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Some metric properties of subsequences

by

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1. Introduction

1.1. Notations and definitions. Metrical properties about independence of subsequences of given sequences in a compact metrizable space X are investigated. The set of X-valued sequences is identified with the compact product space $X^{\mathbf{N}}$. If μ is a Borel probability measure on X, we denote by μ_∞ the infinite product measure induced by μ on $X^{\mathbf{N}}$. Let u be an X-valued sequence and let t be a non-negative integer. Then $u^{(t)}$ denotes the X^t-valued sequence given by

$$u^{(t)}(n) := \big(u(n),\, u(n+1),\, \ldots,\, u(n+t-1)\big).$$

Let $\mathcal{U}$ be a finite family of sequences $u\colon \mathbf{N}\to X_u$ where all X_u, $u\in\mathcal{U}$, are compact metrizable spaces. We recall [16] that $\mathcal{U}$ is said to be *statistically independent* if for all continuous functions $f_u\colon X_u\to\mathbf{C}$, $u\in\mathcal{U}$, one has

$$\lim_{N\to\infty}\left[\left(\frac{1}{N}\sum_{n<N}\Big(\prod_{u\in\mathcal{U}} f_u(u(n))\Big)\right)-\prod_{u\in\mathcal{U}}\left(\frac{1}{N}\sum_{n<N} f_u(u(n))\right)\right]=0.$$

The family $\mathcal{U}$ is said to be *completely statistically independent* if $\mathcal{U}^{(t)} := \{u^{(t)};\, u\in\mathcal{U}\}$ is statistically independent for all positive integers t. Now a family of sequences in compact metrizable spaces will be said independent (resp. completely independent) if the corresponding property holds for all finite sub-families.

Let $\mathcal{F}$ be a family of $\mathbf{N}$-valued sequences $\sigma\colon \mathbf{N}\to\mathbf{N}$ such that $\lim_{n\to\infty}\sigma(n) = +\infty$.

DEFINITION 1. (a) A sequence $u\colon \mathbf{N}\to X$ is called *$\mathcal{F}$-independent* if the family

$$\mathcal{S}(u, \mathcal{F}) := \{u\circ\sigma;\, \sigma\in\mathcal{F}\}$$

is statistically independent.

(b) The sequence u is said to be *$\mathcal{F}$-independent at rank t* if the family $\{(u\circ\sigma)^{(t)};\, \sigma\in\mathcal{F}\}$ is statistically independent.

(c) If u is $\mathcal{F}$-independent at rank t for all positive integers t then u is said to be *completely* $\mathcal{F}$*-independent*.

From now on, μ denotes a given Borel probability measure on X. Classically the X-valued sequence u is said to be *μ-uniformly distributed* if for all continuous functions $f\colon X \to \mathbf{C}$, one has

$$\lim_{N\to\infty} \frac{1}{N} \sum_{n<N} f(u_n) = \int_X f(x)\,\mu(dx). \tag{1}$$

Recall that a family $\mathcal{U}$ of X-valued sequences u is *equi-μ-uniformly distributed* [9] if the limit in (1) holds uniformly in $u \in \mathcal{U}$.

DEFINITION 2. The sequence u is called *$(\mathcal{F}, \mu)$-independently distributed* if the sequences $u \circ \sigma$, $\sigma \in \mathcal{F}$, are μ-(uniformly) distributed and statistically independent.

Now we introduce the notion of sparse family [4] and regular sequence.

DEFINITION 3. A sequence $\sigma\colon \mathbf{N} \to \mathbf{N}$ is called *regular* if there is a subset A of $\mathbf{N}$ with asymptotical density 1, such that $\sigma|_A$ is one-to-one.

DEFINITION 4. A family $\mathcal{F}$ of sequences of positive integers is called *sparse* if, for all $(\sigma, \tau) \in \mathcal{F} \times \mathcal{F}$, one has

$$\sigma \neq \tau \;\Rightarrow\; \lim_{N\to\infty} (1/N)\,\mathrm{card}\{n \in \mathbf{N};\ n < N \text{ and } \sigma(n) = \tau(n)\} = 0.$$

1.2. Examples. The following standard families are sparse and regular.

$\mathcal{F}_\tau := \{\tau_k;\ k \in \mathbf{N}\}$, with $\tau_k(n) := n + k$,

$\mathcal{F}_\pi := \{\pi_l;\ l \in \mathbf{N}^*\}$, with $\pi_l(n) := ln$,

$\mathcal{F}_{\tau,\pi} := \{\tau_k \circ \pi_l;\ k \in \mathbf{N} \text{ and } l \in \mathbf{N}^*\}$,

$\mathcal{F}_P := \{n \to p(n);\ p \text{ non-constant real polynomial with } p(\mathbf{N}) \subset \mathbf{N}\}$.

Notice that by definition the sequence $u\colon \mathbf{N} \to X$ is $(\mathcal{F}_\tau, \mu)$-independently distributed if and only if u is completely μ-uniformly distributed (cf. [9], p. 204). The corresponding notions where $\mathcal{F}_\tau$ is replaced by $\mathcal{F}_\pi$ or $\mathcal{F}_P$ were studied by J. Coquet [1]. Let $\mathcal{F}$ and $\mathcal{F}'$ be different families among the above ones. The existence of sequences u which are $(\mathcal{F}, \mu)$-independently distributed and such that the family $\mathcal{E}(u, \mathcal{F}')$ is equi-μ-uniformly distributed is investigated in [4]. It is well known that μ_∞-almost all sequences u are $(\mathcal{F}_\tau, \mu)$-independently distributed and if μ is not a Dirac measure, for any such sequence u, the family $\mathcal{E}(u, \mathcal{F}_\tau)$ is never equi-μ-uniformly distributed. The same assertion holds with $\mathcal{F}_\pi$. The set

$$\mathcal{U}(X, \mu, \mathcal{F}) := \{u \in X^{\mathbf{N}};\ \mathcal{E}(u, \mathcal{F}) \text{ is equi-}\mu\text{-uniformly distributed}\}$$

for $\mathcal{F} = \mathcal{F}_\tau$ is not empty and one has (see [9])

$$\mu_\infty(\mathcal{U}(X, \mu, \mathcal{F})) = 0. \tag{2}$$

Due to [4] (Theorem 4), the same properties hold if $\mathcal{F} = \mathcal{F}_\pi$. In the case $\mathcal{F} = \mathcal{F}_{\tau,\pi}$ and more generally for any family $\mathcal{F}$ such that $\mathcal{F} \circ \sigma \subset \mathcal{F}$ for all $\sigma \in \mathcal{F}_{\tau,\pi}$, the set $\mathcal{U}(X, \mu, \mathcal{F})$ is empty ([4], Theorem 6).

We quote a last example. Let $\sigma_k\colon \mathbf{N} \to \mathbf{N}$, $k \in \mathbf{N}$, be strictly increasing sequences with disjoint images, then the family $\mathcal{S} := \{\sigma_k\colon k \in \mathbf{N}\}$ is a sparse family of regular sequences. Remark that we also have (2) for $\mathcal{F} = \mathcal{S}$.

1.3. In Part 2 one first proves that μ_∞-almost all sequences are $(\mathcal{F}, \mu)$-independently distributed whenever $\mathcal{F}$ is a countable sparse family of regular sequences and we discuss some consequences. Despite of (2) there are sparse families $\mathcal{S}$ of regular sequences such that there exist sequences u in $\mathcal{U}(X, \mu, \mathcal{S})$ which are also $(\mathcal{S}, \mu)$-independently distributed. In Part 3 we study the $(\mathcal{F}_P, \lambda)$-independent distribution of $n \to \theta^n$ mod 1 where λ is the Lebesgue measure on the one dimensional torus $\mathbf{T}$ identified with $[0, 1[$.

The last part is devoted to the construction of a *sparse family* $\mathfrak{S}$ which turns any countable family of sequences into a family of independent sequences. Moreover, for each σ in $\mathfrak{S}$ and each μ-well-distributed sequence u, the sequence $u \circ \sigma$ is μ-well-distributed too. In terms of our notation, that means we have:

$$\forall \sigma \in \mathfrak{S},\ \forall u \in \mathcal{U}(X, \mu, \mathcal{F}_\tau)\colon\ u \circ \sigma \in \mathcal{U}(X, \mu, \mathcal{F}_\tau).$$

The set $\mathfrak{S}$ derives from the dyadic expansion of integers. If $s(\cdot)$ denotes the sum of digits to base two then all sequences $n \to s(\sigma(n))$ are 2-additive and are also continuous generalized Morse sequences introduced by M. Keane [7]. Moreover, using previous results ([8], [14]), we prove that the spectral measures of these sequences are mutually singular.

2. $\mathcal{F}$-Independent sequences

2.1. THEOREM 1. *Let $\mathcal{F}$ be a countable sparse family of regular sequences of integers and let μ be a Borel probability measure on X, then μ_∞-almost all sequences u are $\mathcal{F}$-independent, each $u \circ \sigma$, $\sigma \in \mathcal{F}$, being μ-distributed.*

Proof. Let $f_1, \ldots, f_h$ be in $\mathcal{C}(X)$ satisfying $\int_X f_i\,d\mu = 0$ for all i, $1 \leqslant i \leqslant h$ and $(f_i|f_j)_\mu = \delta_{ij}$ for all $(i, j) \in \{1, \ldots, h\}^2$, where $\delta_{ij} = 0$ or 1 according as $i \neq j$ or $i = j$. Let $(\sigma_i)_{i \in \mathbf{N}^*}$ be an indexing of $\mathcal{F}$ by $\mathbf{N}^*$ and let A be a subset of $\mathbf{N}$ satisfying:

$$(\forall (i, j) \in \{1, \ldots, h\}^2)\ \ (i \neq j \;\Rightarrow\; \forall n \in A,\ \sigma_i(n) \neq \sigma_j(n)),$$

and such that each $\sigma_{i|A}$ is one-to-one. The set A can be chosen of asymptotical density one, in other words:

$$A(N) := \operatorname{card}(A \cap [0, N]) \sim N, \quad N \to +\infty.$$

We set

$$S_N = \int_{X^N} \left| \frac{1}{A(N)} \sum_{n<N, n\in A} f_1\big(u(\sigma_1(n))\big) \dots f_h\big(u(\sigma_h(n))\big) \right|^2 \mu_\infty(du)$$

$$= \frac{1}{(A(N))^2} \sum_{\substack{m,n<N \\ m,n\in A}} \int_{X^N} f_1\big(u(\sigma_1(m))\big) \overline{f_1\big(u(\sigma_1(n))\big)} \dots f_h\big(u(\sigma_h(m))\big) \overline{f_h\big(u(\sigma_h(n))\big)} \mu_\infty(du).$$

If n belongs to A, all the sets $E_n := \{\sigma_1(n), \dots, \sigma_h(n)\}$ have h elements. From Fubini's theorem and the choice of f_i, the corresponding term in the sum is zero whenever $E_m \neq E_n$.

If n and m belong to A, $E_m = E_n$ implies the existence of a permutation π of $\{1, \dots, h\}$ such that for all $k \leqslant h$ one has, say, $\sigma_k(m) = \sigma_{\pi(k)}(n) = l_k$. Then

$$\int_{X^N} f_1\big(u(\sigma_1(m))\big) \overline{f_1\big(u(\sigma_1(n))\big)} \dots f_h\big(u(\sigma_h(m))\big) \overline{f_h\big(u(\sigma_h(n))\big)} \mu_\infty(du)$$

$$= \int_{X^N} f_1(u(l_1)) \overline{f_{\pi(1)}(u(l_1))} \dots f_h(u(l_h)) \overline{f_{\pi(h)}(u(l_h))} \mu_\infty(du) = \prod_{k=1}^{h} (f_k | f_{\pi(k)}),$$

a product which vanishes if $\pi \neq \mathrm{id}$ and equals 1 if $\pi = \mathrm{id}$. The last case corresponds to $m = n$ if m and n belong to A. Hence we obtain $S_N = 1/A(N)$.

For N large enough, $A(N) \geqslant \frac{1}{2}N$, thus $\sum_{N=1}^{\infty} S_{N^2} < +\infty$ and from the Fatou–Karo lemma, for μ_∞-almost every u, one has

$$\operatorname*{Lim}_{N\to\infty} \frac{1}{A(N^2)} \Big| \sum_{\substack{n<N^2 \\ n\in A}} f_1\big(u(\sigma_1(n))\big) \dots f_h\big(u(\sigma_h(n))\big) \Big| = 0.$$

Moreover,

$$\left| \frac{1}{N^2} \sum_{n<N^2} f_1\big(u(\sigma_1(n))\big) \dots f_h\big(u(\sigma_h(n))\big) \right|$$

$$\leqslant \Big(\prod_{k=1}^{h} \|f_k\|_\infty\Big) \frac{N^2 - A(N^2)}{N^2} + \frac{1}{A(N^2)} \Big| \sum_{\substack{n<N^2 \\ n\in A}} f_1\big(u(\sigma_1(n))\big) \dots f_h\big(u(\sigma_h(n))\big) \Big|.$$

Finally, consider

$$\theta(N) := \frac{1}{N} \sum_{n<N} f_1\big(u(\sigma_1(n))\big) \dots f_h\big(u(\sigma_h(n))\big),$$

then $\operatorname*{Lim}_{N\to\infty} \theta(N^2) = 0$, μ_∞-a.e. and thus $\operatorname*{Lim}_{N\to\infty} \theta(N) = 0$, μ_∞-a.e. We conclude by remarking that an at most countable family of f_k satisfying the previous conditions exists which spans, with the constant functions, a $\mathbf{C}$-vector space dense in $\mathscr{C}(X)$. ∎

2.2. THEOREM 2. *Let σ_k: $\mathbf{N} \to \mathbf{N}$, $k \in \mathbf{N}$, be strictly increasing sequences with disjoint images. Let $\mathscr{S}$ be the family $\{\sigma_k\colon k \in \mathbf{N}\}$ and let μ be a Borel probability measure on X. Then there exists a sequence u: $\mathbf{N} \to X$ such that the family $\mathscr{E}(u, \mathscr{S})$ is both statistically independent and equi-μ-distributed.*

Proof. There is no loss of generality if we assume that the family $E := \{\sigma_k(\mathbf{N});\ k \in \mathbf{N}\}$ forms a partition of $\mathbf{N}$. Let $(q_k)_{k\geqslant 0}$ be an increasing sequence of integers such that if we define

$$M_k := \sum_{m<k} q_m m^m,$$

then

$$\operatorname*{Lim}_{k\to\infty} k^k / M_k = 0.$$

We denote by I_k the interval $[M_k, M_{k+1}[$ and for any integer n in I_k let $\tilde{n}$ be the remainder in the euclidian division of $n - M_k$ by k^k. We write

$$\sum_{j<k} e_j(\tilde{n}) k^j, \quad 0 \leqslant e_j(\tilde{n}) < k,$$

the expansion of $\tilde{n}$ to base k. Now we consider any μ-distributed sequence w and we define the family of X-valued sequences v_j, $j \in \mathbf{N}$, by

$$v_j(n) := \begin{cases} w(e_j(\tilde{n})) & \text{if } n \in I_k \text{ and } j < k; \\ w(e_{k-1}(\tilde{n})) & \text{if } n \in I_k \text{ and } j \geqslant k. \end{cases}$$

Finally, we take the sequence u: $\mathbf{N} \to X$ defined by

$$u(m) := v_j(n) \Leftrightarrow m = \sigma_j(n).$$

We claim that u has the required properties. Let f: $X \to \mathbf{C}$ be a continuous function and assume that $|f|$ is bounded by 1. For simplicity, we write ω instead of $\int f\,d\mu$ and for all integers $k > 0$, we write ω_k instead of $\frac{1}{k} \sum_{m<k} f(w(m))$. Note that $|\omega| \leqslant 1$ and $|\omega_k| \leqslant 1$. By construction we have

$$\sum_{M_k \leqslant n < M_k + ck^k} f(v_j(n)) = ck^k \omega_k,$$

for all integers $j \geqslant 0$ and $c \in \{0, \dots, q_k - 1\}$. Now let N, r and a be positive integers satisfying the following inequalities:

$$M_r + ar^r \leqslant N < M_r + (a+1)r^r \quad \text{and} \quad a < q_r.$$

Then we have

$$\Big|\sum_{n<N} f(v_j(n)) - \sum_{k<r} q_k k^k \omega_k\Big| \leqslant r^r.$$

But $\operatorname{Lim}_{k\to\infty} \omega_k = \omega$ so that $\operatorname{Lim}_{K\to\infty} M_K^{-1} \sum_{k<K} q_k k^k \omega_k = \omega$. Choose $\varepsilon > 0$; there exists an integer L, independent of j, such that we have both

$$k^k/M_k \leqslant \varepsilon/4 \quad \text{and} \quad |\omega_k - \omega| \leqslant \varepsilon/4$$

for all $k \geqslant L$. A straightforward computation gives for all $N \geqslant M_L$

$$\Big|\sum_{n<N} f(v_j(n)) - N\omega\Big| \leqslant 2r^r + \Big|\sum_{k<r} q_k k^k(\omega_k - \omega)\Big| \leqslant (\varepsilon/2) M_r + 2M_L + (\varepsilon/4) M_r.$$

Therefore,

$$\Big|\sum_{n<N} f(v_j(n)) - N\omega\Big| \leqslant \varepsilon N$$

holds for all $j \geqslant 0$ and all $N \geqslant \operatorname{Max}\{8M_L\varepsilon^{-1}, M_L\}$. We have thus established that the family $\mathscr{E}(u, \mathscr{S}) (= \{v_j;\ j \in \mathbf{N}\})$ is equi-μ-distributed. It remains to prove that u is $\mathscr{S}$-independent. For all positive integers J and all continuous functions f_j: $X \to \mathbf{C}$, $0 \leqslant j < J$, we easily verify that

$$\sum_{M_k \leqslant n < M_k + ck^k} \Big(\prod_{j<J} f_j(v_j(n))\Big) = ck^{k-J} \prod_{j<J} \Big(\sum_{m<k} f_j(w(m))\Big)$$

whenever $k \geqslant J$ and $c < q_k$. But this is the crucial step which yields the desired result. We leave the details to the reader. ■

2.3. Remarks.

1. Different notions of regular sequences can be defined which ensure the conclusion of Theorem 1. We only quote two such definitions which are independent.

The mapping σ: $\mathbf{N} \to \mathbf{N}$ is called *D-regular* if $\bar{d}(B_k) = 0$ where $\bar{d}(B_k)$ is the upper asymptotical density of $B_k := \{n \in \mathbf{N};\ \operatorname{card}\{\sigma^{-1}(\sigma(n))\} \geqslant k\}$.

The second notion will be useful in the next section. The mapping σ: $\mathbf{N} \to \mathbf{N}$ is called *M-regular* if the series $\sum_{N=1}^{\infty} \Delta(N) N^{-3}$ converges, where

$$\Delta(N) := \operatorname{card}\{(n, m) \in \mathbf{N}^2;\ m < N,\ n < N,\ \sigma(m) = \sigma(n)\}.$$

For $\alpha > 2$, the sequence σ_α given by $\sigma_\alpha(n) := [(\operatorname{Log} n)^\alpha]$ is M-regular but not D-regular ($[\cdot]$, as usual, denotes the integer part). Let $(\alpha_k)_k$ be a strictly increasing sequence of real numbers > 2 and write σ_k instead of σ_{α_k}. Then the family $\{\sigma_k;\ k \in \mathbf{N}\}$ is sparse and M-regular (but also regular).

Now let A be the subset of $\mathbf{N}$ subjected to:

$$\operatorname{card}\{n < N;\ n \in A\} := [N/\sqrt{\operatorname{Log} N}] \quad \text{for } N \geqslant 2.$$

The sequence defined by $\sigma(n) = 0$ if $n \in A$ and $\sigma(n) = n$ if $n \notin A$ is D-regular, regular but not M-regular.

2. Taking $\mathscr{F} = \mathscr{F}_\tau$, we see that the conclusion of Theorem 1 fails if $\mathscr{F}$-independence is replaced by $\mathscr{F}$-independence at rank $t > 1$. But for "very well sparse" families, Theorem 1 could be strengthened. We do not examine this problem in detail but only claim that if $\mathscr{G} := \{\sigma \circ \tau_k;\ \sigma \in \mathscr{F},\ k < t\}$ is a sparse family of regular sequences, μ_∞-almost every sequence u is $\mathscr{F}$-independent at rank t, each $u \circ \sigma$, $\sigma \in \mathscr{F}$, being μ-distributed. In particular:

COROLLARY 1. *Let μ be a Borel measure on X. For μ_∞-almost every sequence, the family of sequences $\{u \circ \sigma;\ \sigma \in \mathscr{F}_\pi\}$ is completely statistically independent, each of the $u \circ \sigma$ ($\sigma \in \mathscr{F}_\pi$) being completely μ-distributed.*

2.4. We mention two other consequences of Theorem 1.

COROLLARY 2. *Let $\mathscr{F}$ be a countable sparse family of regular sequences and let q be a natural integer $\geqslant 2$. Then there is a $\mathscr{F}$-independent sequence u: $\mathbf{N} \to \{0, 1, \ldots, q-1\}$, each sequence $u \circ \sigma$, $\sigma \in \mathscr{F}$, being uniformly distributed* mod q.

In fact, from Theorem 1, almost all sequences with respect to the infinite equidistributed measure has the required property. Moreover, such a sequence u can be given using an explicit construction when $\mathscr{F}$ is asymptotically ordered, that is, totally ordered by means of the relation

$$\sigma \ll \tau \Leftrightarrow \exists N,\ \forall n \geqslant N:\ \sigma(n) < \tau(n).$$

We refer to [15] for definition and characterization of normal sets. The following result extends Theorem 4 in [1], its proof is similar and makes use of the preceding corollary.

COROLLARY 3. *Let $\mathscr{F}$ be an asymptotically ordered countable sparse family of regular non-decreasing sequences of positive integers. For all normal subsets A of $\mathbf{R}^*$ there is a sequence Λ of real numbers such that:*

(i) *If $x \in A$ then $x\Lambda$ is $\mathscr{F}$-independent, each $x\Lambda \circ \sigma$, $\sigma \in \mathscr{F}$, being uniformly distributed* mod 1.

(ii) *If $x \notin A$ then $x\Lambda$ is not uniformly distributed* mod 1.

3. Subsequences of θ^n

3.1. Let θ be a real number > 1. It is well known [9] that, for almost every real number x, the sequence $n \to x\theta^n$ is uniformly distributed mod 1, and even completely uniformly distributed mod 1 if θ is a transcendental number. This means that such sequences are $\mathscr{F}_\tau$-independent. In fact, from Corollary 4.3, page 35, [9], we can derive:

PROPOSITION 1. *If $\theta > 1$ is transcendental, the sequence $n \to x\theta^n$ is $\mathscr{F}_P$-independent ($\mathscr{F}_P$ is the family of non-constant polynomial functions p such that $p(N) \subset N$) for almost every real number x, each of the sequences $n \to x\theta^{p(n)}$, $p \in \mathscr{F}_P$, being uniformly distributed* mod 1.

The complete proof is left to the reader, we only quote that if $p_1, \ldots, p_s$ are different polynomials in $\mathscr{F}_P$ and $(a_1, \ldots, a_s) \in \mathbf{Z}^s \backslash \{0, \ldots, 0\}$, there exist $\delta > 0$ and $n \in N$ such that

$$(n \geqslant N,\ m \geqslant N,\ n \neq m) \Rightarrow \Big|\sum_{k=1}^{s} a_k \theta^{p_k(n)} - \sum_{k=1}^{s} a_k \theta^{p_k(m)}\Big| \geqslant \delta.$$

H. Niederreiter and R. Tichy proved [12] that for any sequence $n \to a_n$ of distinct positive integers, for almost every $\theta > 1$ the sequence $n \to \theta^{a_n}$ is completely uniformly distributed (see also [13] for a more general result). Let us consider now all the polynomial sequences simultaneously.

THEOREM 3. *For almost every real number $\theta > 1$, the sequence $n \to \theta^n$ is $\mathscr{F}_P$-independent (the corresponding sequence $n \to \theta^{p(n)}$, $p \in \mathscr{F}_P$, being uniformly distributed* mod 1).

3.2. Proof of Theorem 3.

3.2.1. We note that θ is necessarily transcendental, which we assume from now on. Let $p_1, \ldots, p_s$ be different elements of $\mathscr{F}_P$. We assume that

$$0 < p_1(n) < \ldots < p_s(n) \qquad \text{for } n \geqslant N_0.$$

Let $a_1, \ldots, a_s$ be rational integers with $a_s > 0$. We set

$$u_n(\theta) := \sum_{k=1}^{s} a_k \theta^{p_k(n)}$$

so that

$$u'_m(\theta) - u'_n(\theta) = \sum_{k=1}^{s} a_k\big(p_k(m)\theta^{p_k(m)-1} - p_k(n)\theta^{p_k(n)-1}\big)$$

and

$$u''_m(\theta) - u''_n(\theta) = \sum_{k=1}^{s} a_k\big(p_k(m)(p_k(m)-1)\theta^{p_k(m)-2} - p_k(n)(p_k(n)-1)\theta^{p_k(n)-2}\big).$$

Let t be the element of $\{1, \ldots, s\}$ defined by $t := \inf\{j;\ \forall k \geqslant j,\ \deg(p_s - p_k) = 0\}$, and let $\delta_k = p_s - p_k$ for $k \geqslant t$ ($\delta_k \in N^*$). We denote by $\alpha_1, \ldots, \alpha_l$ the possible roots $\geqslant 1$ of the polynomial

$$Q(\theta) = \sum_{t \leqslant k \leqslant s} a_k \theta^{\delta_t - \delta_k}.$$

We may assume that $a_t \neq 0$. Fix $\mu > 0$ and choose $\gamma > 0$ such that $|Q(\theta)| \geqslant \mu$ provided that

(3) $\qquad \theta \geqslant 1 + \gamma \quad$ and $\quad |\theta - \alpha_j| \geqslant \gamma \quad$ for all $j \in \{1, \ldots, l\}$.

Let E_μ be the set of all numbers θ satisfying (3). We will verify that:

(4) $\quad \exists \lambda > 0,\ \exists N_1 \in N,\ \forall (m, n) \in N^2$:

$$(m > n \geqslant N_1) \Rightarrow \big(|u'_m(\theta) - u'_n(\theta)| \geqslant \lambda,\ \forall \theta \in E_\mu\big),$$

and

(5) $\quad \exists N_2 \in N,\ \forall (m, n) \in N^2$:

$(m > n \geqslant N_2) \Rightarrow \big((u''_m(\theta) - u''_n(\theta))$ has a constant sign on each interval included in $E_\mu\big)$.

3.2.2. Verification of (4). One has:

$$\sum_{t \leqslant k \leqslant s} a_k\big(p_k(m)\theta^{p_k(m)-1} - p_k(n)\theta^{p_k(n)-1}\big)$$
$$= Q(\theta)\theta^{-1-\delta_t}\big(p_s(m)\theta^{p_s(m)} - p_s(n)\theta^{p_s(n)}\big) - \Big(\sum_{t \leqslant k \leqslant s} a_k \delta_k \theta^{-1-\delta_k}\Big)\big(\theta^{p_s(m)} - \theta^{p_s(n)}\big),$$

thus

$$u'_m(\theta) - u'_n(\theta) = \Sigma_1 + \Sigma_2$$

with

$$\Sigma_1 := Q(\theta)\theta^{-\delta_t}\big(p_s(m)\theta^{p_s(m)-1} - p_s(n)\theta^{p_s(n)-1}\big)$$

and

$$\Sigma_2 := -\Big(\sum_{t \leqslant k \leqslant s} a_k \delta_k \theta^{-1-\delta_k}\Big)\big(\theta^{p_s(m)} - \theta^{p_s(n)}\big)$$
$$+ \sum_{k < t} a_k\big(p_k(m)\theta^{p_k(m)-1} - p_k(n)\theta^{p_k(n)-1}\big).$$

Then

(6) $$|\Sigma_2| \leqslant \theta^{p_s(m)-1} \sum_{t \leqslant k \leqslant s} |a_k \delta_k| + \sum_{k<t} |a_k|\, p_k(m)\theta^{p_k(m)-1} \leqslant K_2\, \theta^{p_s(m)-1}$$

for sufficiently large m. On the other hand:

$$p_s(m)\theta^{p_s(m)-1} - p_s(n)\theta^{p_s(n)-1} \geqslant p_s(m)\big(\theta^{p_s(m)-1} - \theta^{p_s(m-1)-1}\big)$$
$$\geqslant p_s(m)\theta^{p_s(m)-1}(1 - 1/\theta)$$

and since $|Q(\theta)| \geqslant \mu$ and $\lim_{\theta \to \infty} Q(\theta)\theta^{-\delta_t} = a_t \neq 0$,

(7) $$|\Sigma_1| \geqslant K_1\, p_s(m)\theta^{p_s(m)-1}$$

where K_1 depends on γ. From (6) and (7) we derive (4).

3.2.3. Verification of (5) and conclusion. The map $u''_m - u''_n$ is continuous, thus we only have to give a lower bound for $|u''_m(\theta) - u''_n(\theta)|$. The calculation is as above.

From Koksma's theorem (Theorem 4.3, p. 34, [9]), the sequence $n \to e(u_n(\theta))$ has a zero mean-value for almost every $\theta \in E_\mu$. Since $]1, +\infty[= \bigcup_{n \geq 1} E_{1/n}$, this is true for almost every $\theta > 1$. This finishes the proof. ∎

3.3. Remarks.

1. The family $\mathscr{F}_P$ could be replaced by a sparse family $\mathscr{F}$ of one-to-one sequences such that:

$$(\forall \sigma \in \mathscr{F},\ \forall \tau \in \mathscr{F})\ (\sigma \neq \tau \Rightarrow \sigma - \tau \text{ is monotonic}).$$

In the proof, we take $\sigma_1, \ldots, \sigma_s$ in $\mathscr{F}$ such that $0 < \sigma_1(n) < \ldots < \sigma_s(n)$ for $n \geq N$ and $t := \inf\{j;\ \forall k \geq j,\ \sigma_j - \sigma_k$ is bounded$\}$.

2. Theorem 3 can be generalized to sequences $n \to \sum_{r=1}^{R} x_r(\theta_r)^n = V(n)$ where $x_1, \ldots, x_R$ are fixed real numbers different from 0. For almost every $\{\theta_1, \ldots, \theta_R\} \in]1, +\infty[^R$, the corresponding sequence V is $\mathscr{F}_P$-independent and for all $p \in \mathscr{F}_P$, the subsequences $V \circ p$ are uniformly distributed mod 1.

4. Construction of independent sequences

4.1. Construction. Let E be an infinite part of $\boldsymbol{N}$. Let θ: $\boldsymbol{N} \to E$ be the increasing one-to-one mapping of $\boldsymbol{N}$ onto E and let σ_E: $\boldsymbol{N} \to \boldsymbol{N}$ be given by

$$(*) \qquad \sigma_E(n) := \sum_{k=0}^{\infty} \varepsilon_{\theta(k)}(n) 2^k,$$

whenever $n = \sum_{r=0}^{\infty} \varepsilon_r(n) 2^r$ is the binary expansion of n. According to the definition of Gel'fond [5] the sequence σ_E is 2-additive.

Now let $\{E_j;\ j \in \boldsymbol{N}^*\}$ be a partition of $\boldsymbol{N}$ into infinite subsets E_j and let θ_j: $\boldsymbol{N} \to E_j$ be the increasing bijection of $\boldsymbol{N}$ onto E_j. We write σ_j instead of σ_{E_j}.

PROPOSITION 2. *The set $\mathfrak{S} := \{\sigma_j;\ j \in \boldsymbol{N}^*\}$ is a sparse family of M-regular sequences.*

Proof. Let σ be the sequence $(*)$ derived from the increasing one-to-one mapping θ of $\boldsymbol{N}$ onto an infinite part E of $\boldsymbol{N}$ and set $\tau(x) := \operatorname{card}([0, x] \cap E)$. By definition the equality $\sigma(n) = \sigma(m)$ holds if and only if one has $\varepsilon_{\theta(k)}(n) = \varepsilon_{\theta(k)}(m)$ for all integers k. Hence, for $N \geq 1$ and $x = \operatorname{Log} N / \operatorname{Log} 2$ one gets $\Delta(N) \leq 2 \cdot 2^{\tau(x)} \cdot 2^{x - \tau(x)} \leq 2N$ so that the series $\sum_{N=1}^{\infty} \Delta(N) N^{-3}$ converges and σ is M-regular. Notice that σ is not D-regular. Now let σ' and θ' be given as above but $E \cap E' = \emptyset$. The equality $\sigma(n) = \sigma'(n)$ means that $\varepsilon_{\theta(k)}(n) = \varepsilon_{\theta'(k)}(n)$

for all integers k. Consider $\tau'(x) := \operatorname{card}([0, x] \cap E')$ and choose $x = \dfrac{\operatorname{Log} N}{\operatorname{Log} 2}$. Let z be an integer such that

$$(8) \qquad z = \sigma(n) = \sigma'(n)$$

for an integer $n < N$. Then at least $\tau(x) + \tau'(x)$ digits of n are fixed. Hence, the number of solutions n of (8) is at most $2^{x - \tau(x) - \tau'(x)}$. Assume that $\tau(x) \leq \tau'(x)$, then the number of different z is $\operatorname{card}\{\sigma(\{0, \ldots, N-1\})\} \leq 2^{\tau(x)}$. Therefore, the number of n such that $\sigma(n) = \sigma'(n)$ and $n < N$, is at most $2^{x - \tau'(x)}$. Due to this we get

$$\operatorname{card}\{n < N;\ \sigma(n) = \sigma'(n)\} \leq N 2^{-\tau(x)}$$

with $\lim_{x \to \infty} \tau(x) = +\infty$ and the proof is complete. ∎

We now quote two simple lemmata:

LEMMA 1. *Let $\Omega_t := \{0, 1, \ldots, 2^t - 1\}$ be endowed with the equiprobability λ_t and let X_j be the restriction of σ_j to Ω_t. Then the random variables X_j, $j = 1, 2, \ldots$ are independent and equidistributed.*

Proof. Let t_j be the number of elements in $\Omega_t \cap E_j$ so that $X_j(\Omega_t) = \Omega_{t_j}$. For any m_j in Ω_{t_j}, an easy computation gives

$$\lambda_t(\{X_j = m_j\}) = 2^{-t_j}.$$

But the events $\{X_j = m_j\}$ are independent because of the disjointness of the sets E_j. ∎

The proof of the next lemma is straightforward and we leave the details to the reader.

LEMMA 2. *Let $n \to x_n$ be a complex valued sequence and let $n \to a_n$ be an increasing sequence of positive integers such that $a_n \in O(n)$. Then*

$$\sum_{n<N} x_n \in o(N) \Rightarrow \sum_{n<N} x_{n+a_N} \in o(N).$$

4.2. Universal properties. We first give a universal property of a topological nature satisfied by all sequences σ_E whenever E is an infinite part of $\boldsymbol{N}$. After that, we prove metrical properties of the above family $\mathfrak{S}$.

THEOREM 4. *Let μ be a Borel probability measure on X and let σ $(= \sigma_E)$ be any sequence defined by $(*)$, the set E being an infinite part of $\boldsymbol{N}$. Then for all μ-well-distributed sequences u: $\boldsymbol{N} \to X$ the sequence $u \circ \sigma$ is also μ-well-distributed.*

Proof. Let g: $X \to \boldsymbol{C}$ be a continuous map such that $\int g\, d\mu = 0$. We have to prove that

$$(9) \qquad \operatorname*{Lim}_{N \to \infty} \left(\operatorname*{Sup}_{s \in \boldsymbol{N}} \left| \frac{1}{N} \sum_{n<N} g(u \circ \sigma(n+s)) \right| \right) = 0.$$

By assumption, there exists a sequence $(\varepsilon_r)_{r\geqslant 0}$ of non-negative real numbers ε_r such that

$$\text{(10)}\qquad \forall s\in N:\ \Big|\sum_{n<2^r} g\big(u(n+s)\big)\Big| \leqslant \varepsilon_r 2^r \quad\text{and}\quad \operatorname*{Lim}_{r\to\infty}\varepsilon_r = 0.$$

Notice that (10) is equivalent to the μ-well-distribution of u. We may assume $|g(\cdot)|\leqslant 1$. Let $\varepsilon>0$ be given and choose r such that $\varepsilon_{r'}\leqslant\varepsilon/2$ for $r' := \operatorname{card}(E\cap[0, r[)$. Let N, t be positive integers and define integers a and b by the inequalities

$$(a-1)2^r\leqslant t<a2^r \quad\text{and}\quad b2^r\leqslant N+t<(b+1)2^r.$$

Then

$$\Big|\sum_{t\leqslant n<N+t} g\big(u\circ\sigma(n)\big)\Big| \leqslant 2^{r+1}+\sum_{a\leqslant n<b}\Big|\sum_{0\leqslant m<2^r} g\big(u(\sigma(n2^r)+\sigma(m))\big)\Big|$$
$$\leqslant 2^{r+1}+\sum_{a\leqslant n<b}\Big(2^{r-r'}\Big|\sum_{0\leqslant m'<2^{r'}} g\big(u(\sigma(n2^r)+\sigma(m'))\big)\Big|\Big).$$

Hence

$$\Big|\sum_{t\leqslant n<N+t} g\big(u\circ\sigma(n)\big)\Big| \leqslant 2^{r+1}+(b-a)2^r\varepsilon_{r'}\leqslant\Big(\frac{2}{b-a}+\varepsilon_{r'}\Big)N.$$

Now for $N\geqslant 2^r(2+4/\varepsilon)$ we have $(b-a)\geqslant 4/\varepsilon$ so that we obtain

$$\Big|\sum_{t\leqslant n<N+t} g\big(u\circ\sigma(n)\big)\Big|\leqslant\varepsilon N. \ ∎$$

THEOREM 5. *Let $(X_j)_{j>0}$ be a sequence of compact metrizable spaces. For each $j>0$, let μ_j be a Borel probability measure on X_j and let u_j: $N\to X_j$ be a μ_j-distributed sequence. Then the family $\mathscr{U}_{\Xi} := \{u_j\circ\sigma_j;\ j\in N^*\}$ is statistically independent, each of the sequence $u_j\circ\sigma_j$ being μ_j-distributed.*

Proof. From the definition, we have to show that, given an integer $d\geqslant 1$ and $f_j\in\mathscr{C}(X_j)$ for all $j\leqslant d$, if we put

$$\omega := \prod_{j=1}^{d}\int_{X_j} f_j\,d\mu_j \quad\text{and}\quad G(n) := \prod_{j=1}^{d} f_j\big(u_j\circ\sigma_j(n)\big),$$

then

$$\omega = \operatorname*{Lim}_{N\to\infty}(1/N)\sum_{n<N} G(n).$$

If $N=\sum_{r=0}^{v} a_r 2^r$ is the dyadic expansion of N, with $a_v\neq 0$, we put $N_c = \sum_{c\leqslant r\leqslant v} a_r 2^r$ for $c\leqslant v$. Moreover, let t_j be the counting function of $\theta_j(N)$, i.e.:

$$t_j(m) = \operatorname{card}\{r<m;\ r\in\theta_j(N)\}.$$

Fixing $\varrho\in N^*$, $\varrho<v$, we have

$$\text{(11)}\qquad \sum_{n<N} G(n) = \sum_{n<2^v} G(n)+\sum_{c=v-\varrho}^{v-1}\Big(\sum_{N_{c+1}\leqslant n<N_c} G(n)\Big)+O(N\cdot 2^{-\varrho})$$

because G is bounded. On the other hand, due to Lemma 1, we get

$$\sum_{n<2^v} G(n) = 2^{(v-\sum_{j=1}^{d} t_j(v))}\prod_{j=1}^{d}\Big(\sum_{m_j<2^{t_j(v)}} f_j\big(u_j(m_j)\big)\Big).$$

Choose $\varepsilon>0$; the hypothesis concerning u_j leads to

$$\text{(12)}\qquad \Big|\sum_{n<2^v} G(n)-\omega 2^v\Big|\leqslant\varepsilon 2^v$$

for v (i.e. for N) sufficiently large.

In the same way, if $a_c\neq 0$, and $c\geqslant v-\varrho$:

$$\sum_{N_{c+1}\leqslant n<N_c} G(n) = \sum_{n<2^c} G(N_{c+1}+n) = \sum_{n<2^c}\prod_{j=1}^{d} f_j\big(u_j(\sigma_j(N_{c+1})+\sigma_j(n))\big)$$

because $\sigma_j(N_{c+1}+n) = \sigma_j(N_{c+1})+\sigma_j(n)$ for all $n<2^c$.

As above, we get

$$\text{(13)}\qquad \sum_{N_{c+1}\leqslant n<N_c} G(n) = 2^{(c-\sum_{j=1}^{d} t_j(c))}\prod_{j=1}^{d}\Big(\sum_{m_j<2^{t_j(c)}} f_j\big(u_j(m_j+\sigma_j(N_{c+1}))\big)\Big).$$

But one has

$$\frac{\sigma_j(N_{c+1})}{2^{t_j(c)}}\leqslant\frac{\sigma_j(N)}{2^{t_j(v-\varrho)}}\leqslant\frac{\sigma_j(1+2+\ldots+2^v)}{2^{t_j(v-\varrho)}}\leqslant\frac{1+2+\ldots+2^{t_j(v+1)}}{2^{t_j(v-\varrho)}}<2^{\varrho+2}$$

so that

$$\text{(14)}\qquad \sigma_j(N_{c+1})\leqslant 2^{\varrho+2}\,2^{t_j(c)}.$$

Now, ϱ being fixed, we then derive from (13), (14) and Lemma 2 that:

$$\text{(15)}\qquad \Big|\sum_{N_{c+1}\leqslant n<N_c} G(n)-\omega 2^c\Big|\leqslant\varepsilon 2^c$$

for c (i.e. for N) large enough. Joining (11) to (12) and (15), we obtain

$$\Big|\sum_{n<N} G(n)-\omega N\Big|\leqslant\varepsilon N+O(N\cdot 2^{-\varrho})$$

for sufficiently large N. Thus

$$\operatorname*{Lim\,sup}_{N\to\infty}\Big|(1/N)\sum_{n<N} G(n)-\omega\Big|\leqslant C\cdot 2^{-\varrho}$$

where C is an absolute constant and ϱ is arbitrary. Therefore ω is the mean value of G. ∎

THEOREM 6. *Let $\mathscr{U}$ be a family of sequences u: $N \to X_u$ where X_u denotes a compact metric space. Assume that each sequence u is μ_u-distributed with respect to a Borel measure μ_u on X_u. Let σ $(=\sigma_E)$ be any sequence defined by (*) (E being infinite). If $\mathscr{U}$ is statistically independent then the family $\mathscr{U}\circ\sigma := \{u\circ\sigma;\ u\in\mathscr{U}\}$ is also statistically independent.*

Proof. Without loss of generality, we may assume that $\mathscr{U}$ is finite. For each u in $\mathscr{U}$, let g_u: $X_u \to C$ be continuous and set

$$\tilde{\omega} := \prod_{u\in\mathscr{U}} \int_{X_u} g_u\,d\mu_u \quad \text{and} \quad G(n) := \prod_{u\in\mathscr{U}} g_u(u\circ\sigma(n)).$$

By Theorem 5 the sequence $u\circ\sigma$ is also μ_u-distributed in X_u, hence we have to show that

$$\tilde{\omega} = \operatorname*{Lim}_{N\to\infty} (1/N) \sum_{n<N} G(n).$$

Let $N = \sum_{r=0}^{v} a_r 2^r$ be the dyadic expansion of N. Use N_c and ϱ as in the proof of Theorem 5 and let $t(\cdot)$ be the counting map of E. To estimate $\sum_{n<N} G(n)$ we start from equality (11). By Lemma 1, we obtain

$$\sum_{n<2^v} G(n) = 2^{v-t(v)} \sum_{m<2^{t(v)}} \prod_{u\in\mathscr{U}} g_u(u(m)).$$

Let $\varepsilon > 0$; by assumption on $\mathscr{U}$ there is v_0 such that

$$\Big|2^{-t(v)} \sum_{m<2^{t(v)}} \prod_{u\in\mathscr{U}} g_u(u(m)) - \prod_{u\in\mathscr{U}} \Big(2^{-t(v)} \sum_{m<2^{t(v)}} g_u(u(m))\Big)\Big| \leqslant \varepsilon/2$$

whenever $v \geqslant v_0$. But we may choose $v_1 \geqslant v_0$ such that for all $M \geqslant 2^{t(v_1)}$ one has

$$\left|\prod_{u\in\mathscr{U}} \left(\frac{1}{M} \sum_{m<M} g_u(u(m))\right) - \tilde{\omega}\right| \leqslant \varepsilon/2.$$

Therefore,

$$\Big|\sum_{n<2^v} G(n) - \tilde{\omega} 2^v\Big| \leqslant \varepsilon 2^v \tag{16}$$

whenever $v \geqslant v_1$.

Now we consider the sum $\Sigma_c := \sum_{N_{c+1}\leqslant n<N_c} G(n)$, with $a_c \neq 0$. As above we get

$$\Sigma_c = 2^{c-t(c)} \sum_{m<2^{t(c)}} \prod_{u\in\mathscr{U}} g_u\big(u(\sigma(N_{c+1})+m)\big).$$

On the other hand, we have by assumption $\operatorname*{Lim}_{M\to\infty} (1/M) \sum_{m<M} \prod_{u\in\mathscr{U}} g_u(u(m)) = \tilde{\omega}$ and since inequality (14) holds, it follows from Lemma 2 that we also have $\operatorname*{Lim}_{c\to\infty} 2^{-c}\Sigma_c = \tilde{\omega}$. Therefore, there exists v_2 $(\geqslant v_1 - \varrho)$ such that $c \geqslant v_2$ implies

$$|\Sigma_c - 2^c\tilde{\omega}| \leqslant \varepsilon 2^c. \tag{17}$$

Using (11), (16) and (17) we derive a constant (which only depends on the functions g_u, $u\in\mathscr{U}$) such that

$$\Big|\sum_{n<N} G(n) - \tilde{\omega} N\Big| \leqslant \varepsilon N + CN\cdot 2^{-\varrho}$$

for sufficiently large N. Since ϱ is arbitrary, the desired result follows. ■

4.3. Spectral properties. Recall that $s(\cdot)$ denotes the sum of digits to base two. Let E be any nonempty subset of N, let θ be the increasing counting map of E and let χ_E: $N \to \{+1, -1\}$ be the 2-multiplicative sequence defined by $\chi_E(n) := (-1)^{s(\sigma_E(n))}$ where σ_E is still given by (*). Now, we endow N with the group law $\oplus$ corresponding to the addition to base two without carry. Let $\varepsilon_k(n)$ be the kth digit in the dyadic expansion of n. By definition, for all integers n and m one has

$$\varepsilon_k(n\oplus m) \equiv \varepsilon_k(n) + \varepsilon_k(m) \bmod 2, \qquad k = 0, 1, 2, \ldots$$

Now we remark that χ_E is a character on $(N, \oplus)$. Conversely, for any character χ on $(N, \oplus)$ (also called Walsh character) one has $\chi = \chi_E$ where $E := \{t\in N;\ \chi(2^t) = -1\}$. Clearly, χ_E is periodical whenever E is finite. Spectral properties of χ_E was studied by M. Mendès France [11] and dynamical point of view was first investigated by S. Kakutani [6] in order to give examples of minimal uniquely ergodic discrete symbolic systems. More results and generalizations are due to M. Keane [7], [8]. From now on, we recognize any Walsh character χ as a generalized Morse sequence (to base two) in the terminology of Keane. To see this, we assume familiarity with [7] but change the 0's to +1's and the 1's to −1's. Thus we have

$$(\chi(n))_{n\geqslant 0} = b^{(0)} \times b^{(1)} \times b^{(2)} \times \ldots$$

where $b^{(k)} := (+1, -1)$ for $k\in E$ otherwise $b^{(k)} := (+1, +1)$.

Recall that the Borel measure λ_E on the torus $T = R/Z$ is said to be the spectral measure of χ_E if the Fourier transform $\hat{\lambda}_E$ is the correlation function of χ_E, that is to say:

$$\hat{\lambda}_E(k) := \int_T e^{2i\pi kt}\,\lambda_E(dt) = \operatorname*{Lim}_{N\to\infty} \frac{1}{N} \sum_{\substack{n<N\\ n+k\geqslant 0}} \chi_E(n+k)\overline{\chi_E(n)}, \qquad k\in Z. \tag{18}$$

From basic results [3], the spectral measure λ_E exists and is given by the weak-limit

$$\lambda_E(dt) = *\text{-}\lim_{N\to\infty} \frac{1}{N}\left|\sum_{n<N} \chi_E(n)e^{-2i\pi nt}\right|^2 h(dt).$$

The 2-multiplicativity leads to the product formula

$$\lambda_E(dt) = *\text{-}\lim_{K\to\infty}\Big(\prod_{k<K}(1+\chi_E(2^k)\cos 2^{k+1}\pi t)\Big)h(dt). \tag{19}$$

It is known that λ_E is singular with respect to the Haar measure. Moreover λ_E is continuous if E is infinite. If E is finite then χ_E is periodic with period 2^T where $T = 1+\operatorname{Max} E$ and λ_E corresponds to the Haar measure of the finite sub-group of $\boldsymbol{T}$ generated by 2^{-T}. Now, we shall say that E is *thick* if there exists $K>0$ such that

$$\forall m \geqslant 0, \quad E\cap[m, m+K[\neq \emptyset.$$

THEOREM 7. *Let E and E' be thick subsets of* $\boldsymbol{N}$. *Then the spectral measures λ_E and $\lambda_{E'}$ are equivalent or mutually singular. Moreover the following statements are equivalent*:

(i) *λ_E and $\lambda_{E'}$ are equivalent* ($\lambda_E \sim \lambda_{E'}$).

(ii) *The symmetric difference $E\Delta E'$ is finite.*

(iii) *The series* $\sum_{k=0}^{\infty} |\hat{\lambda}_E(2^k)-\hat{\lambda}_{E'}(2^k)|^2$ *converges.*

(iv) $\hat{\lambda}_E(2^k) = \hat{\lambda}_{E'}(2^k)$ *for sufficiently large k.*

Proof. It is known from [8], Lemma, that either λ_E and $\lambda_{E'}$ are mutually singular or $\lambda_E \sim \lambda_{E'}$. Now we compute $\hat{\lambda}(2^k)$ from (18). The product $\chi_E(n)\chi_E(n+2^k)$ is constant, equal to $\chi_E(2^k)\ldots\chi_E(2^{k+s})$ on the arithmetical progressions $A_k(0) := \{n\in\boldsymbol{N};\ \varepsilon_k(n) = 0\}$ for $s=0$ and

$$A_k(s) := \{n\in\boldsymbol{N};\ \varepsilon_k(n) = 1, \ldots, \varepsilon_{k+s-1}(n) = 1, \varepsilon_{k+s}(n) = 0\}$$

for all $s \geqslant 1$. Hence,

$$\hat{\lambda}_E(2^k) = \frac{1}{2}\sum_{s=0}^{\infty}\frac{\chi_E(2^k)\ldots\chi_E(2^{k+s})}{2^s}.$$

Put $\varepsilon_E^{(k)}(s) := \frac{1}{2}(1+\chi_E(2^k)\ldots\chi_E(2^{k+s}))$ such that $\varepsilon_E^{(k)}(s)\in\{0,1\}$ and

$$1+\hat{\lambda}_E(2^k) = \sum_{s=0}^{\infty}\varepsilon_E^{(k)}(s)2^{-s}. \tag{20}$$

Notice that $\varepsilon_E^{(k)}(s)$ takes the values 0 and 1 infinitely often.

Obviously (iv) implies (iii). Assume property (iii) and use (20). Since E and E' are thick, then for each integer $S\geqslant 0$ there exists $K\geqslant 0$ such that the equalities $\varepsilon_E^{(k)}(s) = \varepsilon_{E'}^{(k)}(s)$ hold for all $k\geqslant K$ and all $s = 0, 1, \ldots, S$. In particular, this implies $\chi_E(2^k) = \chi_{E'}(2^k)$ (and consequently $\hat{\lambda}_E(2^k) = \hat{\lambda}_{E'}(2^k)$) for $k\geqslant K$. Therefore (iii) implies (ii) and (iv).

Assume (ii) (recall that $E\cap E'$ is infinite). Since λ_E and $\lambda_{E'}$ are continuous, formula (19) gives easily

$$\lambda_{E'}(dt) = \prod_{m\in E\setminus E'}(\tan(2^m\pi t))^2 \prod_{n\in E'\setminus E}(\cotan(2^n\pi t))^2\cdot\lambda_E(dt)$$

so that property (i) holds. It remains to prove that (i) implies (iii). We may derive this implication from [14], Lemma 4, using the sequence $n\to X_n$ of complex random variables given on $\boldsymbol{T}$ by $X_n(t) := e^{i\pi 2^{n+1}t}$ (such that the expectation of X_n with respect to λ is $\hat{\lambda}_E(2^n)$). ∎

4.4. Remark. The above construction to base 2 is typical but it also holds to base $g>2$ and Theorems 4, 5, 6 remain valid in this case.

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