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An alternative approach to a theorem of Tom Meurman

by

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§ 1. Introduction. Tom Meurman proved the following theorem which is Corollary 2 on page 352 of his paper [3].

THEOREM 1. *For $3 \leqslant H \leqslant T$, and any $\varepsilon > 0$,*

$$\sum_{\chi \bmod q} \int_{T}^{T+H} |L(\tfrac{1}{2}+it, \chi)|^2\, dt \ll \big(qH+(qT)^{1/3}\big)\big(q(T+H)\big)^{\varepsilon}$$

where $q \geqslant 1$ is any integer and χ runs over all residue class characters mod q *and the constant implied by the Vinogradov symbol $\ll$ depends only on ε.*

Remark. We have stated the condition $3 \leqslant H \leqslant T$ which is not different from his conditions.

The history of this theorem is as follows. By Balasubramanian's result [1] the special case $q = 1$ of this theorem follows as an immediate corollary. A somewhat simpler proof in the case $q = 1$ was given by P. N. Ramachandran in his M. Phil. thesis [7]. The object of this paper is to give a simple proof of this theorem in fact with $(q(T+H))^{\varepsilon}$ replaced by more precise functions on the lines of [5] (see the appendix therein). Our proof resembles to some extent the method of an earlier draft of [6]. The only serious estimate which we use is Theorem 5.9 of [8] due to van der Corput. We also use the functional equation for $L(s, \chi)$ (χ proper) to get an approximate functional equation. Accordingly we prove the following theorem.

THEOREM 2. *Let $\sum_{\chi}^{*}$ denote the sum over all proper characters χ* mod q, *$\varphi(q) = q\prod_{p|q}(1-1/p)$, $d_0(q) = \prod_{p|q}(2-2/p)$, and $T \geqslant T_0$, a large positive constant. Then the following statements hold.*

(a) *Let $q \geqslant 10^8 T$. Then if $0 \leqslant |t| \leqslant T$, we have*

$$\sum_{\chi}^{*} |L(\tfrac{1}{2}+it, \chi)|^2 \ll \varphi(q)\log q. \tag{1}$$

(b) *Let $10^8 T \geqslant q$. Then for $C\log T \leqslant H \leqslant T$ where C is a large positive* *constant, we have*

$$(2)\qquad \sum_{\chi}^{*} \int_{T}^{T+H} |L(\tfrac{1}{2}+it, \chi)|^2\,dt \ll \varphi(q) H \log T + (qT)^{1/4}(\log\log T)^2 + (T/H)^{1/2}(\log T)^{3/2} d_0(q).$$

COROLLARY 1. *For all $H \geqslant 1$ and $H \leqslant T$, we have, for $10^8 T \geqslant q$,*

$$(3)\qquad \sum_{\chi}^{*} \int_{T}^{T+H} |L(\tfrac{1}{2}+it, \chi)|^2\,dt \ll \varphi(q) H \log T + \varphi(q)(\log T)^2 + (\varphi(q)\,T)^{1/3}(\log T)^{4/3}(d_0(q))^{2/3}.$$

COROLLARY 2. *For $t_0 = |t|+2$, $q \geqslant 1$, we have*

$$(4)\qquad \sum_{\chi}^{*} |L(\tfrac{1}{2}+it, \chi)|^2 \ll \varphi(q)\log q + \varphi(q)(\log t_0)^3 + (\varphi(q)\,t_0)^{1/3}(\log t_0)^{7/3}(d_0(q))^{2/3}.$$

Remark. By ignoring all but one character we get an upper bound for $|L(\frac{1}{2}+it, \chi)|$ uniformly in q and t. Some deep estimates for these have been given by D. R. Heath-Brown in his paper [2] by using ideas of van der Corput. However, our upper bound is comparable with Heath-Brown's if $q \leqslant \sqrt{T}$.

§ 2. **Proof of Theorem 2.** We prove this theorem by a series of lemmas. Without loss of generality we can assume that $t \geqslant 0$ in proving part (a) of Theorem 2.

LEMMA 1. *Let $s = \frac{1}{2}+it$, $w = u+iv$, $T/2 \leqslant t \leqslant 3T$, $T \geqslant q^{1/4}$, $L = \log(qT)$, $\tau = L^3$, $h = 10L$ and $X = 10000(qT)^{1/2}$. Let χ be a proper character mod q and let $\psi(s) = \psi(s, \chi)$ be defined by $L(s, \chi) = \psi(s) L(1-s, \bar{\chi})$. Then we have*

$$(5)\qquad L(s, \chi) = \sum_{n=1}^{\infty} \chi(n)\,\mathrm{Exp}(-(n/X)^h)\,n^{-s} + \psi(s) \sum_{n \leqslant X} \bar{\chi}(n)\,n^{s-1} + J(s, \chi) + O((qT)^{-1})$$

where the O-constant is absolute and

$$J(s, \chi) = -\frac{1}{2\pi i}\int_{u=1/4,\,|v|\leqslant\tau} \psi(s+w)\Big(\sum_{n\leqslant X}\bar{\chi}(n)\,n^{s+w-1}\Big)\Gamma\Big(\frac{w}{h}+1\Big)X^w\frac{dw}{w}.$$

Proof. We start with

$$\sum_{n=1}^{\infty} \chi(n)\,\mathrm{Exp}(-(n/X)^h)\,n^{-s} = \frac{1}{2\pi i}\int_{u=2,\,|v|\leqslant\tau} L(s+w, \chi)\,\Gamma\Big(\frac{w}{h}+1\Big)X^w\frac{dw}{w} + O((qT)^{-1}),$$

and then move the line of integration to $u = -5L$. The pole at $w = 0$ contributes $L(s, \chi)$ and the horizontal parts of the contour contribute $O((qT)^{-1})$. We then use the functional equation and the fact that for $\mathrm{Re}(z) \leqslant \frac{3}{4}$, we have

$$(6)\qquad |\psi(z, \chi)| \ll \left(\frac{q(|z|+1)}{2\pi}\right)^{1/2-\mathrm{Re}(z)}.$$

(See the last line of page 340 of [4]. The result stated there is slightly different; but (6) can be proved without much difficulty.) We next break off the portion $\sum_{n\geqslant X}\bar{\chi}(n)\,n^{s+w-1}$ of the series $\sum_{n=1}^{\infty}\bar{\chi}(n)\,n^{s+w-1}$ and estimate this tail portion by $O(X^{(1-h)/2})$. The total contribution from this term is $O((qT)^{-1})$, because there is an extra factor $X^{-h/2}$ coming from X^w. (We have also used the fact that in $-h/2 \leqslant \mathrm{Re}(w) \leqslant 3$ we have uniformly

$$|\Gamma(w/h+1)| \ll \mathrm{Exp}(-|\mathrm{Im}\,w|/h),$$

where the implied constant is absolute.) In the integral containing $\sum_{n\leqslant X}\bar{\chi}(n)\,n^{s+w-1}$, we move the line of integration to $u = \frac{1}{4}$. There will be a pole at $w = 0$ which gives the residue contribution

$$-\psi(s)\sum_{n\leqslant X}\bar{\chi}(n)\,n^{s-1}.$$

This proves the lemma.

Remark. The proof of Lemma 1 is due to M. Jutila who improved on an earlier argument of the second of us. But we cannot give a reference.

LEMMA 2. *Let $q \geqslant 10^8 T$. Then if $T \geqslant q^{1/4}$ and $T/2 \leqslant t \leqslant 3T$, we have*

$$(7)\qquad \sum_{\chi}^{*} |L(\tfrac{1}{2}+it, \chi)|^2 = O(\varphi(q)\log q)$$

where the implied constant is absolute.

Proof. Observe that $X \leqslant q$ and so

$$\sum_{\chi}^{*} \Big|\psi(s)\sum_{n\leqslant X}\bar{\chi}(n)\,n^{s-1}\Big|^2 \leqslant \sum_{\chi}\Big|\sum_{n\leqslant X}\bar{\chi}(n)\,n^{s-1}\Big|^2 \leqslant \varphi(q)\sum_{n\leqslant X}1/n = O(\varphi(q)\log q).$$

Consider the first term on the RHS of (5). The contribution from $n > 2X$ is in absolute value

$$\leqslant \sum_{n>2X} n^{-1/2}\cdot 2(X/n)^h\,\mathrm{Exp}(-\tfrac{1}{2}\cdot 2^h) = O((qT)^{-1}).$$

Hence

$$\sum_{\chi}^{*}\Big|\sum_{n=1}^{\infty}\chi(n)\,\mathrm{Exp}(-(n/X)^h)\,n^{-s}\Big|^2 \ll \sum_{\chi}\Big|\sum_{n\leqslant 2X}\chi(n)\,\mathrm{Exp}(-(n/X)^h)\,n^{-s}\Big|^2 + 1 \ll \varphi(q)\sum_{n\leqslant 2X}1/n \ll \varphi(q)\log q.$$

Next we handle the sum

$$\sum_{\chi}^{*} |J(s,\chi)|^2$$

$$\ll \left(\int_{-\infty}^{\infty} \mathrm{Exp}\left(-\frac{|v|}{h}\right)\frac{dv}{1+|v|}\right) \int_{u=1/4,\,|v|\leqslant\tau} (1+|t+v|)^{-1/2} q^{-1/2} X^{1/2} S \left|\Gamma\left(\frac{w}{h}+1\right)\right| \left|\frac{dw}{w}\right|$$

(where $S = \sum_{\chi} \left|\sum_{n\leqslant X} \bar{\chi}(n) n^{s+w-1}\right|^2 \ll \varphi(q) X^{1/2}$) and so the sum in question is

$$\ll \varphi(q)(\log h)\, T^{1/2} \int_{-\infty}^{\infty} (1+|t+v|)^{-1/2}\, \mathrm{Exp}\left(-\frac{|v|}{h}\right)\frac{dv}{1+|v|}.$$

In the last integral the contribution of the portion $-3t/2 \leqslant v \leqslant -t/2$ is

$$O\left(\mathrm{Exp}\left(-\frac{t}{4h}\right)\int_{-3t/2}^{-t/2} \mathrm{Exp}\left(-\frac{|v|}{2h}\right)\frac{dv}{1+|v|}\right) = O\left(\left(\frac{h}{T}\right)^{1/2}\right)$$

and the contribution of the remaining portion is

$$O\left(T^{-1/2}\int_{-\infty}^{\infty} \mathrm{Exp}\left(-\frac{|v|}{h}\right)\frac{dv}{1+|v|}\right) = O(T^{-1/2}\log h).$$

Hence the sum in question is

$$O\left(\varphi(q)(\log q)^{1/2}\log\log q\right).$$

This proves the lemma.

In an earlier draft of the paper we had in Lemma 1 the condition $T \geqslant 1000$ in place of $T \geqslant q^{1/4}$. It was pointed out by the referee that the argument does not work unless $T \geqslant 2L^3$. So in order to prove part (a) of Theorem 2 we split its proof into two parts according as $T \geqslant q^{1/4}$ or $0 \leqslant t < q^{1/4}$. We show that by a slight modification of the argument used for $T \geqslant q^{1/4}$ we can cover the case $0 \leqslant t < q^{1/4}$ also. (Actually the proof of the case $0 \leqslant t < q^{1/4}$ gives

$$\max_{|t|\leqslant Cq} \sum_{\chi}^{*} |L(\tfrac{1}{2}+it,\chi)|^2 \leqslant D(C)\,\varphi(q)\log q,$$

where $C > 0$ is an arbitrary constant and $D(C)$ depends only on C. But we have retained the proof of the first part since it will be of use in the later sections of the paper.) We give a brief sketch of the proof in this case. Notice that our conditions on T and q imply that q exceeds a large absolute constant. Lemma 1 will now be replaced by

LEMMA 1′. *Let $s = \frac{1}{2}+it$, $w = u+iv$, $0 \leqslant t < q^{1/4}$, $q \geqslant 10^8$, $t_0 = t+2$, and $X = 10000(qt_0)^{1/2}$. Let χ be a proper character* mod q *and let $\psi(s)$ be defined as in Lemma* 1. *Then we have*

(5′)
$$L(s,\chi) = \sum_{n=1}^{\infty} \chi(n) n^{-s}\,\mathrm{Exp}\left(-(n/X)^2\right) + \psi(s)\sum_{n<X}\bar{\chi}(n) n^{s-1} + J_1(s,\chi) + J_2(s,\chi)$$

where

$$J_1(s,\chi) = -\frac{1}{2\pi i}\int_{u=-1} \psi(s+w)\Big(\sum_{n\geqslant X}\bar{\chi}(n) n^{s+w-1}\Big)\Gamma\left(1+\frac{w}{2}\right) X^{w}\frac{dw}{w},$$

$$J_2(s,\chi) = -\frac{1}{2\pi i}\int_{u=1/4} \psi(s+w)\Big(\sum_{n<X}\bar{\chi}(n) n^{s+w-1}\Big)\Gamma\left(1+\frac{w}{2}\right) X^{w}\frac{dw}{w}.$$

Proof. The lemma follows as before, the only difference being that we keep $J_1(s,\chi)$ without replacing it by the estimate $O\left((qT)^{-1}\right)$.

Lemma 2 will now be replaced by

LEMMA 2′. *Let q exceed a large absolute constant and $0 \leqslant t < q^{1/4}$. Then we have*

(7′)
$$\sum_{\chi}^{*} |L(\tfrac{1}{2}+it,\chi)|^2 = O\left(\varphi(q)\log q\right).$$

Proof. We observe that

$$\sum_{\chi}^{*} \Big|\sum_{n\leqslant q}\chi(n) n^{-s}\,\mathrm{Exp}\left(-(n/X)^2\right)\Big|^2 \ll \varphi(q)\log q$$

and that

$$\sum_{\chi}^{*} \Big|\sum_{r=1}^{\infty} \varrho\varrho^{-1} \sum_{rq<n\leqslant(r+1)q} \chi(n) n^{-s}\,\mathrm{Exp}\left(-(n/X)^2\right)\Big|^2$$

($\varrho = \varrho_r > 0$ to be chosen)

$$\leqslant \Big(\sum_{r=1}^{\infty}\varrho^2\Big)\sum_{r=1}^{\infty}\varrho^{-2}\varphi(q)\sum_{rq<n\leqslant(r+1)q}' n^{-1}\,\mathrm{Exp}\left(-(n/X)^2\right)$$

(accent denotes $(n,q)=1$)

$$\leqslant \Big(\sum_{r=1}^{\infty}\varrho^2\Big)\Big(\sum_{r=1}^{\infty}\varrho^{-2}\varphi(q)\sum_{rq<n\leqslant(r+1)q}' X^2 n^{-3}\Big)$$

$\ll \varphi(q)(X/q)^2$ by choosing $\varrho = r^{-1/2}(\log(r+1))^{-1}$

$\ll \varphi(q)$ if $X \ll q$.

Thus

$$\sum_{\chi}^{*} \Big|\sum_{n=1}^{\infty}\chi(n) n^{-s}\,\mathrm{Exp}\left(-(n/X)^2\right)\Big|^2 \ll \varphi(q)\log q \quad \text{if } X \ll q.$$

By a similar reasoning

$$\sum_{\chi}^{*} \Big|\sum_{n<X} \chi(n) n^{-s}\Big|^2 \ll \varphi(q)\log q \quad \text{if } X \ll q.$$

Let $\mathrm{Re}(w) = -1$. Then by the above reasoning

$$\sum_{\chi}^{*} \Big|\sum_{n\geqslant X} \bar{\chi}(n) n^{s+w-1}\Big|^2 \ll \varphi(q)/X^2 \quad \text{if } X \ll q.$$

Also by a suitable application of Hölder's inequality

$$\sum_{\chi}^{*} |J_1(s,\chi)|^2 \ll \int_{-\infty}^{\infty} q^{2}(1+|t+v|)^2 \frac{\varphi(q)}{X^2} \mathrm{Exp}\left(-\frac{|v|}{2}\right)\frac{dv}{X^2}$$

$$\ll \varphi(q)(\sqrt{t_0 q}/X)^4 \ll \varphi(q) \quad \text{if } \sqrt{t_0 q} \ll X.$$

Let $\mathrm{Re}(w) = \frac{1}{4}$. Then (by the above reasoning)

$$\sum_{\chi}^{*} \Big|\sum_{n<X} \bar{\chi}(n) n^{s+w-1}\Big|^2 \ll \varphi(q) X^{1/2} \quad \text{if } X \ll q.$$

Also by a suitable application of Hölder's inequality

$$\sum_{\chi}^{*} |J_2(s,\chi)|^2 \ll \int_{-\infty}^{\infty} q^{-1/2}(1+|t+v|)^{-1/2} \varphi(q) X^{1/2} \mathrm{Exp}(-|v|/2) X^{1/2}\, dv$$

$$\ll \varphi(q)(X/\sqrt{qt_0}) \ll \varphi(q) \quad \text{if } X \ll \sqrt{qt_0}.$$

(We have used $|\psi(z)| \ll \big(q(1+|\mathrm{Im}(z)|)\big)^{1/2-\mathrm{Re}(z)}$ for all z for which $-\frac{1}{2} \leqslant \mathrm{Re}(z) \leqslant \frac{3}{4}$.)

Collecting these results we see that Lemma 2′ is completely proved. Lemma 2 and Lemma 2′ complete the proof of part (a) of Theorem 2.

In the remainder of the proof of Theorem 2 we assume $q \leqslant 10^8 T$.

We now consider $S_1(t) = S_1 = \sum_{n\leqslant X} \bar{\chi}(n) n^{s-1}$ (observe that $|\psi(s)| = 1$). We start with

LEMMA 3. *Let $0 \leqslant u_i \leqslant H/(4\alpha)$ $(i = 1, \ldots, \alpha)$ where α is a natural number at our choice. We put $\alpha = [100 \log T]$ and we have*

$$(8)\quad \sum_{\chi}^{*} \int_{T}^{T+H} |S_1|^2\, dt \leqslant \left(\frac{H}{4\alpha}\right)^{-\alpha} \sum_{\chi} \int_{0}^{H/(4\alpha)} \cdots \int_{0}^{H/(4\alpha)} \Big(\int_{T-V}^{T+H+V} |S_1(t)|^2\, dt\Big) du_1 \ldots du_\alpha$$

where $V = u_1 + \ldots + u_\alpha$ and the integration over $u_1, \ldots, u_\alpha$ is over the α-dimensional cube defined just now.

Proof. This idea has been used extensively by R. Balasubramanian, see [1]. The proof is simple and is left as an exercise to the reader.

We state our result concerning S_1 in Lemma 7. Our treatment of S_1 needs a lengthy but simple discussion based on Lemma 6 to follow. We now take all pairs of integers (m, n) with $1 \leqslant m \leqslant X$, $1 \leqslant n \leqslant X$ and introduce certain squares with sides parallel to the axes in the following way. First we take the square $A_0 = (0 \leqslant x \leqslant q,\ 0 \leqslant y \leqslant q)$. Next we take the squares A_{n+1} of side length $2^{n+1} q$ with the left-hand lower corner (with coordinates) $(\frac{3}{4}q2^n, \frac{3}{4}q2^n)$ for $n = 0, 1, 2, \ldots$ These squares intersect in smaller squares. We stop at the first big square which projects beyond $B = (0 \leqslant x \leqslant X,\ 0 \leqslant y \leqslant X)$. We designate the smaller squares by $B_0, B_1, \ldots$ It is easily seen that the total contribution to (8) from terms of the type $f(m, n)$ defined by

$$(9)\quad f(m,n) = \bar{\chi}(m)\chi(n)(m/n)^{it}$$

with (m, n) in B, but not in A's and B's is

$$(10)\quad \leqslant \left(\frac{H}{4\alpha}\right)^{-\alpha} \varphi(q) \sum{}' \frac{2^{\alpha+1}}{|\log(m/n)|^{\alpha+1}(mn)^{1/2}} = o(1)$$

if $H \geqslant C \log T$ where C is a large positive constant (since $|\log(m/n)|$ is bounded below for the pairs (m, n) in question). The accent denotes the sum restricted by $(m, q) = (n, q) = 1$, and $m \equiv n \pmod q$.

We next consider the contribution to (8) from terms of the type (9) with (m, n) belonging to $A_0, A_1, A_2, \ldots$ and $B_0, B_1, B_2, \ldots$ We denote a typical square (one of A's and B's) by

$$(11)\quad D = (U \leqslant m \leqslant U',\ U \leqslant n \leqslant U')$$

where $(m, q) = (n, q) = 1$, $U' \leqslant \frac{11}{3}U$ and $\frac{3}{4}q \leqslant U \leqslant U' \leqslant X$ except for the square A_0 where this condition is $(1 \leqslant m \leqslant q,\ 1 \leqslant n \leqslant q)$, $(m, q) = (n, q) = 1$. We thus obtain the following.

LEMMA 4. *The right-hand side of (8) is*

$$(12)\quad \leqslant 4\left(\frac{H}{4\alpha}\right)^{-\alpha} \sum_{U}\sum_{\chi} \int_{0}^{H/(4\alpha)} \cdots \int_{0}^{H/(4\alpha)} \int_{T-V}^{T+H+V} |S_2(t)|^2\, dt\, du_1 \ldots du_\alpha + o(1)$$

where $S_2(U, t) = S_2(t) = \sum_{U\leqslant n\leqslant U'} \bar{\chi}(n) n^{s-1}$.

LEMMA 5. *The contribution of the diagonal terms of D defined in (11) to (12) is*

$$(13)\quad O(H\varphi(q)\log T).$$

Proof. We have simply to verify that

$$H\varphi(q) \sum_{n\leqslant X} 1/n = O(H\varphi(q)\log T).$$

This proves the lemma.

The diagonal and the non-diagonal terms of A_0 are disposed of as in Lemma 2. Consider non-diagonal terms coming from D. Since $\left|\log\frac{n}{m}\right| \geqslant \left|\frac{m-n}{m+n}\right|$

the terms with $|m-n| \geqslant \Delta$ (a parameter) contribute to a typical summand coming from U of (12) an amount

$$(14)\qquad \ll \varphi(q)\left(\frac{60\,\alpha U}{H\Delta}\right)^{\alpha}\sum(mn)^{-1/2} \ll (qT)^{-1}$$

provided we choose $\Delta = 100\,\alpha U H^{-1}$. As stated before we impose on H the condition $H \geqslant C\log T$ where C is a large positive constant. Hence the sum of such terms over all U is $O(1)$. Hence it suffices to consider those terms coming from D with

$$(15)\qquad |m-n| \leqslant 100\,\alpha U/H.$$

(Note that the condition $H \geqslant C\log T$ ensures that the RHS of (15) is $\leqslant 10^4 C^{-1} U$ and so $\leqslant \varepsilon U$ for all ε $(0<\varepsilon<1)$ if C is large depending on ε.) Of these we estimate the contributions from those (m, n) with $m > n$. (The terms with $m < n$ can be estimated in a similar way.) We have to estimate

$$(16)\qquad -4i\varphi(q)\sum_{\substack{m\equiv n \,(\mathrm{mod}\, q)\\ m>n,(m,q)=(n,q)=1}}\left(\left(\frac{m}{n}\right)^{i(T+H+V)}-\left(\frac{m}{n}\right)^{i(T-V)}\right)(mn)^{-1/2}\left(\log\frac{m}{n}\right)^{-1}$$

where the sum is subject to (15). We now write $m = n+rq$. Thus it suffices to estimate

$$(17)\qquad 4\varphi(q)\sum_r\sum_n\frac{\mathrm{Exp}\,(it\log(1+rq/n))}{(n(n+rq))^{1/2}\log(1+rq/n)}$$

where the conditions of summation are $(n, q) = 1$, $U \leqslant n \leqslant U'$, $1 \leqslant r \leqslant 100\,\alpha U/(Hq)$, and t denotes a number lying between $T/2$ and $3T$. Because of the summation condition on r it is not hard to verify that for each fixed r, $(n(n+rq))^{1/2}\log(1+rq/n)$ is monotonic in n and lies between two constant multiples of rq. (We can assume that $100\,\alpha U(Hq)^{-1} \geqslant 1$, otherwise there is no r.) The condition $(n, q) = 1$ on n can be replaced by its characteristic function and (17) reads

$$(18)\qquad 4\varphi(q)\sum_r\sum_{d|q}\mu(d)\sum_{U\leqslant nd\leqslant U'}\cdots$$

To estimate the innermost sum in (18) we employ the following important theorem due to van der Corput.

Lemma 6. *If $f(x)$ is twice differentiable and*

$$0<\lambda_2\leqslant f''(x)\leqslant h\lambda_2 \quad (\text{or } \lambda_2\leqslant -f''(x)\leqslant h\lambda_2)$$

throughout the interval (a, b) and $b \geqslant a+1$, then

$$\sum_{a<n\leqslant b}\mathrm{Exp}\,(2\pi i f(n)) = O\left(h(b-a)\lambda_2^{1/2}\right)+O\left(\lambda_2^{-1/2}\right).$$

Remark. We have quoted Theorem 5.9 of [8] in the notation employed there. The h of this lemma should not be confused with the symbol h introduced earlier in this paper.

To estimate the innermost sum in (18) we take

$$f(x) = \frac{t}{2\pi}\log\left(1+\frac{rq}{xd}\right),\qquad a = \frac{U}{d},\qquad b = \frac{U''}{d}$$

where $U \leqslant U'' \leqslant U'$ and we employ partial summation since the quantities multiplying $\mathrm{Exp}\,(2\pi i f(n))$ are monotonic in n. We easily see that $f''(x)$ lies between two positive constant multiples of $Trq(x^3 d)^{-1}$ and so the innermost sum in (18) is $(rq)^{-1}$ times a quantity which is

$$(19)\qquad \ll \frac{U}{d}\left(\frac{Trqd^2}{U^3}\right)^{1/2}+\left(\frac{U^3}{Trqd^2}\right)^{1/2}.$$

We note that if $d > \frac{11}{3}U$ there is no integer n satisfying $U \leqslant nd \leqslant U'$ and so in (19) we have not written $U/d+1$. Hence the sum (18) is majorized by

$$\ll \frac{\varphi(q)}{q}\sum_{d|q}|\mu(d)|\sum_{r\leqslant 100\alpha U(Hq)^{-1}}\frac{1}{r}\left(\left(\frac{Trq}{U}\right)^{1/2}+\left(\frac{U^3}{Trq}\right)^{1/2}\frac{1}{d}\right)$$

$$\ll \frac{\varphi(q)}{q}\sum_{d|q}|\mu(d)|\left(\left(\frac{\alpha T}{H}\right)^{1/2}+\left(\frac{U^3}{Tq}\right)^{1/2}\frac{1}{d}\right)$$

$$\ll \left(\frac{\alpha T}{H}\right)^{1/2}\frac{\varphi(q)}{q}2^{\omega(q)}+\left(\frac{U^3}{Tq}\right)^{1/2}$$

where $\omega(q) = \sum_{p|q} 1$. Summation over U gives a factor $\log T$ for the first term and the total $O\left((qT)^{1/4}\right)$ for the second. Collecting we have

Lemma 7. *Let $S_1(t) = S_1 = \sum_{n\leqslant X}\bar{\chi}(n)n^{s-1}$, $H \geqslant C\log T$ where C is a large positive constant. Then*

$$(20)\qquad \sum_{\chi}{}^{*}\int_{T}^{T+H}|S_1(t)|^2\,dt \ll \varphi(q)H\log T+(qT)^{1/4}+\frac{\varphi(q)}{q}2^{\omega(q)}(\log T)^{3/2}(T/H)^{1/2}.$$

We now turn to the first sum (say $S_3(t)$) on the right of (5). As before we can break off the portion $n \geqslant 2X$ with an error $O\left((qT)^{-1}\right)$. The treatment of the portion $n < 2X$ is similar to the proof of Lemma 7. We have only to observe that the term

$$\frac{\mathrm{Exp}\left(-\left(\frac{n}{X}\right)^h\right)\mathrm{Exp}\left(-\left(\frac{n+r}{X}\right)^h\right)}{(n(n+r))^{1/2}\log(1+r/n)}$$

is a decreasing function of n (for each fixed r such that rn^{-1} does not exceed a small constant), because the numerator is decreasing and the denominator is

$$r\left(1+\frac{r}{n}\right)^{1/2}\left(1-\frac{1}{2}\left(\frac{r}{n}\right)+\frac{1}{3}\left(\frac{r}{n}\right)^2-\dots\right)$$
$$=r\left(1+\frac{1}{2}\left(\frac{r}{n}\right)-\frac{1}{8}\left(\frac{r}{n}\right)^2+\dots\right)\left(1-\frac{1}{2}\left(\frac{r}{n}\right)+\frac{1}{3}\left(\frac{r}{n}\right)^2-\dots\right)$$
$$=r\left(1-\frac{1}{24}\left(\frac{r}{n}\right)^2+\dots\right).$$

Hence we have

LEMMA 8. *Let* $S_3(t)=\sum_{n=1}^{\infty}\chi(n)\operatorname{Exp}(-(n/X)^h)n^{-s}$. *Then the estimates of Lemma* 7 *hold with* $S_1(t)$ *replaced by* $S_3(t)$.

Finally, we have to consider the term $J(s,\chi)$ in (5). We now go back to the proof of Lemma 2. We have

$$\sum_{\chi}{}^{*}|J(s,\chi)|^2\ll\left(\int_{-\infty}^{\infty}\operatorname{Exp}\left(-\frac{|v|}{h}\right)\frac{dv}{1+|v|}\right)$$
$$\times\left(\int_{u=1/4,|v|\le\tau}(1+|t+v|)^{-1/2}q^{-1/2}X^{1/2}S_4\left|\Gamma\left(\frac{w}{h}+1\right)\frac{dw}{w}\right|\right)$$

where $S_4=S_4(t+v)=\sum_{\chi}|\sum_{n\le X}\bar{\chi}(n)n^{s+w-1}|^2$. Since $q\ll T$, we have in the interval $|v|\le\tau$ the inequality $(1+|t+v|)^{-1/2}\ll T^{-1/2}$. Thus

$$(21)\qquad \sum_{\chi}{}^{*}\int_{T}^{T+H}|J(s,\chi)|^2\,dt$$
$$\ll(qT)^{-1/4}(\log h)\int_{u=1/4,|v|\le\tau}\left|\Gamma\left(\frac{w}{h}+1\right)\right|\left(\int_{T}^{T+H}S_4(t+v)\,dt\right)\left|\frac{dw}{w}\right|.$$

Here the innermost integral on the RHS is

$$(22)\qquad \int_{T+v}^{T+H+v}S_4(t)\,dt.$$

This can be estimated from above (uniformly with respect to v) by a method similar to the one adopted in proving Lemma 7. First of all we can integrate with respect to t and restrict to the squares A's and B's with an error $O((qT)^{-1})$. In a typical square D, we can restrict to

$$(23)\qquad |m-n|\le 100\alpha UH^{-1},\quad m\equiv n\ (\mathrm{mod}\,q),\ (m,q)=(n,q)=1.$$

The diagonal terms contribute to (21) an amount which is

$$(24)\qquad \ll(qT)^{-1/4}(\log h)\int_{u=1/4,|v|\le\tau}\left|\Gamma\left(\frac{w}{h}+1\right)\right|\varphi(q)H\sum_{n\le X}n^{-1/2}\left|\frac{dw}{w}\right|$$
$$\ll\varphi(q)H(\log h)^2\ll\varphi(q)H(\log\log T)^2.$$

In the non-diagonal terms it is enough to consider $m>n$. (The portion of the sum with $m<n$ can be treated similarly.) Thus it suffices to estimate as before the sum (we assume $H\ge C\log T$ as before)

$$(25)\qquad 4\varphi(q)\sum_{r}\sum_{n}\frac{\operatorname{Exp}(it\log(1+rq/n))}{(n(n+rq))^{1/4}\log(1+rq/n)}$$

where the conditions of summation are $(n,q)=1$, $U\le n\le U'$, and $1\le r\le 100\alpha U(Hq)^{-1}$ and t denotes a real number lying between $T/2$ and $3T$. (We get an estimate independent of v and multiply it by $(qT)^{-1/4}(\log h)^2$. This when summed up over U and added to (24) gives an estimate for the quantity (21).) Now we can drop the condition $(n,q)=1$ by introducing the characteristic function and consider

$$(26)\qquad 4\varphi(q)\sum_{r}\sum_{d|q}\mu(d)\sum_{U\le nd\le U'}\dots$$

For each fixed r and d the quantity

$$(nd(nd+rq))^{-1/4}\left(\log\left(1+\frac{rq}{nd}\right)\right)^{-1}$$

is monotonic as n varies and lies between two positive constant multiples of $(nd)^{1/2}(rq)^{-1}$. Thus the quantity is

$$\ll\frac{\varphi(q)}{q}\sum_{r\le 100\alpha U(Hq)^{-1}}r^{-1}\sum_{d|q}|\mu(d)|U^{1/2}\left(\left(\frac{Trq}{U}\right)^{1/2}+\left(\frac{U^3}{Trq}\right)^{1/2}\frac{1}{d}\right)$$
$$\ll\frac{\varphi(q)}{q}\sum_{d|q}|\mu(d)|\left(\left(\frac{T\alpha U}{H}\right)^{1/2}+\left(\frac{U^4}{Tq}\right)^{1/2}\frac{1}{d}\right)$$
$$\ll\frac{\varphi(q)}{q}2^{\omega(q)}(\log T)^{1/2}\left(\frac{TU}{H}\right)^{1/2}+\left(\frac{U^4}{Tq}\right)^{1/2}.$$

Summing over U we get a quantity which is

$$\ll\frac{\varphi(q)}{q}2^{\omega(q)}(\log T)^{1/2}(qT)^{1/4}\left(\frac{T}{H}\right)^{1/2}+(qT)^{1/2}.$$

Hence for the non-diagonal terms arising from (21) we have the total contribution

$$\ll\frac{\varphi(q)}{q}2^{\omega(q)}(\log T)^{1/2}(\log\log T)^2\left(\frac{T}{H}\right)^{1/2}+(qT)^{1/4}(\log\log T)^2.$$

Collecting we state

LEMMA 9. *We have with $H \geqslant C\log T$, where C is a certain positive constant,*

$$\sum_{\chi}{}^{*} \int_{T}^{T+H} |J(s,\chi)|^2\,dt \ll \varphi(q)H\log T + \frac{\varphi(q)}{q}2^{\omega(q)}(\log T)^{1/2}(\log\log T)^2\left(\frac{T}{H}\right)^{1/2} + (qT)^{1/4}(\log\log T)^2.$$

Combining Lemmas 7, 8 and 9, we state

LEMMA 10. *We have, with $H \geqslant C\log T$ and $T \geqslant T_0$,*

$$\sum_{\chi}{}^{*} \int_{T}^{T+H} |L(\tfrac{1}{2}+it,\chi)|^2\,dt \ll \varphi(q)H\log T + (qT)^{1/4}(\log\log T)^2 + \frac{\varphi(q)}{q}2^{\omega(q)}(\log T)^{3/2}\left(\frac{T}{H}\right)^{1/2}.$$

Theorem 2 follows from Lemmas 2 and 10 on observing that

$$\frac{\varphi(q)}{q}2^{\omega(q)} = d_0(q).$$

§ 3. Deduction of the corollaries. Let $H > 0$ and $H_1 = H_0 + H + C\log T$ where H_0 is a positive parameter at our choice. Then an upper bound for the sum

$$S_5(H) = S_5 = \sum_{\chi}{}^{*} \int_{T}^{T+H} |L(\tfrac{1}{2}+it,\chi)|^2\,dt$$

is

$$S_5(H_1) \ll \varphi(q)H\log T + \varphi(q)(\log T)^2 + (qT)^{1/4}(\log\log T)^2 + (T/H_0)^{1/2}(\log T)^{3/2}d_0(q) + \varphi(q)H_0\log T.$$

Choose H_0 such that the last two terms on the right become equal. This gives Corollary 1.

To prove Corollary 2 we note (1) and the fact that Corollary 1 holds uniformly in $|\sigma-\frac{1}{2}| \leqslant (\log T)^{-1}$ instead of $\sigma = \frac{1}{2}$, by an imitation of the proof of part (b) of Theorem 2. Now $|L(\frac{1}{2}+it,\chi)|^2$ is majorized by the average value of $|L(z,\chi)|^2$ over the disc $|\frac{1}{2}+it-z| \leqslant (\log T)^{-1}$. We then replace integration over the disc by the square $(|\frac{1}{2}-\mathrm{Re}(z)| \leqslant (\log T)^{-1},\ |t-\mathrm{Im}(z)| \leqslant (\log T)^{-1})$. This gives Corollary 2 on applying Corollary 1.

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Postscript. One of the results of D. R. Heath-Brown (see [2]) is

$$|L(\tfrac{1}{2}+it,\chi)| \ll \left(q^{1/4}(\log q)^{1/2} + (d(q))^3(qt_0)^{1/6}\right)\log(qt_0).$$

Y. Motohashi has informed the second author that he has proved recently the result

$$\int_0^T |L(\tfrac{1}{2}+it,\chi)|^{12}\,dt \ll (qT)^{2+\varepsilon} + q^{3+\varepsilon}$$

and also an asymptotic formula for

$$\int_0^T |L(\tfrac{1}{2}+it,\chi)|^2\,dt$$

with an error term $O\left((qT)^{1/3+\varepsilon} + q^{1/2+\varepsilon}\right)$.

We can obtain (by the methods of the present paper, without seriously changing its contents) the following result. Let (k, l) be an exponent pair in the sense of A. Ivić's book *The Riemann Zeta-Function* (A Wiley-Interscience Publication, 1985, see pages 72 to 78 for an excellent treatment of exponent pairs([1])). Then

$$\sum_{\chi \bmod q} \int_{T}^{T+H} |L(\tfrac{1}{2}+it,\chi)|^2\,dt \ll \left(qH + (qT)^{\lambda}\right)(qT)^{\varepsilon}$$

where $\lambda = \dfrac{(k+l)}{2(k+1)}$, $3 \leqslant H \leqslant T$ and $\varepsilon > 0$ is an arbitrary constant and the constant implied by the Vinogradov symbol $\ll$ depends only on ε. The exponent pair $\left(\dfrac{97}{251}, \dfrac{132}{251}\right)$ given on page 77 of A. Ivić's book gives $\lambda = \dfrac{229}{696} = 0.32902\ldots$ It is also possible to refine $(qT)^{\varepsilon}$ as done in Theorem 2. In Corollary 2 to Theorem 2, the RHS can accordingly be replaced by $(q^{1/2}+(qt_0)^{\lambda/2})(qt_0)^{\varepsilon}$.

For this purpose in place of Lemma 6 (from Titchmarsh's book [8]) we have only to use the definition of the exponent pairs. Accordingly the estimate for (17) will be

$$\ll \frac{\varphi(q)}{q}\sum_{d|q}|\mu(d)| \sum_{r \leqslant 100\alpha U(Hq)^{-1}} \left(\frac{trqd}{U^2}\right)^k \left(\frac{U}{d}\right)^l.$$

Similar treatment for $S_3(t)$ on the right of (5). The estimate for (25) will be

$$\ll \frac{\varphi(q)}{q}\sum_{d|q}|\mu(d)| \sum_{r \leqslant 100\alpha U(Hq)^{-1}} \frac{U^{1/2}}{r}\left(\frac{trqd}{U^2}\right)^k \left(\frac{U}{d}\right)^l.$$

The rest of the proof runs without any serious modification.

([1]) See, however, the review by Heath-Brown, Zbl. 556 # 10026.

Added in proof. We are indebted to the following mathematicians for their friendly help in computing upper bounds for the minimum value of λ as (k, l) runs over all exponent pairs: Professor D. R. Heath-Brown (for his letter deted 25.2.1989), Professor A. Ivić (for his letter dated 8.3.1989), Professor H. L. Montgomery (for his letter dated 17.3.1989, communicating the result of his student Professor S. Graham), Professor M. Jutila (for his letter dated 31.3.1989), Professor M. N. Huxley (for his letters dated 11.4.1989 and 11.5.1989).

The best known upper bound for $\min\lambda$ is $\leqslant \frac{681}{2074} = 0.328365101..$ computed by Professor M. N. Huxley (letter dated 11.5.1989).

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Sur la suite des nombres premiers jumeaux

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I. Introduction et résultats

La conjecture des nombres premiers jumeaux consiste à montrer que la fonction $\pi_2(x)$ définie par

$$\pi_2(x) = |\{p \leqslant x;\ p+2 = p'\}|$$

(p désigne toujours un nombre premier) tend vers $+\infty$ lorsque x tend vers $+\infty$. Plus précisément, Hardy et Littlewood ([12]) ont conjecturé l'équivalence

$$\pi_2(x) \sim Cx\log^{-2}x \quad (x \to +\infty) \tag{1.1}$$

où C désigne le produit infini

$$C = 2\prod_{p\geqslant 3}\left(1-\frac{1}{(p-1)^2}\right).$$

1. Majoration asymptotique de $\pi_2(x)$. L'équivalence (1.1) semble extrêmement difficile à démontrer, il est donc déjà très intéressant de rechercher les constantes b telles qu'on ait l'inégalité

$$\pi_2(x) \leqslant (b+\varepsilon)Cx\log^{-2}x \tag{1.2}$$

pour tout $\varepsilon > 0$ et tout $x > x_0(\varepsilon)$.

On doit à Brun d'avoir démontré l'existence de telles constantes ([3]), les étapes les plus significatives furent ensuite $b = 8$ (Selberg [16]) et $b = 4$ (Bombieri et Davenport [1]). Ce dernier résultat dépend du crible de Selberg en dimension 1 et du théorème de Bombieri–Vinogradov sur la répartition en moyenne des nombres premiers dans les progressions arithmétiques (se reporter à [11] pour une présentation complète).

On sait que, dans les formules générales du crible linéaire (Lemme 2), les fonctions F et f sont optimales; mais Chen ([6]) a remarqué que, dans ce problème, les suites criblées sont très particulières: elles satisfont le principe d'inversion du rôle des variables (*switching principle*) — Chen avait déjà profité de cette propriété pour démontrer que tout entier pair assez grand est somme d'un nombre premier et d'un entier ayant au plus deux facteurs premiers