo* N. *Let* χ^* *be the primitive character with conductor* f_χ *corresponding to* χ. *Then*

$$\sum_{r=1}^{[N/2]} (-1)^r \chi(r) = \pm\prod_{p|N} (\chi^*(p)-(-1)^{(p-1)/2}) \sum_{r=1}^{[f_\chi/2]} (-1)^r \chi^*(r),$$

$$\sum_{|r|<N/2} (-1)^r \chi^{\wedge}(r) = \frac{\chi(2)}{\sqrt{N}} \sum_{r=1}^{N-1} \chi(r)\sec\frac{2r\pi}{N}.$$

Proof. The lemma is proved by modifying the proof of Lemmas 7 and 16.

LEMMA 23. *Let* N *be odd. For any even Dirichlet character* χ *modulo* N, *there exists an odd Dirichlet character* ξ *modulo* $4N$ *such that*

$$2\sum_{r=1}^{N-1} \chi(r)\sec\frac{2r\pi}{N} = \sum_{r=1}^{4N} \xi(r)\cot\frac{r\pi}{4N}.$$

Proof. Since $2\sec 2x = \cot(x+\pi/4) - \cot(x-\pi/4)$, the left-hand side is equal to

$$\sum_{\substack{1 \leq r \leq 4N \\ r \equiv 1 \bmod 4}} \chi(r)\cot\frac{r\pi}{4N} - \sum_{\substack{1 \leq r \leq 4N \\ r \equiv 3 \bmod 4}} \chi(r)\cot\frac{r\pi}{4N}.$$

We define ξ by $\xi(x) = \chi(x)$ if $x \equiv 1 \bmod 4$, $-\chi(x)$ if $x \equiv 3 \bmod 4$, and 0 if $x \equiv 0 \bmod 4$. It is easily proved that ξ is an odd Dirichlet character modulo $4N$.

THEOREM 24. *Assume that one of the assertions in Theorem* 20-I *holds. Then the numbers* $\sec(2r\pi/N)$ *with* $r = 1, \ldots, [N/2]$ *and* $(r, N) = 1$ *are linearly independent over* $\boldsymbol{Q}$.

Proof. Lemma 23 leads to the theorem in the case of N odd. For N even, set $N' = N/2$. If N' is odd then the numbers $\sec(2r\pi/N')$ with $r = 1, \ldots, [N'/2]$ and $(r, N') = 1$ are linearly independent over $\boldsymbol{Q}$, so that the theorem follows from $\sec(4r\pi/N) = -\sec(2(N'-2r)\pi/N)$. If N' is even then by $\sec(2r\pi/N) = \operatorname{cosec}((N'-2r)\pi/N)$, the theorem reduces to Theorem 21.

Note that by Theorem 24, the numbers $\sec(2\pi/N)$ are always a normal basis of the maximal real subfield K^+ of the Nth cyclotomic field K over $\boldsymbol{Q}$, while their reciprocal numbers $\cos(2\pi/N)$ are a normal basis if and only if N is square free.

Compare the assertion (II-5) in Theorem 20-II with the following theorem.

THEOREM 25. *Assume that one of the assertions in Theorem* 20-I *holds and further that* N *is odd. In order that the square matrix* $((-1)^{d(a,b)})$ *with* $\varphi(N)/2$ *rows is regular, where* a *as row and* b *as column run over all positive integers prime to* N *and less than* $N/2$, *and where* $d(a, b)$ *means the unique integer with* $a \equiv d(a, b)b \bmod N$ *and* $-(N-1)/2 \leq d(a, b) \leq (N-1)/2$, *it is necessary and sufficient that all prime divisors of* N *are congruent to* 3 mod 4.

Proof. Necessity. We see from Lemma 13 that the left-hand side of the first formula of Lemma 22 does not vanish. Therefore taking the principal character for χ, we obtain

$$\prod_{p|N} (1-(-1)^{(p-1)/2}) \neq 0,$$

and so $p \equiv 3 \bmod 4$.

Sufficiency. By Theorem 11 and Lemmas 22 and 23, there exists an odd Dirichlet character ξ modulo $4f_\chi$ such that

$$\sum_{r=1}^{(N-1)/2} (-1)^r \chi(r) = \pm \prod_{p|N} (\chi^*(p)+1)\, c(\chi)\, L(1, \xi),$$

where

$$c(\chi) = \frac{2}{\pi} \sum_{r=1}^{f_\chi} \chi^*(r)\, e(-2r/f_\chi).$$

From $L(1, \xi) \neq 0$, it is sufficient to show that $\chi^*(p) \neq -1$ when all prime divisors are congruent to 3 modulo 4. Then the multiplicative order $M(p)$ of p mod N' for $N' = N/p^a$ with $p^a \| N$ is either odd or twice an odd number since $\varphi(q^m) \equiv 2$ mod 4 for every prime q with $q \equiv 3$ mod 4. If $M(p)$ is odd then $\chi^*(p) \neq -1$ by $\chi^*(p)^{M(p)} = 1$. If $M(p)$ is twice an odd number, then $p^{M(p)/2} \equiv -1$. Therefore $\chi^*(p)^{M(p)/2} = \chi^*(-1) = 1$, since χ is even, so that $\chi^*(p) \neq -1$. This completes the proof of the sufficiency.

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