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Bounds for the degrees in the membership test for a polynomial ideal

by

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0. Introduction. Let $Q=(Q_1,\ldots,Q_m)$ be an m-tuple of polynomials in $C[x_1,\ldots,x_n]$ and let r be a positive integer. We define $D(Q,r)$ as the first integer such that for any polynomial P of degree $\leqslant r$ which belongs to the ideal generated by $Q_1,\ldots,Q_m$ there exist polynomials $A_1,\ldots,A_m$ such that

$$P=A_1Q_1+\ldots+A_mQ_m \quad \text{and} \quad \max_i \deg A_i \leqslant D.$$

Let $d=\max_i \deg Q_i$; a well-known result of G. Hermann [H] shows that

$$D(Q,r)<2(2d)^{2^{n-1}}+r.$$

Moreover, E. Mayr and A. Meyer [M-M] prove that this doubly exponential growth is in general unavoidable. In spite of this, a recent result of W. D. Brownawell [B] suggests the possibility that, under suitable "smoothness" hypotheses, a polynomial bound for the growth of $D(Q,r)$ is available. In fact, W. D. Brownawell shows that in the Nullstellensatz case (i.e. $r=0$) we have the upper bound

$$D(Q,0)\leqslant \max_i \deg A_i \leqslant \min\{n,m\}(nd^{\min\{n,m\}}+d).$$

Let I be the ideal of $C[x_1,\ldots,x_n]$ generated by the polynomials Q_i; let $\bar{I}$ be the homogeneous ideal of $C[x_0,\ldots,x_n]$ generated by the homogenizations ${}^hQ_i$ of the polynomials Q_i. We denote by V the affine variety $\{Q_1=\ldots=Q_m=0\}$ and by $\bar{V}$ the projective variety $\{{}^hQ_1=\ldots={}^hQ_m=0\}$.

In the present paper, using some ideas of W. D. Brownawell and a powerful result from several complex variables developed by H. Skoda, we prove the following result:

THEOREM 1. *If V is a smooth affine variety of dimension $n-m$ and I is the ideal of V, then*

$$D(Q,r)\leqslant (m-1)d+(n-1)m^{n-m}(d-1)^{n-m}d^m+r.$$

If $\bar{V}$ is a smooth projective variety of dimension $n-m$ and $\bar{I}$ is the ideal of $\bar{V}$, then

$$D(Q, r) \leqslant (m-1)(2r-1)+r.$$

1. Lower bounds for $\max_i |Q_i(x)|$. Let $y \in C^s$. We denote by $\|y\|$ the euclidean norm of y and by $\|y\|_*$ the euclidean norm of $(1, y) \in C^{s+1}$. Let

$$JQ(x) = \max_{A \subset [1,\ldots,n], |A|=m} \left| \operatorname{Det}\left(\frac{\partial Q_i}{\partial x_j}\right)_{\substack{i=1,\ldots,m \\ j \in A}} \right|.$$

Our strategy for the proof of Theorem 1 will be to use the main result of H. Skoda ([S], Theorem 1). To do so we shall have to check the convergence of

$$(1) \qquad \int_{C^n} |P(x)|^2 \, \|Q(x)\|^{-2\alpha p-2} \, \|x\|_*^{-2K} \, d\lambda$$

where $p = \min\{m-1, n\}$. Hence it will be necessary to find some lower bounds for $\|Q(x)\|$. Let us consider the following hypotheses on $Q_1, \ldots, Q_m$:

$$(H_1) \qquad \omega \in V \Rightarrow \operatorname{rank} \operatorname{Jac}(Q_i)(\omega) = m,$$

$$(H_2) \qquad \omega' \in \bar{V} \Rightarrow \operatorname{rank} \operatorname{Jac}({}^hQ_i)(\omega') = m.$$

We observe that (H_1) holds if and only if V is a smooth affine variety of codimension m and I is the ideal of V. Likewise (H_2) holds if and only if $\bar{V}$ is a smooth projective variety of codimension m and $\bar{I}$ is the ideal of $\bar{V}$. Notice that under one of the hypotheses (H_i) we have $m \leqslant n$, hence $p = m-1$. In what follows we assume $m \geqslant 2$ and $d \geqslant 2$, since otherwise the assertion of Theorem 1 is obvious. We shall denote by $c_1, c_2, \ldots, c_6$ positive constants depending only on $Q_1, \ldots, Q_m$.

LEMMA 1. *There exists a positive constant c_1 with the following properties. Suppose*

$$\|Q(x)\| < c_1 \|x\|_*^{-T}.$$

Then:

(i) *If (H_1) holds and $T \geqslant (n-1)m^{n-m}(d-1)^{n-m}d^m - 1$, we have*

$$JQ(x) \geqslant c_1 \|x\|_*^{-[(n-1)m^{n-m}(d-1)^{n-m}d^m-1]}.$$

(ii) *If (H_2) holds and $T \geqslant (m-1)(d-1)$, we have*

$$JQ(x) \geqslant c_1 \|x\|_*^{-1}.$$

Proof. (i) Let $\Delta_1, \ldots, \Delta_N$ be the determinants of the $m \times m$ minors of $\operatorname{Jac}(Q)$. The polynomials $Q_1, \ldots, Q_m, \Delta_1, \ldots, \Delta_N$ have no common zero in C^n. By Lemma 2, p. 438 of [M-W], we can find a regular sequence $P_1, \ldots, P_s$ in $(Q_1, \ldots, Q_m, \Delta_1, \ldots, \Delta_N)$ such that:

(a) $P_1, \ldots, P_s$ have no common zero in C^n;

(b) $\max_{j=1,\ldots,s} |P_j(x)| \leqslant c_2 \max(\|Q(x)\|, JQ(x))$;

(c) $\deg P_i \leqslant d$ for $i = 1, \ldots, m$ and $\deg P_i \leqslant m(d-1)$ for $i = m+1, \ldots, s$.

Proposition 8 of [B] ensures

$$\max_{j=1,\ldots,s} |P_j(x)| \geqslant c_3 \|x\|_*^{-t}, \qquad t = [(n-1)m^{n-m}(d-1)^{n-m}d^m - 1].$$

Hence, if $\|Q(x)\| < (c_3/c_2)\|x\|_*^{-t}$ we have $JQ(x) > (c_3/c_2)\|x\|_*^{-t}$.

(ii) Let $\Delta_1, \ldots, \Delta_N$ be the determinants of the $m \times m$ minors of $\operatorname{Jac}({}^hQ)$. The polynomials ${}^hQ_1, \ldots, {}^hQ_m, \Delta_1, \ldots, \Delta_N$ have no common zero in $P(C^n)$. Hence by the compactness of P^n there exists a positive constant c_4 such that

$$\max\left\{\max_{1 \leqslant i \leqslant m} \frac{|Q_i(x)|}{\|x\|_*^{\deg Q_i}}, \max_{1 \leqslant h \leqslant N} \frac{|\Delta_h(1, x)|}{\|x\|_*^{\deg \Delta_h}}\right\} \geqslant c_4.$$

If $\|Q(x)\| < c_4 \|x\|_*^{-1}$, we can find a minor Δ of $\operatorname{Jac}({}^hQ)$ such that

$$|\Delta(1, x)| \geqslant c_4.$$

If $\partial/\partial x_0$ does not appear in Δ we have $JQ(x) > c_4$. Assume

$$\Delta = \operatorname{Det}\left(\frac{\partial\, {}^hQ_i}{\partial x_j}\right)_{\substack{i=1,\ldots,m \\ j=0,\ldots,m-1}}.$$

Using Euler's formula we find

$$\Delta(1, x) = \operatorname{Det}\left((\deg Q_i \cdot Q_i(x))_{i=1,\ldots,m}, \left(\frac{\partial\, {}^hQ_i}{\partial x_j}\right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,m-1}}\right)$$
$$- \sum_{l=1}^{n} x_l \operatorname{Det}\left(\left(\frac{\partial\, {}^hQ_i}{\partial x_l}\right)_{i=1,\ldots,m}, \left(\frac{\partial\, {}^hQ_i}{\partial x_j}\right)_{\substack{i=1,\ldots,m \\ j=1,\ldots,m-1}}\right).$$

Hence

$$|\Delta(1, x)| \leqslant c_5 (\|Q(x)\| \, \|x\|_*^{(m-1)(d-1)} + \|x\|_* JQ(x)).$$

Thus, for $\|Q(x)\| < (c_4/(c_5+1))\|x\|_*^{-(m-1)(d-1)}$ we find $JQ(x) > (c_4/(c_5+1)) \times \|x\|_*^{-1}$. Lemma 1 is proved with $c_1 = \min\{c_3/c_2, c_4, c_4/(c_5+1)\}$. ■

Let $B = L = (n-1)m^{n-m}d^n$ if (H_1) holds and let $B = (n-1)(d-1)$, $L = 1$ if (H_2) holds. Let us define

$$\Omega = \{x \in C^n : \|Q(x)\| < c_1 \|x\|_*^{-B}\}.$$

By Lemma 1 for any $x \in \Omega$ we have $JQ(x) \geqslant c_1 \|x\|_*^{-L}$.

2. Proof of Theorem 1. The results of the previous section allow us to check the convergence of (1).

PROPOSITION 1. *The integral*

$$\int_{C^n} \|Q(x)\|^{-2\gamma} \, \|x\|_*^{-2K} \, d\lambda$$

converges for

$$(2) \qquad \gamma < m, \quad K > n + \max(\gamma B, L-m).$$

Proof. For $K > \gamma B+n$ we have

$$\int_{\Omega^c} \|Q(x)\|^{-2\gamma} \|x\|_*^{-2K} d\lambda \leqslant \int_{C^n} \|x\|_*^{-2[K-\gamma B]} d\lambda < +\infty.$$

Let $\Delta_1, \ldots, \Delta_N$ be as in the proof of Lemma 1(i). For $h = 1, \ldots, n$ let

$$A_h = \{x \in \Omega:\ |\Delta_h(x)| \geqslant c_1 \|x\|_*^{-L}\}.$$

It is enough to show that

$$\int_{A_h} \|Q(x)\|^{-2\gamma} \|x\|_*^{-2K} d\lambda$$

converges for γ and K as in (2) and $h = 1, \ldots, n$. Let $h = 1$, say, and consider

$$f\colon C^n = R^{2n} \to R^{2n} = R^{2m} \times R^{2(n-m)}$$

defined by

$$f(x) = (\Re Q_1(x), \Im Q_1(x), \ldots, \Re Q_m(x), \Im Q_m(x), \Re x_{m+1}, \Im x_{m+1}, \ldots, \Re x_n, \Im x_n).$$

For any $x \in A_1$ we have

$$|\mathrm{Det\,Jac}\, f(x)| \geqslant c_1^2 \|x\|_*^{-2L}. \tag{3}$$

If $(y, w) \in f(A_1)$ the fiber $f^{-1}(y, w)$ is finite and of cardinality $\leqslant \deg({}^hI) \leqslant d^m$: in fact, if $W = \{Q(x)-y = x_{m+1}-w_1 = \ldots = x_n - w_{n-m} = 0\}$, by inequality (3) we have

$$\dim_x W \leqslant \dim \Theta_{x,W} \leqslant 0$$

for any $x \in f^{-1}(y, w)$; hence no component of W of dimension greater than zero intersects A_1.

Thus, using Theorem 3.2.3 on p. 243 of [F] (with $u(x) = \|Q(x)\|^{-2\gamma} \times \|x\|_*^{-2K+2L} \chi_{A_1}$ where χ_{A_1} is the characteristic function of A_1) we find

$$\begin{aligned}\int_{A_1} \|Q(x)\|^{-2\gamma} \|x\|_*^{-2K} d\lambda(x) &\leqslant c_6 \int_{A_1} \|Q(x)\|^{-2\gamma} \|x\|_*^{-2K+2L} |\mathrm{Det\,Jac}\, f(x)|\, d\lambda(x)\\ &= c_6 \int_{f(A_1)} \Big[\sum_{f(x)=(y,w)} \|Q(x)\|^{-2\gamma} \|x\|_*^{-2K+2L}\Big] d\lambda(y, w)\\ &\leqslant d^m c_6 \Big[\int_{\|y\| \leqslant c_1} \|y\|^{-2\gamma} d\lambda(y)\Big]\Big[\int_{C^{n-m}} \|w\|_*^{-2K+2L} d\lambda(w)\Big] < +\infty\end{aligned}$$

for $\gamma < m$ and $K-L > n-m$. ∎

For the reader's convenience, we combine the main result of H. Skoda ([S], Theorem 1) and its application to polynomial ideals due to W. D. Brownawell (see [B], p. 590) into the following proposition:

PROPOSITION 2. *Let $Q_1, \ldots, Q_m$, P be polynomials in $C[x_1, \ldots, x_n]$ with $\max_i \deg Q_i = d$ and $\deg P = r$. If (1) converges for some $K > 0$ and $\alpha > 1$, then there exist $A_1, \ldots, A_m \in C[x_1, \ldots, x_n]$ such that*

$$P = A_1 Q_1 + \ldots + A_m Q_m \tag{4}$$

and

$$\max_i \deg A_i < \alpha(m-1)d + K - n.$$

Proof of Theorem 1. Let $P \in (Q_1, \ldots, Q_m)$ with degree $\leqslant r$; we can find m polynomials $B_1, \ldots, B_m \in C[x_1, \ldots, x_n]$ with degree $\leqslant \delta = D(Q, r)$ such that $P = B_1 Q_1 + \ldots + B_m Q_m$. Hence

$$|P(x)| \leqslant c_7 \|x\|_*^{\delta} \|Q(x)\|$$

where c_7 is some positive constant depending only on $Q_1, \ldots, Q_m$, P. Thus

$$\begin{aligned}&\int_{C^n} |P(x)|^2 \|Q(x)\|^{-2\alpha(m-1)-2} \|x\|_*^{-2K} d\lambda\\ &\qquad\leqslant \int_{C^n} |P(x)|^{2(\alpha-1)m} \|Q(x)\|^{-2\alpha(m-1)-2} \|x\|_*^{-2[K-(1-(\alpha-1)m)r]} d\lambda\\ &\qquad\leqslant c_7^{2(\alpha-1)m} \int_{C^n} \|Q(x)\|^{-2(m+1-\alpha)} \|x\|_*^{-2[K-(\alpha-1)\delta m-(1-(\alpha-1)m)r]} d\lambda.\end{aligned}$$

By Proposition 1, the last integral converges for

$$K > (\alpha-1)\delta m + r[1-(\alpha-1)m] + n + \max((m+1-\alpha)B, L-m).$$

Proposition 2 ensures the existence of polynomials $A_1, \ldots, A_m$ such that (1) holds, with

$$\begin{aligned}\max \deg A_i \leqslant (\alpha-1)\delta m + r[1-(\alpha-1)m]&\\ + \max((m+1-\alpha)B, L-m) + \alpha(m-1)d&.\end{aligned}$$

Hence, for $\alpha \downarrow 1$,

$$\delta \leqslant r + (m-1)d + \max(mB, L-m)$$

and the assertion follows easily. ∎

Remark. Using the new version of Proposition 8 of [B] which has been announced at the Ulm 1987 Journées Arithmétiques we can improve our result to give, under the assumption (H_1), the better bound

$$D(Q, r) \leqslant (m-1)d + m^{n-m}(d-1)^{n-m} d^m + r.$$

Addendum. While revising this paper taking into account the referee's notes, I have had the opportunity of reading a manuscript of C. A. Berenstein and A. Yger (see [B-Y]) in which these authors propose a result stronger than mine and available under the only assumption that the ideal $(Q_1, \ldots, Q_m)$ has codimension m.

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Une nouvelle mesure d'indépendance algébrique pour $(\alpha^\beta, \alpha^{\beta^2})$

par

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1. Introduction, théorème. Soit α un nombre algébrique non nul, pour lequel on a choisi une détermination du logarithme, $\log\alpha$, que l'on suppose non nulle; et soit β un nombre algébrique de degré 3. En 1948, A. O. Gel'fond a démontré l'indépendance algébrique des nombres α^β, α^{β^2}; on s'intéresse ici à leurs mesures d'indépendance algébrique.

Soit K un corps de nombres. Pour un polynôme $P\in K[X_1, X_2]$ on note $d(P)$ son degré total, $H(P)$ sa hauteur naïve, $h(P)$ sa hauteur logarithmique (voir [P2], § I.3, pour la définition et les propriétés de h); les mesures $\log H$ et h sont équivalentes si on se limite à des polynômes à coefficients dans $\mathbf{Z}$.

La première mesure d'indépendance algébrique de $(\alpha^\beta, \alpha^{\beta^2})$ est obtenue en 1950 par A. O. Gel'fond et N. I. Fel'dman (voir [G-F]):

Pour tout $\varepsilon>0$, il existe $t(\varepsilon)>0$ tel que pour tout polynôme $P\in\mathbf{Z}[X_1, X_2]$ vérifiant $d(P)+\log H(P)>t(\varepsilon)$ on ait

$$\log|P(\alpha^\beta, \alpha^{\beta^2})| > -\exp\left((d(P)+\log H(P))^{4+\varepsilon}\right).$$

Ce résultat est amélioré en 1979 par W. D. Brownawell (voir [B], théorème 1) qui remplace dans la minoration ci-dessus $(d(P)+\log H(P))^{4+\varepsilon}$ par $cd(P)^3(d(P)+\log H(P))$. Puis nous avons montré en 1987 en utilisant un critère de P. Philippon ([P1], théorème 2) que l'on peut remplacer $(d(P)+\log H(P))^{4+\varepsilon}$ par $(d(P)+\log H(P))^{2+\varepsilon}$ (voir [D1]).

Ce critère utilisait comme mesure d'une polynôme P la taille $d(P)+h(P)$ et ne permettait pas de séparer degré et hauteur dans les mesures d'indépendance algébrique. Il a été récemment amélioré par P. Philippon et E. M. Jabbouri (voir [J]) et cette nouvelle version permet de démontrer le résultat suivant.

Théorème. *Soient α, β deux nombres algébriques ($\alpha\neq 0$, $\log\alpha\neq 0$, β de degré 3). Il existe deux réels positifs c_1, c_2 ne dépendant que des données α, β tels que pour tout polynôme $P\in\mathbf{Z}[X_1, X_2]$ on ait*

$$\log|P(\alpha^\beta, \alpha^{\beta^2})| \geqslant -c_1\exp\left[c_2 d(P)\left(d(P)+\log H(P)\right)\right].$$