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Representation of real numbers as sums of U_2-numbers

by

K. ALNIAÇIK * (Istanbul)

In [2] P. Erdős proved that every real number can be represented as a sum and product of two Liouville (U_1) numbers. In this paper we prove the

THEOREM. *Every real number — except Liouville numbers — can be represented as a sum of two U_2-numbers.*

First let us recall the

DEFINITION (1). The complex number ξ is called a *U_2-number* if for every $w > 0$ there are infinitely many algebraic numbers γ of degree 2 with

$$|\xi-\gamma| < H(\gamma)^{-w}$$

and if there exist positive constants c, m depending only on ξ such that

$$|\xi-p/q| > cq^{-m}$$

holds for every rational number p/q.

Let α be a real number with the continued fraction expansion

$$\alpha = \langle a_0, a_1, \ldots, a_n, \ldots\rangle. \tag{1}$$

In the sequel we shall write for simplicity $\alpha = \langle a_n\rangle$ instead of (1). In the proof we shall use some lemmas:

LEMMA 1. *Let α be an algebraic number of degree 2 and let a, b, c, d $(ad-bc \neq 0)$ be integers (2). Then we have*

$$H\left(\frac{a\alpha+b}{c\alpha+d}\right) \leqslant 6\max(|a|, |b|, |c|, |d|)^2 H(\alpha).$$

LEMMA 2. *Let $r_1 = \langle a_n\rangle$, $r_2 = \langle b_n\rangle$ be positive real numbers and let p_n/q_n denote the n-th convergent of r_1. Next assume that*

$$0 < |r_1-r_2| < 72^{-1} q_{s+3}^{-3} \tag{2}$$

holds for some integer $s \geqslant 0$. Then $a_i = b_i$ for $0 \leqslant i \leqslant s$.

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(1) We note that we have, in fact, defined a Koksma U_2^*-number instead of a Mahler U_2-number. However, it is known that they are the same (see [3], [4]).

(2) In this paper the word "integer" will be used as an abbreviation for "rational integer".

Proof. First we show that $a_0 = b_0$. Assume that $a_0 \neq b_0$, then $|a_0 - b_0| > 1$ and this gives us $a_0 + 1 \leqslant r_2$ or $r_2 \leqslant a_0$. On the other hand, it is clear that $a_0 + (2q_1)^{-1} < r_1 < a_0 + 1 - (2q_2)^{-1}$. Combining these inequalities we obtain $|r_1 - r_2| \geqslant \min\{(2q_1)^{-1}, (2q_2)^{-1}\} = (2q_2)^{-1}$, which contradicts (2). Hence we may assume that there is an integer $k \geqslant 0$ such that $a_i = b_i$ for $i = 0, 1, \ldots, k$ and $a_{k+1} \neq b_{k+1}$.

Thus it follows from the theory of continued fractions that

$$|r_1 - r_2| \geqslant |r'_1 - r'_2| (9a_{k+1} b_{k+1} q_k^2)^{-1} \tag{3}$$

where $r'_1 = \langle a_{k+1}, a_{k+2}, \ldots \rangle$, $r'_2 = \langle b_{k+1}, b_{k+2}, \ldots \rangle$.

Now if $b_{k+1} > 4a_{k+1}$, then $|r'_1 - r'_2| > 2^{-1} b_{k+1}$ and by (3) we get $|r_1 - r_2| > 18^{-1} q_{k+1}^{-2}$. Next assume that $b_{k+1} \leqslant 4a_{k+1}$ and $|r'_1 - r'_2| > 1$. Then by (3), $|r_1 - r_2| > 36^{-1} q_{k+1}^{-2}$. Finally, let $b_{k+1} \leqslant 4a_{k+1}$ and $|r'_1 - r'_2| \leqslant 1$. We note that in this case $|a_{k+1} - b_{k+1}| = 1$ and so

$$a_{k+1} \geqslant r'_2 \quad \text{or} \quad r'_2 \geqslant a_{k+1} + 1. \tag{4}$$

On the other hand, using the properties of continued fractions we see that $a_{k+1} + (2q_{k+2})^{-1} < r'_1 < a_{k+1} + 1 - (2q_{k+3})^{-1}$. By combining this with (4) we obtain

$$|r'_1 - r'_2| \geqslant \min\{|a_{k+1} - (a_{k+1} + (2q_{k+2})^{-1})|, |a_{k+1} + 1 - (a_{k+1} + 1 - (2q_{k+3})^{-1})|\}$$
$$= (2q_{k+3})^{-1}.$$

Using this and $b_{k+1} \leqslant 4a_{k+1}$ in (3) we have $|r_1 - r_2| \geqslant 72^{-1} q_{k+3}^{-3}$, and so by (2) we have $k \geqslant s$, which completes the proof.

LEMMA 3. *Let $r_1 = \langle a_n \rangle = \langle a_0, a_1, \ldots, a_m, 1, 1, \ldots \rangle$ and $r_2 = \langle b_n \rangle$ be positive real numbers and let p_i/q_i denote the i-th convergent of r_1. Assume that $|r_i - p/q| > \lambda_i^{-1} q^{-m_i}$ $(i = 1, 2)$, where λ_i $(\geqslant 2)$ and m_i $(\geqslant 2)$ are constants not depending on q. Furthermore, assume that*

$$0 < |r_1 - r_2| < N^{-1} \quad \text{where} \quad N > 72q_{m+4}^3. \tag{A}$$

Then we have:

(a) *There is an integer k such that $a_i = b_i$ $(i = 0, 1, \ldots, k)$, $a_{k+1} \neq b_{k+1}$, $k \geqslant m+1$ and $11q_k > \sqrt{N}$.*

(b) $|r_1 - p/q| > (2\lambda_2)^{-1} q^{-m_2}$.

(c) $|r_2 - p/q| > (2\lambda_1)^{-1} q^{-m_1}$ *if $q \leqslant q_{k-1}$.*

Proof. (a) Since $N > 72q_{m+4}^3$, by Lemma 2 we have $a_i = b_i$ for $i = 0, 1, \ldots, m+1$. Hence there is an integer $k \geqslant m+1$ with $a_i = b_i$ for $0 \leqslant i \leqslant k$ and $a_{k+1} \neq b_{k+1}$. Thus

$$N^{-1} > |r_1 - r_2| > |r'_1 - r'_2| (9b_{k+1} q_k^2)^{-1}, \tag{5}$$

where $r'_1 = (1 + \sqrt{5})/2$ and $r'_2 = \langle b_{k+1}, b_{k+2}, \ldots \rangle$.

It is clear that $b_{k+1} \geqslant 2$. Now if $2 \leqslant b_{k+1} \leqslant 4$, then $|r'_1 - r'_2| > 3^{-1}$ and by (5) we get $108q_k^2 > N$. Next assume that $b_{k+1} > 4$. Since $r'_1 < 2$ we see that $|r'_1 - r'_2| > 2^{-1} b_{k+1}$, and so by (5) we obtain $18q_k^2 > N$, which completes the proof of (a).

(b) It is enough to show that (b) is true for $p/q = p_n/q_n$, where p_n/q_n is the nth convergent of r_1. For $n < m-1$ we have

$$|r_1 - p_n/q_n| = q_n^{-1} (\xi_1 q_n + q_{n-1})^{-1} > q_n^{-1} (2\xi_2 q_n + q_{n-1})^{-1}$$
$$> (2q_n)^{-1} (\xi_2 q_n + q_{n-1})^{-1} = 2^{-1} |r_2 - p_n/q_n| > (2\lambda_2)^{-1} q_n^{-m_2},$$

where $\xi_1 = \langle a_{n+1}, a_{n+2}, \ldots \rangle$, $\xi_2 = \langle b_{n+1}, b_{n+2}, \ldots \rangle$. If $n > m$, then

$$|r_1 - p_n/q_n| > 3^{-1} q_n^{-2} > (2\lambda_2)^{-1} q_n^{-m_2},$$

and this completes the proof.

(c) The proof is similar to the proof in case (b).

LEMMA 4. *Let $r_1 = \langle a_n \rangle$, $r_2 = \langle b_n \rangle$ be positive real numbers with $a_i = b_i$ $(i = 0, 1, \ldots, k)$, $a_{i+k} = 1$ $(i = 1, 2, \ldots)$ and $b_{k+1} > 1$ for some integer $k > 0$. Then we have*

$$\text{(a)} \quad |r_1 - r_2| > 121 q_k^{-2}, \qquad \text{(b)} \quad |r_1 - r_2| < q_k^{-2},$$

where q_k is the denominator of the k-th convergent of r_1.

The proof of (a) is similar to the proof of Lemma 3(a) and the proof of (b) follows from the well-known properties of continued fractions.

Proof of the Theorem. Let r be a real number with $r \notin U_1$ where U_1 is the set of all Liouville numbers. It is clear that if $r \in \boldsymbol{Q}$ or r is a U_2-number, then there is nothing to prove. Next if r is a real algebraic number of degree 2, then by Corollary 2 in [1] we have the Theorem. Hence we assume that $r \notin U_1 \cup U_2$ and r is not an algebraic number of degree $\leqslant 2$. Furthermore, we may assume that $3 < r < 4$.

For the proof, we shall construct algebraic numbers α_n, β_n $(n = 1, 2, \ldots)$ in the order $\alpha_1, \beta_1, \alpha_2, \beta_2, \ldots$ with the following properties:

(a) $\lim_{n \to \infty} \alpha_n$ and $\lim_{n \to \infty} \beta_n$ exist. Set $\gamma_1 = \lim_{n \to \infty} \alpha_n$, $\gamma_2 = \lim_{n \to \infty} \beta_n$.

(b) $r = \gamma_1 + \gamma_2$.

(c) There are constants $C, M > 0$, depending on γ_i $(i = 1, 2)$ and r only, such that $|\gamma_i - p/q| > Cq^{-M}$ $(i = 1, 2)$.

(d) $|\gamma_1 - \alpha_n| \leqslant H(\alpha_n)^{-n}$, $|\gamma_2 - \beta_n| \leqslant H(\beta_n)^{-n}$ $(n = 1, 2, \ldots)$.

We note that (c) and (d) show that $\gamma_i \in U_2$ $(i = 1, 2)$ and we obtain the assertion of the Theorem by (b).

Since $r \notin U_1 \cup U_2$, there are constants $c_0 = c_0(r) > 1$, $t = t(r) > 0$ such that

$$|r - \beta| > c_0^{-1} H(\beta)^{-t} \tag{6}$$

holds for algebraic numbers β of degree 2. Put $L = c_0 (96 \cdot 384 \cdot 121)^t$.

In the sequel $p_n^{(i)}/q_n^{(i)}$, $R_n^{(i)}/S_n^{(i)}$, $r_n^{(i)}/s_n^{(i)}$, $P_n^{(i)}/Q_n^{(i)}$ ($n = 0, 1, \ldots$) will denote the nth convergents of $\alpha_i, r-\alpha_i, \beta_i, r-\beta_i$ ($i = 1, 2, \ldots$) in their respective continued fraction expansions, and we set $r-\alpha_i = \langle a_n^{(i)} \rangle$, $r-\beta_i = \langle b_n^{(i)} \rangle$ ($i = 1, 2, \ldots$). Finally, let $\{\varepsilon_i\}$ ($\varepsilon_1 = 0$, $\varepsilon_{i+1} > \varepsilon_i$, $\varepsilon_i < 1$) be a sequence of real numbers.

Now we define algebraic numbers α_ν, β_ν as follows: $\alpha_1 = (1+\sqrt{5})/2$,

$$(7)_\nu \quad \beta_\nu = \langle d_i^{(\nu)} \rangle, \quad \text{where} \quad d_i^{(\nu)} = \begin{cases} a_i^{(\nu)}, & 0 \leqslant i \leqslant l_\nu \\ 1, & i > l_\nu \end{cases} \qquad (\nu = 1, 2, \ldots),$$

$$(8)_\nu \quad \alpha_\nu = \langle c_i^{(\nu)} \rangle, \quad \text{where} \quad c_i^{(\nu)} = \begin{cases} b_i^{(\nu-1)}, & 0 \leqslant i \leqslant k_\nu \\ 1, & i > k_\nu \end{cases} \qquad (\nu = 2, 3, \ldots),$$

where l_ν ($\nu = 1, 2, \ldots$) and k_ν ($\nu = 1, 2, \ldots$) are positive integers satisfying

$$(9) \qquad S_{l_1}^{(1)} \geqslant \max\{22L^{1/\varepsilon_2},\ 22\cdot 2^{1/(\varepsilon_3-\varepsilon_2)},\ H(\beta_1)\}, \qquad a_{l_1+1}^{(1)} \neq 1,$$

$$(10)_\nu \qquad S_{l_\nu}^{(\nu)} \geqslant \max\{72(Q_{k_\nu+4}^{(\nu)})^{3/2},\ 22\cdot 2^{1/(\varepsilon_{\nu+2}-\varepsilon_{\nu+1})},\ 22L^{1/\varepsilon_{\nu+1}},\ H(\beta_\nu)^\nu\},$$
$$a_{l_\nu+1}^{(\nu)} \neq 1,$$

$$(11)_\nu \qquad Q_{k_{\nu+1}}^{(\nu)} \geqslant \max\{72(S_{l_\nu+4}^{(\nu)})^{3/2},\ 22\cdot 2^{1/(\varepsilon_{\nu+2}-\varepsilon_{\nu+1})},\ 22L^{1/\varepsilon_{\nu+1}},\ H(\alpha_{\nu+1})^{\nu+1}\},$$
$$k_1 = 0, \qquad b_{k_\nu+1}^{(\nu)} \neq 1.$$

It is clear that, since α_1 is given, the real numbers α_n, β_n have been defined in the following order: $\alpha_1, \beta_1, \alpha_2, \beta_2, \alpha_3, \ldots$ We have by $(7)_\nu$,

$$(12) \qquad s_i^{(n)} = S_i^{(n)}, \quad r_i^{(n)} = R_i^{(n)} \quad (i = 0, 1, \ldots, l_n,\ n \geqslant 1).$$

Proof of (a). It follows from $(7)_\nu$ and $(8)_\nu$ and Lemma 4(b) that

$$(13) \qquad \begin{aligned} |\beta_\nu - (r-\alpha_\nu)| &\leqslant (S_{l_\nu}^{(\nu)})^{-2} && (\nu = 1, 2, \ldots), \\ |\alpha_\nu - (r-\beta_{\nu-1})| &\leqslant (Q_{k_\nu}^{(\nu-1)})^{-2} && (\nu = 2, 3, \ldots). \end{aligned}$$

Let $j \geqslant 2$ be an integer. Then using (13) we get

$$|\alpha_j - \alpha_{j-1}| \leqslant |\alpha_j - (r-\beta_{j-1})| + |\beta_{j-1} - (r-\alpha_{j-1})| \leqslant (Q_{k_j}^{(j-1)})^{-2} + (S_{l_{j-1}}^{(j-1)})^{-2}.$$

On the other hand, $(11)_{\nu=j-1}$ gives us $Q_{k_j}^{(j-1)} > S_{l_{j-1}}^{(j-1)}$. Therefore

$$(14) \qquad |\alpha_j - \alpha_{j-1}| < 2(S_{l_{j-1}}^{(j-1)})^{-2},$$

and using (14) for integers m, n ($m > n > 0$) we have

$$(*) \qquad |\alpha_m - \alpha_n| \leqslant \sum_{j=n}^{m-1} |\alpha_{j+1} - \alpha_j| \leqslant 2 \sum_{j=n}^{m-1} (S_{l_j}^{(j)})^{-2} < 4(S_{l_n}^{(n)})^{-1}.$$

Since $S_{l_n}^{(n)} \to \infty$ as $n \to \infty$, $(*)$ shows that $\{\alpha_\nu\}$ is a Cauchy sequence and so $\lim_{n\to\infty} \alpha_n$ exists. Similarly, one can show that $\{\beta_\nu\}$ is also a Cauchy sequence, and this completes the proof of (a).

Proof of (b). Putting $\gamma_1 = \lim_{n\to\infty} \alpha_n$ and $\gamma_2 = \lim_{n\to\infty} \alpha_n$, we see from (13) that $r = \gamma_1 + \gamma_2$.

Proof of (c). First, by induction, we are going to show that

$$(15)_\nu \qquad |\alpha_\nu - p/q| > 6^{-1} q^{-4t-\varepsilon_\nu} \qquad (\nu = 1, 2, \ldots),$$

$$(16)_\nu \qquad |\beta_\nu - p/q| > (2c_1)^{-1} q^{-4t-\varepsilon_\nu} \qquad (\nu = 1, 2, \ldots),$$

where $c_1 = 2c_0 96^t$.

Let p/q ($p, q > 0$) be a rational number with $p \geqslant 4q$. Then since $\alpha_1 = (1+\sqrt{5})/2 < 2$, we have $|\alpha_1 - p/q| > 2$. If $\nu > 1$ is an integer then by $(*)_{m=\nu, n=1}$ we have $|\alpha_\nu - \alpha_1| < 4(S_{l_1}^{(1)})^{-1} < 1$, and so $|\alpha_\nu - p/q| \geqslant |\alpha_1 - p/q| - |\alpha_\nu - \alpha_1| > 1$. Similarly, we have $|\beta_1 - p/q| > 1$, since $\beta_1 < 3$. Next for $\nu > 1$ we have $|\beta_\nu - \beta_1| < \frac{1}{2}$, and therefore $|\beta_\nu - p/q| \geqslant |\beta_1 - p/q| - |\beta_\nu - \beta_1| > \frac{1}{2}$. Thus $(15)_\nu$ and $(16)_\nu$ are true for p/q with $p \geqslant 4q > 0$. Therefore we may assume that the rational numbers p/q in $(15)_\nu$, $(16)_\nu$ satisfy $p < 4q$.

Proof of $(15)_\nu$. Since $\alpha_1 = (1+\sqrt{5})/2 = \langle 1, 1, 1, \ldots \rangle$ we have

$$(17) \qquad |\alpha_1 - p/q| > 3^{-1} q^{-2} > 3^{-1} q^{-4t},$$

that is, $(15)_{\nu=1}$ is true. Now assume that α_ν satisfies $(15)_\nu$ for $\nu = 1, \ldots, n$.

Proof of $(15)_{\nu=n+1}$. By $(13)_{\nu=n}$ we have $|(r-\beta_n) - \alpha_n| = |(r-\alpha_n) - \beta_n| < (S_{l_n}^{(n)})^{-2}$. Next by $(10)_{\nu=n}$ (for $n = 1$, we use (9)), we have $(S_{l_n}^{(n)})^2 > 72(Q_{k_n+4}^{(n)})^3 \geqslant 72(q_{k_n+4}^{(n)})^3$. Thus applying Lemma 3 to $r_1 = \alpha_n$, $r_2 = r-\beta_n$, we see that there is a positive integer t_n such that

$$(18) \qquad b_i^{(n)} = c_i^{(n)} \quad \text{for } 0 \leqslant i \leqslant t_n, \qquad b_{t_n+1}^{(n)} \neq 1,$$

$$(19) \qquad t_n > k_n + 1, \qquad 11q_{t_n}^{(n)} = 11Q_{t_n}^{(n)} > S_{l_n}^{(n)}.$$

Now we show that $t_n < k_{n+1}$: It follows from the definition of β_n that $d_i^{(n)} = a_i^{(n)}$ ($i = 0, 1, \ldots, l_n$) and $d_{l_n+1}^{(n)} = 1$. On the other hand, by (9) and $(10)_{\nu=n}$ we have $a_{l_n+1}^{(n)} \neq 1$. Therefore we can apply Lemma 4(a) with $r_1 = \beta_n$, $r_2 = r-\alpha_n$ to obtain $|(r-\alpha_n) - \beta_n| > (11S_{l_n}^{(n)})^{-2}$. Next applying Lemma 4(b) to $r_1 = \alpha_n$, $r_2 = r-\beta_n$ (using (18)) we get $|(r-\beta_n) - \alpha_n| < (Q_{t_n}^{(n)})^{-2}$, and combining these we obtain

$$(20) \qquad 11S_{l_n}^{(n)} > Q_{t_n}^{(n)}.$$

On the other hand, by $(13)_{\nu=n+1}$ we have

$$(21) \qquad |(r-\beta_n) - \alpha_{n+1}| < (Q_{k_{n+1}}^{(n)})^{-2}.$$

Now assume that $t_n \geqslant k_{n+1}$. Then we have $c_{t_n+1}^{(n+1)} = 1$ by $(8)_{\nu=n+1}$. Combining this with (18) we see that $b_{t_n+1}^{(n)} \neq c_{t_n+1}^{(n+1)}$. Thus we can apply Lemma 4(a) with

$r_1 = \alpha_{n+1}$, $r_2 = r - \beta_n$ to find

(22)
$$|(r-\beta_n)-\alpha_{n+1}| > (11 Q_{t_n}^{(n)})^{-2},$$

and so combining (20)–(22) we obtain $Q_{k_{n+1}}^{(n)} < 121 S_{t_n}^{(n)}$, which contradicts $(11)_{v=n}$.

Now we have some remarks:

1. Using the above argument we find a sequence of integers $t_1, t_2, t_3, \ldots$ such that $k_i < t_i < k_{i+1}$ $(i = 1, 2, \ldots)$. Furthermore, by $k_i < k_{i+1}$ $(i = 1, 2, \ldots)$ we see that

(23)
$$t_i < t_{i+1} \quad (i = 1, 2, \ldots).$$

2. It follows from $b_i^{(n)} = c_i^{(n)}$ $(0 \leqslant i \leqslant t_n)$, $b_i^{(n)} = c_i^{(n+1)}$ $(0 \leqslant i \leqslant k_{n+1})$ and $t_n < k_{n+1}$ that $c_i^{(n)} = c_i^{(n+1)}$ $(0 \leqslant i \leqslant t_n)$. Next using this and (23), one can show that

(24)
$$c_i^{(s)} = c_i^{(m)} \quad (0 \leqslant i \leqslant t_s),$$

where s, m are positive integers with $s < m$.

3. Let n, m be positive integers with $n < m$. Since $t_n + 1 > k_n$, it follows from the definition of α_n that $c_{t_n+1}^{(n)} = 1$. On the other hand, by (18) we have $b_{t_n+1}^{(n)} \neq 1$. Next it follows from $t_n + 1 \leqslant k_{n+1}$ and the definition of α_{n+1} that $c_{t_n+1}^{(n+1)} = b_{t_n+1}^{(n)} \neq 1$. Finally, by (24) and $t_n + 1 < t_{n+1}$, we have $c_{t_n+1}^{(m)} = c_{t_n+1}^{(n+1)} = b_{t_n+1}^{(n)} \neq 1$. Thus $c_{t_n+1}^{(n)} \neq c_{t_n+1}^{(m)}$, which shows that if $n \neq m$ then $\alpha_n \neq \alpha_m$.

4. It is clear that in the proof of 3(b) we do not use the condition (A) of Lemma 3, that is: if $r_1 = \langle a_0, a_1, \ldots, a_m, 1, 1, \ldots \rangle$ and $r_2 = \langle b_n \rangle$ are real numbers with $a_i = b_i$ $(0 \leqslant i \leqslant k)$ and $k > m+1$, then 3(b) and 3(c) are true.

Now we continue the proof. Applying Lemma 3(c) to $r_1 = \alpha_n$, $r_2 = \alpha_{n+1}$ (using $(15)_{v=n}$) we obtain

(25)
$$|\alpha_{n+1} - p/q| > 2^{-1} q^{-4t-\varepsilon_n}$$

whenever

(26)
$$0 < q \leqslant q_{t_n-1}^{(n)} = Q_{t_n-1}^{(n)}.$$

We note that the equality in (26) follows from (18).

We consider two cases in (26):

Case 1. Let $0 < q < q_{t_1-1}^{(1)}$. It follows from (24) that $c_i^{(1)} = c_i^{(n+1)}$ for $0 \leqslant i \leqslant t_1$. Thus we can apply Lemma 3(c) with $r_1 = \alpha_1$, $r_2 = \alpha_{n+1}$ (using $(15)_{v=1}$) to obtain $|\alpha_{n+1} - p/q| > 6^{-1} q^{-4t}$.

Case 2. Let $q_{t_{j-1}-1}^{(j-1)} < q < q_{t_j-1}^{(j)}$ for some integer j with $2 \leqslant j \leqslant n$. By (24) we have $c_i^{(j)} = c_i^{(n+1)}$ for $0 \leqslant i \leqslant t_j$. Therefore applying Lemma 3(c) to $r_1 = \alpha_j$, $r_2 = \alpha_{n+1}$ (using $(15)_{v=j}$) we obtain

(27)
$$|\alpha_{n+1} - p/q| > 12^{-1} q^{-4t-\varepsilon_j}.$$

On the other hand, by (9), $(10)_{v=j-1}$ and (19) we get

$$q > Q_{t_{j-1}-1}^{(j-1)} > 2^{-1} Q_{t_{j-1}-1}^{(j-1)} > 22^{-1} S_{t_{j-1}}^{(j-1)} > 2^{1/(\varepsilon_{j+1}-\varepsilon_j)}.$$

Thus (27) yields

(28)
$$|\alpha_{n+1} - p/q| > 6^{-1} q^{-4t-\varepsilon_{j+1}}.$$

Secondly, contrary to (26), suppose that

(29)
$$q > Q_{t_n-1}^{(n)}.$$

It follows from (6) and Lemma 1 that

(30)
$$|(r-\beta_n) - p/q| > c_0^{-1} 96^{-t} q^{-2t} H(\beta_n)^{-t}.$$

Now we give an upper bound for $H(\beta_n)$. It follows from the definition of β_n and (12) that

$$\beta_n = \frac{r_{t_n}^{(n)} \alpha_1 + r_{t_n-1}^{(n)}}{s_{t_n}^{(n)} \alpha_1 + s_{t_n-1}^{(n)}} = \frac{R_{t_n}^{(n)} \alpha_1 + R_{t_n-1}^{(n)}}{S_{t_n}^{(n)} \alpha_1 + S_{t_n-1}^{(n)}}.$$

Using $R_{t_n}^{(n)}$, $R_{t_n-1}^{(n)}$, $S_{t_n-1}^{(n)} < 4 S_{t_n}^{(n)}$ in Lemma 1 we therefore get

(31)
$$H(\beta_n) < 96 (S_{t_n}^{(n)})^2.$$

On the other hand, since $b_{t_n}^{(n)} = c_{t_n}^{(n)} = 1$ we have $Q_{t_n}^{(n)} = b_{t_n}^{(n)} Q_{t_n-1}^{(n)} + Q_{t_n-2}^{(n)} < 2 Q_{t_n-1}^{(n)}$. Using this together with (20), (29) in (31) we obtain

$$H(\beta_n) < 96 (S_{t_n}^{(n)})^2 < 96 \cdot 121 (Q_{t_n}^{(n)})^2 < 384 \cdot 121 (Q_{t_n-1}^{(n)})^2 < 384 \cdot 121 q^2.$$

Hence by (30)

(32)
$$|(r-\beta_n) - p/q| > L^{-1} q^{-4t}.$$

Next, combination of $(10)_{v=n}$ (if $n = 1$, we use (9)), (20) and (29) gives us

$$q > Q_{t_n-1}^{(n)} > 2^{-1} Q_{t_n}^{(n)} > 22^{-1} S_{t_n}^{(n)} > L^{1/\varepsilon_{n+1}}.$$

Thus (32) yields $|(r-\beta_n) - p/q| > q^{-4t-\varepsilon_{n+1}}$. Finally, applying Lemma 3(b) to $r_1 = \alpha_{n+1}$, $r_2 = r - \beta_n$ (using the above inequality) we get

(33)
$$|\alpha_{n+1} - p/q| > 2^{-1} q^{-4t-\varepsilon_{n+1}}.$$

Thus (28), (33) give us $|\alpha_{n+1} - p/q| > 6^{-1} q^{-4t-\varepsilon_{n+1}}$, that is, we have $(15)_{v=n+1}$ in this case.

Proof of $(16)_v$. Using (6),

$$|(r-\alpha_1) - p/q| = |r - (\alpha_1 + p/q)| > c_0^{-1} H(\alpha_1 + p/q)^{-t}.$$

By Lemma 1 we find $H(\alpha_1 + p/q) \leqslant 96 q^2$, and so $|(r-\alpha_1) - p/q| > c_0^{-1} 96^{-t} q^{-2t}$, and applying Lemma 3(b) with $r_1 = \beta_1$, $r_2 = r - \alpha_1$ we obtain

(34)
$$|\beta_1 - p/q| > c_1^{-1} q^{-4t},$$

that is, $(16)_{\nu=1}$ is true. Now assume that $(16)_\nu$ holds for $\nu = 1, \ldots, n-1$ $(n \geqslant 2)$.

Proof of $(16)_{\nu=n}$. It follows from $(13)_{\nu=n}$ that $|(r-\alpha_n)-\beta_{n-1}| = |(r-\beta_{n-1})-\alpha_n| \leqslant (Q_{k_n}^{(n-1)})^{-2}$. Next by $(11)_{\nu=n-1}$ we have $(Q_{k_n}^{(n-1)})^2 \geqslant 72(S_{l_{n-1}+4}^{(n-1)})^3$. Hence we can apply Lemma 3(a) with $r_1 = \beta_{n-1}$, $r_2 = r-\alpha_n$ to find a positive integer m_{n-1} such that

$$a_i^{(n)} = d_i^{(n-1)} \quad \text{for } 0 \leqslant i \leqslant m_{n-1}, \quad a_{m_{n-1}+1}^{(n)} \neq 1, \tag{35}$$

$$m_{n-1} > l_{n-1}+1, \quad 11 s_{m_{n-1}}^{(n-1)} = 11 S_{m_{n-1}}^{(n)} > Q_{k_n}^{(n-1)}. \tag{36}$$

Furthermore, one can show that $m_{n-1} > l_n$, so it follows from (35) and $a_i^{(n)} = d_i^{(n)}$ $(0 \leqslant i \leqslant l_n)$ that $d_i^{(n-1)} = d_i^{(n)}$ for $i = 0, \ldots, m_{n-1}$. Therefore considering $l_1 < m_1 < l_2 < m_2 < l_3 < \ldots$ we see that

$$d_i^{(s)} = d_i^{(t)} \quad \text{for } i = 0, 1, \ldots, m_s \tag{37}$$

where s, t are positive integers with $s < t$.

Now we claim that $\beta_s \neq \beta_t$ for $s < t$. It follows from $m_{t-1}+1 \leqslant l_t$ and $a_i^{(t)} = d_i^{(t)}$ $(0 \leqslant i \leqslant l_t)$ that $d_{m_{t-1}+1}^{(t)} = a_{m_{t-1}+1}^{(t)}$. But by (35) we have $a_{m_{t-1}+1}^{(t)} \neq 1$. Thus $d_{m_{t-1}+1}^{(t)} \neq 1$. On the other hand, it follows from $(7)_{\nu=s}$ and $m_{t-1}+1 > l_s$ that $d_{m_{t-1}+1}^{(s)} = 1$. Therefore $d_{m_{t-1}+1}^{(s)} \neq d_{m_{t-1}+1}^{(t)}$, that is, if $s \neq t$ then $\beta_s \neq \beta_t$.

Now since $d_i^{(n)} = d_i^{(n-1)}$ for $i = 0, 1, \ldots, m_{n-1}$, we can apply Lemma 3(c) with $r_1 = \beta_{n-1}$, $r_2 = \beta_n$ (using $(16)_{\nu=n-1}$) to get

$$|\beta_n - p/q| > (4c_1)^{-1} q^{-4t-\varepsilon_{n-1}} \tag{38}$$

if

$$0 < q \leqslant s_{m_{n-1}-1}^{(n-1)}. \tag{39}$$

We consider two cases in (39):

Case 1. Let $0 < q \leqslant s_{m_1-1}^{(1)}$. Since $d_i^{(1)} = d_i^{(n)}$ for $i = 0, 1, \ldots, m_1$, we can apply Lemma 3(c) with $r_1 = \beta_1$, $r_2 = \beta_n$ (using (34)) to obtain $|\beta_n - p/q| > (2c_1)^{-1} q^{-4t}$.

Case 2. Suppose that $s_{m_1-1}^{(1)} < q \leqslant s_{m_{n-1}-1}^{(n-1)}$. Then there is a positive integer j $(1 \leqslant j \leqslant n-2)$ such that

$$s_{m_j-1}^{(j)} < q < s_{m_{j+1}-1}^{(j+1)}. \tag{40}$$

It follows from (37) that $d_i^{(j+1)} = d_i^{(n)}$ for $i = 0, 1, \ldots, m_{j+1}$. Hence applying Lemma 3(c) to $r_1 = \beta_{j+1}$, $r_2 = \beta_n$ (using $(16)_{\nu=j+1}$) we see that

$$|\beta_n - p/q| > (4c_1)^{-1} q^{-4t-\varepsilon_{j+1}} \tag{41}$$

holds for q in (40). On the other hand, (35) gives us $s_{m_j-1}^{(j)} = S_{m_j-1}^{(j+1)}$. Combining this, the inequality $S_{m_j-1}^{(j+1)} > 2^{-1} S_{m_j}^{(j+1)}$ and (40), (36), $(11)_{\nu=j}$ we get $q > 2^{1/(\varepsilon_{j+2}-\varepsilon_{j+1})}$ and by (41)

$$|\beta_n - p/q| > (2c_1)^{-1} q^{-4t-\varepsilon_{j+2}}.$$

Now, contrary to (39), assume that

$$q > s_{m_{n-1}-1}^{(n-1)}. \tag{42}$$

By (6), we have $|(r-\alpha_n)-p/q| = |r-(\alpha_n - p/q)| > c_0^{-1} H(\alpha_n - p/q)^{-t}$ or

$$|(r-\alpha_n)-p/q| > c_0^{-1} 96^{-t} q^{-2t} H(\alpha_n)^{-t}, \tag{43}$$

since $H(\alpha_n - p/q) < 96q^2 H(\alpha_n)$ because of Lemma 1. Next using Lemma 1 one can show that $H(\alpha_n) < 96(Q_{k_n}^{(n-1)})^2$. Combining this with (36), (42), $s_{m_{n-1}}^{(n-1)} = S_{m_{n-1}}^{(n)}$ we obtain

$$H(\alpha_n) < 96(Q_{k_n}^{(n-1)}) < 96\cdot 121(S_{m_{n-1}}^{(n)})^2 = 96\cdot 121(s_{m_{n-1}}^{(n-1)})^2 < 384\cdot 121(s_{m_{n-1}-1}^{(n-1)})^2$$
$$< 384\cdot 121 q^2$$

and so by (43)

$$|(r-\alpha_n)-p/q| > L^{-1} q^{-4t}. \tag{44}$$

On the other hand, by (42), (36) and $(11)_{\nu=n-1}$ we have

$$q > S_{m_{n-1}-1}^{(n)} > 2^{-1} S_{m_{n-1}}^{(n)} > 22^{-1} Q_{k_n}^{(n-1)} > L^{1/\varepsilon_n}.$$

Now (44) yields $|(r-\alpha_n)-p/q| > q^{-4t-\varepsilon_n}$, and by Lemma 3(c) $|\beta_n - p/q| > 2^{-1} q^{-4t-\varepsilon_n}$, that is, we have $(16)_{\nu=n}$.

It follows from the definitions of γ_1, γ_2, α_n, β_n and the relations (9), $(10)_\nu$, $(11)_\nu$ that

$$|\gamma_1 - \alpha_n| \leqslant (Q_{k_n}^{(n-1)})^{-1} < H(\alpha_n)^{-n} \quad (n \geqslant 1), \tag{45}$$

$$|\gamma_2 - \beta_n| \leqslant (S_{l_n}^{(n)})^{-1} < H(\beta_n)^{-n} \quad (n \geqslant 1). \tag{46}$$

Finally, let p/q be a rational number. Then using $(15)_\nu$, $(16)_\nu$ for sufficiently large ν $(= n)$ and (45), (46) in the inequalities

$$|\gamma_1 - p/q| \geqslant |\alpha_n - p/q| - |\gamma_1 - \alpha_n|, \quad |\gamma_2 - p/q| \geqslant |\beta_n - p/q| - |\gamma_2 - \beta_n|,$$

we obtain

$$|\gamma_1 - p/q| > 12^{-1} q^{-4t-1}, \quad |\gamma_2 - p/q| > (4c_1)^{-1} q^{-4t-1},$$

that is, we have (c). Furthermore, we have (d) by (45) and (46), and this completes the proof of the Theorem.

We note that, since the measure of the S-numbers in the interval $[0, 1]$ is 1, every real number can be written as a sum of two S-numbers. So if r is a Liouville number, then there are two S-numbers, say s_1, s_2, such that $r = s_1 + s_2$. On the other hand, by the above theorem we have $s_1 = \gamma_1 + \gamma_2$, $s_2 = \gamma_3 + \gamma_4$ where $\gamma_i \in U_2$ $(i = 1, 2, 3, 4)$, and so $r = \gamma_1 + \gamma_2 + \gamma_3 + \gamma_4$. Therefore we also obtain the

COROLLARY. *Every real number can be written as a sum of four U_2-numbers.*

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DEPARTMENT OF MATHEMATICS
FACULTY OF SCIENCE
ISTANBUL UNIVERSITY
34459 Vezneciler, Istanbul
Turkey

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Lüroth-type alternating series representations for real numbers

by

Sofia Kalpazidou (Thessaloniki),
Arnold Knopfmacher (Johannesburg)
and John Knopfmacher (Johannesburg)

Introduction. In this paper, we introduce an algorithm that leads to a general alternating series expansion for real numbers in terms of rationals. In particular, this algorithm is used to show the existence and uniqueness of two alternating series expansions which are analogous to the positive series of *Lüroth*, and to a modified *Engel* expansion, respectively. In addition, the representation of rational numbers by means of these algorithms is investigated. Thereafter, stochastic properties of the sequence of digits in the Lüroth-type alternating representation are studied. In particular, we solve the *Gauss-type measure problem* for this expansion.

1. A general alternating series algorithm. We first define a general alternating series algorithm, analogous to a positive one of Oppenheim [7], as follows:

Given any real number A, let $a_0 = [A]$, $A_1 = A - a_0$. Then recursively define

$$a_n \doteq [1/A_n] \geqslant 1 \quad \text{for } n \geqslant 1,\ A_n > 0,$$

where $A_{n+1} = (1/a_n - A_n)(c_n/b_n)$ for $a_n > 0$. Here

$$b_i = b_i(a_1, \ldots, a_i), \quad c_i = c_i(a_1, \ldots, a_i)$$

are positive numbers (usually integers), chosen so that $A_n \leqslant 1$ for $n \geqslant 1$. Note that $A_{n+1} \geqslant 0$, since $a_n \leqslant 1/A_n$ for $A_n > 0$.

Using this algorithm we now prove:

THEOREM 1. *Every real number has unique representations in the forms*

$$\text{(i)}\ A = a_0 + \frac{1}{a_1} - \frac{1}{(a_1+1)a_1}\cdot\frac{1}{a_2} + \frac{1}{(a_1+1)a_1(a_2+1)a_2}\cdot\frac{1}{a_3} - \ldots$$

$$= ((a_0, a_1, \ldots, a_n, \ldots)), \quad \textit{say, where } a_n \geqslant 1\ (n \geqslant 1), \textit{ and}$$