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A generalization of a theorem of Euler for regular chains of complex quadratic irrationalities

by

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The classical theory of continued fractions for real numbers contains many beautiful results and serves as a powerful tool in the study of diophantine approximation. There have been several extensions of this theory to the complex field, in particular to the approximation of complex irrationalities by elements of $\boldsymbol{Q}(i)$. Notable are Hurwitz [4], Auric [1], Ford [3], LeVeque [6] and Poitou [8]. The work of Cassels, Ledermann and Mahler [2] should also be mentioned, although their work did not directly treat the development of a continued fraction. While these papers contain many good results—for example, periodicity of quadratics and expansions for equivalent numbers in [1], and metric theorems for partial quotients in [6]—none of these extensions carry through a substantial part of the real theory.

The regular chains developed by A. Schmidt in [13], [14] offer a more substantial carry-over from the real theory to the complex case. This expansion builds on work in [12] and is discussed in [9]–[11]. In the current paper, results analogous to those of Euler (cf. Perron [7]) on the continued fraction expansion of $\sqrt{d}$ are discussed. These results serve as another illustration of the power of Schmidt's method.

To a complex bilinear map m: $z \mapsto (az+b)/(cz+d)$ associate a matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, and let $\det m$ mean $\det M$. Then m is called *unimodular* if a, b, c, $d \in \boldsymbol{Z}[i]$ and $|\det m| = 1$. Hereafter, the term map refers to a unimodular map. Two numbers ξ and η are called *equivalent* if there exists a map m with $\eta = m(\xi)$.

Regular chains depend on seven special maps: v_1, v_2, v_3, e_1, e_2, e_3, and c. Their matrix forms are

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the selection made for Farey sets. Hence if $F^*(m)$ is circular then $F^*(m) \subset F(m)$ is triangular; if $F(m)$ is triangular, then $F^*(m) \supset F(m)$ is circular, and $\partial F^*(m)$ is the circumscribed circle of $F(m)$. The fundamental dual Farey sets are indicated in Fig. 2.

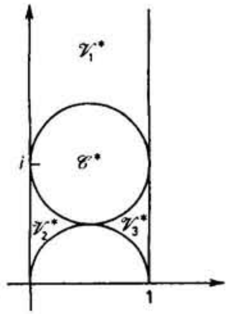

Fig. 2

Then if $\mathscr{F}^* = \{F^*(m) \subset \{z|0 \leqslant \Re z \leqslant 1\}\}$, and $\mathscr{F}_n^*$ is defined analogously to $\mathscr{F}_n$, then $\mathscr{F}^* = \bigcup_{n=0}^{\infty} \mathscr{F}_n^*$, where the union is a disjoint one [13, Lemma 1.6]. Corresponding to each dually regular chain is a *dually regular chain of dual Farey sets* and a unique point of intersection. Conversely, every number in $(\boldsymbol{C} \setminus \boldsymbol{Q}(i)) \cap \{z|0 \leqslant \Re z \leqslant 1\}$ has one (two) dually regular chain(s), denoted by $\operatorname{ch}^* \xi$.

If $\operatorname{ch} \xi = T_0 T_1 T_2 \ldots$, let ξ_k be defined by

$$T_k T_{k+1} \ldots = \begin{cases} \operatorname{ch} \xi_k & \text{if a regular product,} \\ \operatorname{ch}^* \xi_k & \text{if a dually regular product.} \end{cases}$$

Then [13, (2.5)]

$$\xi = \xi_0 = m_{k-1}(\xi_k) = t_0 \circ t_1 \circ \ldots \circ t_{k-1}(\xi_k),$$

which generalizes the real case.

The result of Euler for real quadratics (cf. Perron [7, Satz 3.19]) is as follows.

THEOREM 2. *Let ξ_0 be given by the regular continued fraction*

$$\xi_0 = [a_0; \overline{a_1, a_2, \ldots, a_2, a_1, 2a_0}],$$

with period k. Let A_1/B_1 and A_2/B_2 be the convergents of order $k-3$ and $k-2$, respectively, of $\xi_1 = [a_1; a_2, \ldots, a_1, 2a_0, a_1]$. Then a necessary and sufficient condition that $\xi_0 = \sqrt{d}$, with $d \in \boldsymbol{Z}^+$, is that

$$a_0 = \frac{mA_2 - (-1)^k A_1 B_1}{2},$$

where $m \in \boldsymbol{Z}$. In that case

$$\xi_0 = \sqrt{d} = \sqrt{a_0^2 + mA_1 - (-1)^k B_1^2}.$$

The generalization of this theorem to regular chains is the primary goal of this paper.

Attention will be restricted to nonsquare D in the first quadrant. Then, except for $D \in \{i, 1+i, 1+2i, 2+i, 3i, 4i\}$,

$$\operatorname{ch} \sqrt{D} = V_1^{b_0} E_2 \overrightarrow{T_2 \ldots T_{k+1}}, \tag{4}$$

where $b_0 = [\Im \sqrt{D}]$ and the arrow denotes periodicity [13, 3.4]. The *period* $\overrightarrow{T_2 \ldots T_{k+1}}$ is the shortest repeated product such that $\det T_2 \ldots T_{k+1} = \pm 1$.

The next two lemmas are the analogues for chains of the form

$$\sqrt{d} = [a_0; \overline{a_1, a_2, \ldots, a_{k-1}, 2a_0}]$$

for the real continued fraction expansion [7, Satz 3.9]. This form is derived by showing that $\sqrt{d} + a_0$ is reduced (ξ is reduced if $\xi > 1$ and $-(\xi') \in (0, 1)$). In the complex case, a quadratic ξ is *reduced*, and has a purely periodic regular chain, if $\xi \in \mathscr{I}$ and $\bar{\xi}' \in \mathscr{I}^*$, where the prime denotes algebraic conjugation. The lemmas will employ the fact [13, Theorem 3.5] that the purely periodic regular chain for a reduced ξ is inverse to the purely periodic dually regular chain for $\bar{\xi}'$. What is needed for regular chains is a $P \in \boldsymbol{Z}[i]$ such that $\sqrt{D} + P$ is reduced. The value of P, a kind of "integer part", is determined by the portion of the lattice square in which $\sqrt{D}$ lies. Divide each square into two regions, $\mathscr{A}$ and $\mathscr{B}$, as in Figure 3. No $\sqrt{D}$ lies on the boundary between $\mathscr{A}$ and $\mathscr{B}$, or on the boundaries of the lattice square, since it would then be equivalent to a real number [13, p. 6]. If D is real, then the boundary points are rational integers.

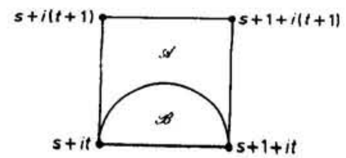

Fig. 3

LEMMA 3. (a) *Suppose $\xi = \sqrt{D} = x + iy$ is in region $\mathscr{A}$. Let $P = [x] + 1 + i[y]$. Then $\xi + P$ is reduced. If $\operatorname{ch} \xi = V_1^{[y]} E_2 (\operatorname{ch} \xi_2)$, then $\operatorname{ch}(\xi + P) = V_1^{2[y]} E_2 V_1^{[x]+1} (\operatorname{ch} \xi_2)$. Hence*

$$\operatorname{ch} \xi_2 = \overrightarrow{T_2 T_3 \ldots V_1^{2[y]} E_2 V_1^{[x]+1}}.$$

(b) *Suppose ξ is in region $\mathscr{B}$.*

(i) *If* $y>1$ *let* $P=[x]+1+i([y]-1)$. *Then* $\xi+P$ *is reduced. If* $\operatorname{ch}\xi = V_1^{[y]}E_2(\operatorname{ch}\xi_2)$, *then* $\operatorname{ch}(\xi+P) = V_1^{2[y]-1}E_2V_1^{[x]+1}(\operatorname{ch}\xi_2)$. *Hence*

$$\operatorname{ch}\xi_2 = \overrightarrow{T_2T_3\dots V_1^{2[y]-1}E_2V_1^{[x]+1}}.$$

(ii) *If* $y<1$, *then there is no* $P\in \mathbf{Z}[i]$ *such that* $\xi+P$ *is reduced. However, if* $P=[x]+1$ *and* $\eta=\xi+P$, *then* $\eta_2=\xi_2+iP$ *is dually reduced,* $\operatorname{ch}\eta = V_1^{[x]+1}(\operatorname{ch}\xi_2)$, *and hence*

$$\operatorname{ch}\xi = E_2\overrightarrow{T_2T_3\dots V_3V_1^{[x]+1}}.$$

Proof. Let $P=a+bi$, so $\xi+P=(x+a)+i(y+b)$ and $\overline{(\xi+P)'} = (-x+a)+i(y-b)$. Then $\xi+P\in\mathcal{I}$ iff $b\geqslant -y$ and $\overline{(\xi+P)'}\in\mathcal{I}^*$ only if $a=[x]+1$ and $b\leqslant y$. If P satisfies these restrictions, then if ξ is in an $\mathcal{A}$ region then the map $z\mapsto\overline{-z+P}$ leaves the collection of $\mathcal{A}$ regions and the collection of $\mathcal{B}$ regions fixed, so $\overline{(\xi+P)'}\in\mathcal{I}^*$. If ξ is in a $\mathcal{B}$ region, then $b\leqslant[y]-1$ is forced, but then if $y<1$, $b\geqslant -y$ cannot be satisfied.

The form for $\operatorname{ch}\xi_2$ in (a) follows from the fact that ξ_2 is dually reduced, and hence purely periodic, which is true for every D. For let $\xi=\xi_0=x+iy$. Then

$$\xi_1 = v_1^{-[y]}(\xi_0) = x+i(y-[y]),$$
$$\xi_2 = e_2^{-1}(\xi_1) = i\xi_1+1-i = ([y]-y+1)+i(x-1)\in\mathcal{I}^*,\quad\text{and}$$
$$\overline{\xi_2'} = ([\bar{y}]+y+1)+i(x+1)\in\mathcal{I}.$$

Part (b)(i) is proved similarly. If $y<1$ and ξ is in $\mathcal{B}$, let $\eta=\xi+[x]+1$, so $\operatorname{ch}\eta = E_2V_1^{[x]+1}(\operatorname{ch}\xi_2)$. Then η_2 is dually reduced, so it has a purely periodic dually regular chain. Since $\overline{\eta_2'} = (1+y)+i(x-[x])\in\mathcal{V}_3$,

$$\operatorname{ch}\eta_2 = \overrightarrow{U_2U_3\dots V_3^l}.$$

But $\operatorname{ch}\eta_2 = V_1^{[x]+1}(\operatorname{ch}\xi_2)$, so

$$\operatorname{ch}\xi_2 = \overrightarrow{T_2T_3\dots V_3^lV_1^{[x]+1}}. \ \blacksquare$$

One may view the first quadrant as being (partially) tiled by translations of the 4-sided region in Fig. 4.

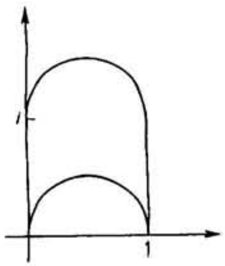

Fig. 4

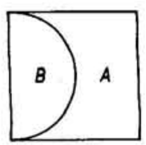

Fig. 5

Then P is the lower right vertex of the tile in which ξ falls. The numbers of part (b)(ii) are in that portion of the first quadrant which is not tiled. The numbers in the tiled portion – those covered by Lemma 3(a) and (b)(i) – will be called *type* (1). The numbers in the untiled portion of the first quadrant will be called *type* (2). Note that all real quadratics in the first quadrant are type (2).

For further information on the form of $\operatorname{ch}\sqrt{D}$, divide each lattice square into two regions, A and B, as in Fig 5. As in the case of regions $\mathcal{A}$ and $\mathcal{B}$, $\sqrt{D}$ cannot lie on the boundaries.

LEMMA 4. *Let* $\operatorname{ch}\sqrt{D} = V_1^{[y]}E_2\overrightarrow{T_2\dots T_kV_1^{[x]+1}}$ *and let*

$$h = \begin{cases}[x]-1 = \Re P-2 & \text{if } \xi\in A,\\ [x]-2=\Re P-3 & \text{if } \xi\in B.\end{cases}$$

Then

$$T_2T_3T_4\dots = \begin{cases}UT_3\dots, & h=0,\\ V_1^hUT_4\dots, & h>0,\end{cases}$$

where U *is of the form:*

$$U\begin{cases}\in\{C\}\cup\big(V_F\setminus\{\prod_{l=1}^{n}V_1\}_{n=1}^{\infty}\big) & \text{if } \xi\in A,\\ =C & \text{if } \xi\in B.\end{cases}$$

Proof. If $\sqrt{D}=x+iy$, then

$$\xi_2 = (v_1^{[y]}\circ e_2)^{-1}(\xi) = (\sqrt{D}-i[y]-1-i)/(-i)$$
$$= (1+[y]-y)+i(x-1).$$

From $0\leqslant 1+[y]-y\leqslant 1$ it follows that if $\xi\in B$, then

$$v_1^{-[x]+2}(\xi_2) = (1+[y]-y)+i(x+1-[x])\in\mathcal{C}^*$$

and if $\xi\in A$, then $v_1^{-[x]+1}(\xi_2)\in\mathcal{C}^*\cup\mathcal{V}_2^*\cup\mathcal{V}_3^*$. $\blacksquare$

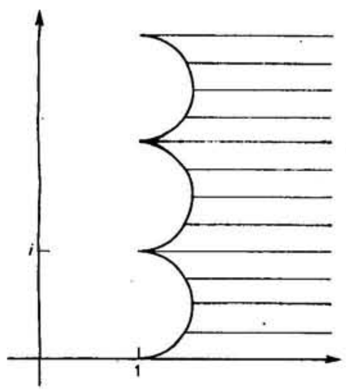

Fig. 6

While attention has been restricted to D in the first quadrant, Lemmas 3 and 4 and the results that follow apply to all D for which $\operatorname{ch}\sqrt{D}$ has the form of (4). These are all D such that $\sqrt{D}$ is in the region of Fig. 6.

Using the powers of V_1 given by Lemmas 3 and 4, (4) can be rewritten as

$$\operatorname{ch}\sqrt{D} = \begin{cases} V_1^{b_0} E_2 V_1^h \overrightarrow{(M_c V_1^l E_2 V_1^m)}. & \text{for type (1) numbers;} \\ E_2 V_1^h \overrightarrow{(M_c V_1^m)}, & \text{for type (2) numbers,} \end{cases}$$

where M_c denotes the "central" portion of the period. Note that it follows from Lemma 4 that M_c does not begin with V_1 and from the proof of Lemma 3 that it does not end with V_1. Although the periodicity in $\operatorname{ch}\sqrt{D}$ begins with ξ_2, it will be useful in the results that follow to consider the period commencing with M_c. In what follows, chains will be indexed slightly differently than in (2). To highlight the similarity to the continued fraction expansions of real quadratics, these chains will be indexed as

$$\operatorname{ch}\sqrt{D} = T_0 \overrightarrow{M_c T_k},$$

where for type (1) numbers $T_0 = V_1^{b_0} E_2 V_1^h$ and $T_k = V_1^l E_2 V_1^m$, and for type (2) numbers $T_0 = E_2 V_1^h$ and $T_k = V_1^m$. In both cases $M_c = T_1 \dots T_{k-1}$.

Summarizing Lemmas 3 and 4, for ξ of type (1), $P = a+ib = [x]+1+ib$, there are 4 possible forms for $\operatorname{ch}\xi$, corresponding to 4 regions:

$$\operatorname{ch}\xi = \begin{cases} V_1^b E_2 V_1^{a-2} \overrightarrow{(M_c V_1^{2b} E_2 V_1^{2a-2})}, \ b = [y], & \xi \in \mathscr{A}A, \\ V_1^b E_2 V_1^{a-3} \overrightarrow{(M_c V_1^{2b} E_2 V_1^{2a-3})}, \ b = [y], & \xi \in \mathscr{A}B, \\ V_1^{b+1} E_2 V_1^{a-2} \overrightarrow{(M_c V_1^{2b+1} E_2 V_1^{2a-2})}, \ b = [y]-1, & \xi \in \mathscr{B}A, \\ V_1^{b+1} E_2 V_1^{a-3} \overrightarrow{(M_c V_1^{2b+1} E_2 V_1^{2a-3})}, \ b = [y]-1, & \xi \in \mathscr{B}B. \end{cases}$$

Since

$$V_1^l E_2 V_1^m = \begin{bmatrix} 1 & -(l+1)+(m+1)i \\ 0 & i \end{bmatrix},$$

this can be written as

$$\operatorname{ch}\xi = \begin{bmatrix} 1 & iP-\delta \\ 0 & i \end{bmatrix} \overrightarrow{M_c \begin{bmatrix} 1 & 2iP-\delta \\ 0 & i \end{bmatrix}}, \tag{5}$$

where δ is given by

$$\delta = \begin{cases} 1+i, & \xi \in \mathscr{A}A, \\ 1+2i, & \xi \in \mathscr{A}B, \\ 2+i, & \xi \in \mathscr{B}A, \\ 2+2i, & \xi \in \mathscr{B}B. \end{cases} \tag{6}$$

For ξ of type (2), $P = a = [x]+1$, there are two cases:

$$\operatorname{ch}\xi = \begin{cases} E_2 V_1^{a-2} \overrightarrow{(M_c V_1^{2a-2})}, & \xi \in \mathscr{B}A, \\ E_2 V_1^{a-3} \overrightarrow{(M_c V_1^{2a-3})}, & \xi \in \mathscr{B}B, \end{cases}$$

where M_c ends in a power of V_3. This can be written as

$$\operatorname{ch}\xi = E_2 \begin{bmatrix} 1 & iP-\delta \\ 0 & 1 \end{bmatrix} \overrightarrow{M_c \begin{bmatrix} 1 & 2iP-\delta \\ 0 & 1 \end{bmatrix}}. \tag{7}$$

where δ is given by

$$\delta = \begin{cases} 2i, & \xi \in A, \\ 3i, & \xi \in B. \end{cases}$$

The form of (7) and role of δ is slightly different than in (5) because of the different role played by P.

The symmetry of the central part of the period of the expansion of $\sqrt{d}$ [7, Satz 3.9, for example] does not generalize to regular chains. This central symmetry for the continued fraction expansion follows from the form for inverse periods and the nature of reduced numbers. The conjugation, in the complex case, rules out the kind of simple connection holding between $1/(\sqrt{d}-a_0)$ and $\sqrt{d}+a_0$ in the real case, from which the central symmetry follows.

The task now is to find the relationship between M_c and D, generalizing Theorem 2. This is contained in the following two results.

THEOREM 5. *Suppose that $\xi = \sqrt{D}$ is of type* (1). *Let $P = a+ib$ be as in Lemma* 3(a), (b)(1). *Then a necessary and sufficient condition that*

$$\operatorname{ch}\xi = \begin{bmatrix} 1 & iP-\delta \\ 0 & i \end{bmatrix} \overrightarrow{M_c \begin{bmatrix} 1 & 2iP-\delta \\ 0 & i \end{bmatrix}}$$

be the regular chain for ξ is that P have the form

$$P = \tfrac{1}{2}[id^{-1} A_1 (A_2 - \delta B_2) + m B_1], \tag{8}$$

where $m \in \mathbf{Z}[i]$, $d = \det M_c$, $P \in \{z \mid \Re z > 0,\ \Im z \geqslant 0\}$ and

$$M_c = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}, \quad \delta = \frac{A_1 - iB_2}{B_1} \in \{1+i,\ 1+2i,\ 2+i,\ 2+2i\}.$$

In that case,

$$D = P^2 + id^{-1} A_2 (A_2 - \delta B_2) + m B_2. \tag{9}$$

Proof. From Lemmas 3 and 4 it follows that $\xi_k = \xi_0 + P$. Then from (5) it follows that

$$M_{k-1} = \begin{bmatrix} p_1^{(k-1)} & p_2^{(k-1)} \\ q_1^{(k-1)} & q_2^{(k-1)} \end{bmatrix} = \begin{bmatrix} 1 & iP-\delta \\ 0 & i \end{bmatrix} M_c = \begin{bmatrix} A_1+(iP-\delta)B_1 & A_2+(iP-\delta)B_2 \\ iB_1 & iB_2 \end{bmatrix}. \tag{10}$$

The identity

$$\xi_0 = m_{k-1}(\xi_k) = \frac{p_1^{(k-1)}(\xi_0+P)+p_2^{(k-1)}}{q_1^{(k-1)}(\xi_0+P)+q_2^{(k-1)}}$$

yields

$$q_1^{(k-1)}\xi_0^2 + (Pq_1^{(k-1)}+q_2^{(k-1)})\xi_0 = p_1^{(k-1)}\xi_0 + Pp_1^{(k-1)} + p_2^{(k-1)}. \tag{11}$$

Since $\xi_0^2 = D$, it follows that the linear term in (11) is 0, hence

$$Pq_1^{(k-1)} + q_2^{(k-1)} = p_1^{(k-1)}, \tag{12}$$

and (11) reduces to

$$q_1^{(k-1)} D = Pp_1^{(k-1)} + p_2^{(k-1)}. \tag{13}$$

However, $q_1^{(k-1)}$, $p_1^{(k-1)}$ and $p_2^{(k-1)}$ depend on P, so (13) is rewritten using (10):

$$D(iB_1) = P(A_1+(iP-\delta)B_1) + A_2 + (iP-\delta)B_2,$$

and thus

$$iB_1(D-P^2) - (A_1-\delta B_1+iB_2)P = A_2-\delta B_2. \tag{14}$$

Similarly, it follows from (12) and (10) that $P(iB_1)+iB_2 = A_1+(iP-\delta)B_1$, and thus

$$iB_2 = A_1-\delta B_1, \tag{15}$$

which together with (14) yields

$$B_1(D-P^2) - B_2(2P) = -i(A_2-\delta B_2). \tag{16}$$

The solution of this diophantine equation in $D-P^2$ and $2P$ will yield values of D and P for given M_c. The necessary condition on M_c, namely,

$$\delta = \frac{A_1-iB_2}{B_1} \in \{1+i,\ 1+2i,\ 2+i,\ 2+2i\},$$

follows from (5), (6) and (15). A particular solution of (16) is

$$D-P^2 = id^{-1}A_2(A_2-\delta B_2), \quad 2P = id^{-1}A_1(A_2-\delta B_2),$$

from which the general solution in (8) and (9) follows. ■

THEOREM 6. *Suppose* $\xi = \sqrt{D}$ *is of type* (2). *Let* $P = [x]+1$. *Then a necessary and sufficient condition that*

$$\operatorname{ch}\xi = E_2 \begin{bmatrix} 1 & iP-\delta \\ 0 & 1 \end{bmatrix} \overrightarrow{M_c \begin{bmatrix} 1 & 2iP-\delta \\ 0 & 1 \end{bmatrix}}$$

be the regular chain for ξ *is that* P *have the form*

$$P = -\tfrac{1}{2} i[-d^{-1}A_1(A_2-\delta B_2)+mB_1] \in \mathbf{Z}^+,$$

where $m \in \mathbf{Z}[i]$, $d = \det M_c$, *and*

$$M_c = \begin{bmatrix} A_1 & A_2 \\ B_1 & B_2 \end{bmatrix}, \quad \delta = \frac{A_1-B_2}{B_1} - 2(1-i) \in \{2i, 3i\}.$$

In that case,

$$D = P^2 + 2i + d^{-1}[(1-i)A_1+A_2](A_2-\delta B_2) - m[(1-i)B_1+B_2].$$

Proof. Return, for this proof, to the chain indexing of Lemma 3. In this case, $\xi_k = \xi_2 + iP$. Then $\xi_1 = \xi_0$,

$$\xi_2 = e_2^{-1}(\xi_1) = (\xi_1-1-i)/(-i) = (\sqrt{D}-1-i)/(-i),$$

so $\xi_2\xi_2' = D-2i$, $\xi_2+\xi_2' = 2(1-i)$, and ξ_2 satisfies

$$\xi_2^2 - 2(1-i)\xi_2 + D - 2i = 0. \tag{17}$$

From (7) it follows that

$$M' = \begin{bmatrix} p_1 & p_2 \\ q_1 & q_2 \end{bmatrix} = \begin{bmatrix} 1 & iP-\delta \\ 0 & 1 \end{bmatrix} M_c = \begin{bmatrix} A_1+(iP-\delta)B_1 & A_2+(iP-\delta)B_2 \\ B_1 & B_2 \end{bmatrix}. \tag{18}$$

From

$$\xi_2 = m'(\xi_k) = m'(\xi_2+iP) = \frac{p_1(\xi_2+iP)+p_2}{q_1(\xi_2+iP)+q_2}$$

follows

$$q_1\xi_2^2 + (iPq_1+q_2-p_1)\xi_2 = iPp_1+p_2,$$

from which it follows, using (17), that

$$-q_1(D-2i) = iPp_1+p_2, \tag{19}$$

$$iPq_1+q_2-p_1 = -2(1-i)q_1. \tag{20}$$

Substituting in (19) and (20) using (18) yields

$$-B_1(D-2i-P^2) - iP(A_1-\delta B_1+B_2) = A_2-\delta B_2, \tag{21}$$

$$A_1-\delta B_1 = 2(1-i)B_1+B_2, \tag{22}$$

respectively. Substituting (22) in (21) yields a diophantine equation in $D-2i-P^2$ and $2iP$:

$$B_1(D-2i-P^2)+[(1-i)B_1+B_2](2iP) = -(A_2-\delta B_2). \tag{23}$$

The form for δ follows from (20). The equation (23) is solved as in Theorem 5. ∎

Theorems 5 and 6 are the analogue for regular chains of Theorem 2. Whereas examples of Theorem 2 can be organized fairly naturally by the size of k, things are a bit more complicated here. Symmetry of M_c is not available in general, and the choice of a partial quotient corresponds to the choice among 7 maps, followed, in the case of the V_j, by a choice of exponent.

For both type (1) and type (2) numbers, M_c is a dually regular product. In the type (1) case, $\det M_c = \pm i$, whereas in the type (2) case, $\det M_c = \pm 1$. The case of $\operatorname{len}(M_c) = 0$ is ruled out for type (1), since $\det I = 1$, and for type (2), since M_c must end in a power of V_3, so $\operatorname{len}(M_c) \geqslant 1$.

Consider now the case of $\operatorname{len}(M_c) = 1$. Let ξ be of type (1). Then $M_c = C$, since the form of T_k rules out E_j. Thus $d = -i$, $\delta = 1+i$, so all these numbers are in $\mathscr{A}A$, and

$$P = \tfrac{1}{2}m(1-i) = \frac{m}{1+i}, \quad D = P^2+mi = \left(\frac{m}{1+i}\right)^2+mi.$$

Let $m = \hat{m}(1+i)$, $\hat{m} = a+bi \in \mathbf{Z}[i]$. Then

$$P = \hat{m} = a+bi, \quad D = \hat{m}^2+(-1+i)\hat{m},$$

$$\operatorname{ch}\sqrt{D} = V_1^b E_2 V_1^{a-2}\overrightarrow{CV_1^{2b}E_2V_1^{2a-2}};$$

for example,

$$P = 2, \qquad \operatorname{ch}\sqrt{2+2i} = E_2\overrightarrow{CE_2V_1^2},$$

$$P = 3+i, \qquad \operatorname{ch}\sqrt{4+8i} = V_1E_2V_1\overrightarrow{CV_1^2E_2V_1^4}.$$

Now consider the type (2) case with $\operatorname{len}(M_c) = 1$. The only length 1 possibility is $M_c = V_3^h$, since M_c must end in V_3. Consider first $h = 1$. Then $d = 1$, $\delta = 2i$, and

$$P = -\tfrac{1}{2}i(-1+3i-mi), \quad D = P^2-1.$$

Since P is real in the type (2) case, $m = n+i$, for $n \in \mathbf{Z}$, so $P = \frac{1}{2}(3-n)$. Choose n so that P is positive to get

$$\operatorname{ch}\sqrt{P^2-1} = E_2V_1^{P-2}\overrightarrow{V_3V_1^{2(P-1)}}.$$

This is to be compared with the continued fraction expansion $\sqrt{P^2-1} = [P-1; \overrightarrow{1, 2(P-1)}]$ and (3). More generally, let $M_c = V_3^h$. Then

$$V_3^h = \begin{bmatrix} 1-hi & hi \\ -hi & 1+hi \end{bmatrix},$$

$\delta = 2i$, $d = 1$ and $P = -\frac{1}{2}i[-h^2+i(2h^2-h+2)-mhi]$. Since P must be real, let $m = n+hi$, so $2P = 2h^2-(n+1)h+2$. Let $m' = 2(P-1)/h = 2h-(n+1) \in \mathbf{Z}$, so $D = (P-1)^2+m'$. Hence

$$\operatorname{ch}\sqrt{(P-1)^2+m'} = E_2V_1^{P-2}\overrightarrow{V_3^hV_1^{2(P-1)}},$$

where $h = 2(P-1)/m'$. This is exactly the case of period length $k = 2$ in Perron [7, p. 89], which, in fact, includes case $k = 1$ as a degenerate case.

Although Theorems 5 and 6 treat all numbers of the form $\sqrt{D}$, some of these chains have additional structure, which simplifies their analysis. A regular chain $T_1T_2\ldots$ is said to be of *type* (a) if there exists an $l \geqslant 1$ such that $T_{l+1}T_{l+2}\ldots$ is obtained from $T_1T_2\ldots$ by a cyclic permutation of subscripts on the V_j, E_j. If l is the smallest such index, then $T_1T_2\ldots$ is periodic with period length $k = 3l$. Hence all such numbers are quadratic. The portion of the period $T_1T_2\ldots T_l$ will be called the *generating product* of the period.

Let

$$S = \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix};$$

then $S^3 = I$, $SV_jS^2 = V_{j+1}$, $SE_jS^2 = E_{j+1}$ and $SCS^2 = C$, where the subscripts are considered (mod 3). Then $\overrightarrow{T_1T_2\ldots T_k}$ is of type (a) if there exist l, k and $j \in \{1, 2\}$ such that

$$S^jT_nS^{-j} = T_{l+n}, \quad n = 1, 2, \ldots,$$

and hence

$$S^jT_1T_2\ldots T_lS^{-j} = T_{l+1}\ldots T_{2l}, \quad S^jT_{l+1}\ldots T_{2l}S^{-j} = T_{2l+1}\ldots T_k.$$

For $j = 1$, the permutation is (1 2 3); call these *type* (a1); for $j = 2$ the permutation is (1 3 2); call these *type* (a2). All other quadratics will be called *type* (b).

For type (a2) numbers, if $\xi_k = \xi_0+P$ is as in Theorem 5, and l is the length of the generating product, then $s(\xi_l) = \xi_k$, so $s(\xi_l) = \xi_0+P$ and $s^2(\xi_0+P) = \xi_l$. In vector form this is

$$\begin{bmatrix} \xi_l \\ 1 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \xi_0+P \\ 1 \end{bmatrix}$$

so the analogue of (10) is

$$\begin{bmatrix} \xi_0 \\ 1 \end{bmatrix} = \begin{bmatrix} p_1^{(l-1)} & p_2^{(l-1)} \\ q_1^{(l-1)} & q_2^{(l-1)} \end{bmatrix}\begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}\begin{bmatrix} \xi_0+P \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & iP-\delta \\ 0 & i \end{bmatrix}\begin{bmatrix} -(A_1+A_2) & A_1 \\ -(B_1+B_2) & B_1 \end{bmatrix}\begin{bmatrix} \xi_0+P \\ 1 \end{bmatrix}.$$

Thus Theorem 5 can be redone with the replacements

$$A_1 \mapsto -(A_1+A_2), \quad A_2 \mapsto A_1,$$
$$B_1 \mapsto -(B_1+B_2), \quad B_2 \mapsto B_1,$$

where now

$$M_c S^2 \begin{bmatrix} 1 & 2iP-\delta \\ 0 & i \end{bmatrix} S$$

is the generating product, and the full period is

$$M_c S^2 \begin{bmatrix} 1 & 2iP-\delta \\ 0 & i \end{bmatrix} S \times S^2 M_c SS \begin{bmatrix} 1 & 2iP-\delta \\ 0 & i \end{bmatrix} S^2 \times SM_c S^2 \begin{bmatrix} 1 & 2iP-\delta \\ 0 & i \end{bmatrix}.$$

The situation is similar for type (a1) numbers. These results can be summarized in the following.

THEOREM 7. *Let ξ, P and M_c be as in Theorem 5. Then necessary and sufficient conditions that M_c be the central portion of the generating product for the period of ξ are as follows.*

For type (a2) *numbers*:

$$P = \tfrac{1}{2}[-id^{-1}(A_1+A_2)(A_1-\delta B_1)-m(B_1+B_2)],$$
$$D = P^2+id^{-1}A_1(A_1-\delta B_1)+mB_1, \quad m \in \mathbf{Z}[i],$$
$$\delta = \frac{A_1+A_2+iB_1}{B_1+B_2} \in \{1+i, 1+2i, 2+i, 2+2i\}.$$

For type (a1) *numbers*:

$$P = \tfrac{1}{2}[-id^{-1}A_2(A_1+A_2-\delta(B_1+B_2))+mB_2],$$
$$D = P^2+id^{-1}(A_1+A_2)(A_1+A_2-\delta(B_1+B_2))-m(B_1+B_2), \quad m \in \mathbf{Z}[i],$$
$$\delta = \frac{A_2+i(B_1+B_2)}{B_2} \in \{1+i, 1+2i, 2+i, 2+2i\}.$$

EXAMPLE. As in the case for type (b) numbers, the only length 1 possibility for either type (a1) or type (a2) numbers is $M_c = C$. In the type (1)(a2) case, $d = -i$, $\delta = 1+2i$. Then

$$P = \tfrac{1}{2}(1-2i-m), \quad D = P^2+2+i+m(1-i).$$

A simple special case is when $m = -2\hat{m}-1$, for $\hat{m} \in \mathbf{Z}[i]$. Then if $\hat{m} = (a-1)+(b+1)i$, we have $P = \hat{m}+1-i = a+bi$, $D = \hat{m}^2+1$, so

$$\operatorname{ch}\sqrt{\hat{m}^2+1} = V_1^b E_2 V_1^{a-3} \overrightarrow{CV_3^{2b} E_1 V_3^{2a-3}}(a2).$$

The arrow of periodicity followed by (an) will denote the generating product for a type (a) number, with $n \in \{1, 2\}$ indicating the subscript permutation.

EXAMPLE. Let $M_c = C$, for $\sqrt{D}$ a type (1)(a1) number, so $d = -i$, $\delta = 2+i$. Then

$$P = 1-i+\tfrac{1}{2}mi, \quad D = P^2+2i-m.$$

A special case is when $m = -2i(\hat{m}-1)$. Let $\hat{m} = a+(b+1)i$, so $P = \hat{m}-i = a+bi$, $D = \hat{m}^2-1$, and thus

$$\operatorname{ch}\sqrt{\hat{m}^2-1} = V_1^{b+1} E_2 V_1^{a-2} \overrightarrow{CV_2^{2b+1} E_1 V_2^{2a-2}}(a1).$$

The form for the generating block of type (2) quadratic numbers is derived in a similar manner.

THEOREM 8. *Let ξ and P be as in Theorem 6. Then necessary and sufficient conditions that M_c be the central portion of the generating product for the period of ξ are as follows.*

For type (a2) *numbers*:

$$P = -\tfrac{1}{2}i[d^{-1}(A_1+A_2)(A_1-\delta B_1)+m(B_1+B_2)],$$
$$D = P^2+2i+d^{-1}[-(1-i)(A_1+A_2)+A_1](A_1-\delta B_1)$$
$$+m[B_1-(1-i)(B_1+B_2)],$$
$$\delta = \frac{(A_1+A_2)+B_1}{B_1+B_2} - 2(1-i) \in \{2i, 3i\}.$$

For type (a1) *numbers*:

$$P = -\tfrac{1}{2}i[d^{-1}A_2((A_1+A_2)-\delta(B_1+B_2))+mB_2],$$
$$D = P^2+2i+d^{-1}[-(1-i)A_2+(A_1+A_2)]$$
$$\times[(A_1+A_2)-\delta(B_1+B_2)]-m[(1-i)B_2-(B_1+B_2)],$$
$$\delta = \frac{A_2+(B_1+B_2)}{B_2} - 2(1-i) \in \{2i, 3i\}.$$

EXAMPLE. For type (2) numbers, there are no type (a1) or type (a2) numbers with $\operatorname{len}(M_c) = 1$, since $\delta \notin \{2i, 3i\}$ for any of the possible M_c. There are no type (2) numbers of types (b), (a1) or (a2) for which $\operatorname{len}(M_c) = 2$. Let

$$M_c = CE_1 V_2 = \begin{bmatrix} 3i & -1-i \\ 2+i & -1 \end{bmatrix},$$

so $\delta = 3i$, $d = 1$. Then for type (2)(a2) numbers,

$$P = -\tfrac{1}{2}i(3+9i+m(1+i)), \quad D = P^2-3+i+mi.$$

If $m = n(1+i)-3$, for $n \in \mathbf{Z}$, then

$$P = n+3, \quad D = (n+2)(n+3+i),$$

so

$$\operatorname{ch}\sqrt{(n+2)(n+3+i)} = E_2 V_1^{n-3} \overrightarrow{CE_1 V_2 V_3^{2n+3}}(a2).$$

For example, for $n = 0$, $P = 3$, $D = 6+2i$,

$$\operatorname{ch}\sqrt{6+2i} = E_2\overrightarrow{CE_1 V_2 V_3^3}\ \text{(a2)}.$$

The possibilities for type (1) numbers with $\operatorname{len}(M_c) = 2$ are: (b): CV_2^h, $V_2^h C$; (a2): CV_2^h, CV_1; (a1): $V_2 C$, $V_3^h C$.

A variety of special cases of greater length can be derived, for numbers of special form. Consider, for example, the type (1)(a1) numbers with

$$M_c = CV_3^{h-2}E_1 V_3^{k-1}C.$$

Among these values are those that correspond to $\xi \in \mathscr{A}B$ and

$$P = d+(1-i), \quad D = d^2+(1-i), \quad d \in Z[i].$$

For example, if $d = 3+2i$, then $P = 4+i$,

$$\operatorname{ch}\sqrt{6+11i} = V_1 E_2 V_1 \overrightarrow{CV_3^3 E_1 CV_2^2 E_3 V_2^5}\ \text{(a1)}.$$

The type (1)(b) numbers with

$$M_c = CV_2^{h-1}E_1 V_2^{k-1}C$$

correspond to numbers satisfying

$$2P = -2[(1+i)\hat{m}^2+\hat{m}-1+i]+m\hat{m},$$

$$D = P^2+4\hat{m}^2+4(1-i)\hat{m}+m[(-1+i)\hat{m}+i],$$

where $\hat{m} = h+ki$. A variety of special forms of D follow from considering $m = 2(1+i)\hat{m}+(3+ni)$, where $n \in Z$. Then P and D simplify to

$$2P = (1+ni)\hat{m}+2(1-i), \quad D = \left(\frac{1+ni}{2}\hat{m}\right)^2+(-n+i).$$

In the following table for this case, $D_1 = r+si$.

n	h	k	P	D
1	$r+s$	$r-s$	$D_1+(1-i)$	$D_1^2+(-1+i)$
0	$2r$	$2s$	$D_1+(1-i)$	D_1^2+i
-1	$r-s$	$r+s$	$D_1+(1-i)$	$D_1^2+(1+i)$
-2	$\frac{1}{5}(2r-4s)$	$\frac{1}{5}(3r+s)$	$D_1+(1-i)$	$D_1^2+(2+i)$
-3	$\frac{1}{5}(r-3s)$	$\frac{1}{5}(3r+s)$	$D_1+(1-i)$	$D_1^2+(3+i)$

It is also possible that the periodic part of $\operatorname{ch}\sqrt{D}$ has a generating product of half the length of the period. These correspond to nine other possible transformations of chain elements discussed in [11, Lemma 2]. They only apply to numbers equivalent to real numbers. One possibility is that the generating product contains only elements of V_F and no powers of V_2. Any number $\sqrt{d}$,

with $d \in Z^+$ and non-square, for which the number of terms in the periodic part of the continued fraction expansion of $\sqrt{d}$ is odd, will have such a chain. For example, $\sqrt{41} = [6; \overline{2, 2, 12}]$ and

$$\operatorname{ch}\sqrt{41} = E_2 V_1^5 \overrightarrow{V_3^2 V_1^2 V_3^{12} V_1^2 V_3^2 V_1^{12}},$$

where in this case the central portion of the generating product is $M_c = V_3^2 V_1^2$ and there is a (1 3) subscript permutation.

These examples illustrate that just as the continued fraction expansion of $\sqrt{d}$ tends to be simplest for d close to a square, particularly for $d = m^2 \pm 1$, the chains for numbers close to Gaussian squares also tend to be simple. Analogous to $m^2 \pm 1$ is the box around c^2, namely $c^2+\varepsilon+\delta i$, with ε, $\delta \in \{-1, 0, 1\}$ and ε, δ not both 0.

In summary, the chains of Schmidt provide expansions of quadratic complex irrationalities that bear a striking resemblance in their simplicity of form to the continued fraction expansions of real quadratic irrationalities. This provides new evidence for the analogy between regular chains and the continued fraction expansion and suggests the appropriateness of pursuing the properties of chains.

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Bounds for the degrees in the membership test for a polynomial ideal

by

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0. Introduction. Let $Q = (Q_1, \ldots, Q_m)$ be an m-tuple of polynomials in $C[x_1, \ldots, x_n]$ and let r be a positive integer. We define $D(Q, r)$ as the first integer such that for any polynomial P of degree $\leqslant r$ which belongs to the ideal generated by $Q_1, \ldots, Q_m$ there exist polynomials $A_1, \ldots, A_m$ such that

$$P = A_1 Q_1 + \ldots + A_m Q_m \quad \text{and} \quad \max_i \deg A_i \leqslant D.$$

Let $d = \max_i \deg Q_i$; a well-known result of G. Hermann [H] shows that

$$D(Q, r) < 2(2d)^{2^{n-1}} + r.$$

Moreover, E. Mayr and A. Meyer [M-M] prove that this doubly exponential growth is in general unavoidable. In spite of this, a recent result of W. D. Brownawell [B] suggests the possibility that, under suitable "smoothness" hypotheses, a polynomial bound for the growth of $D(Q, r)$ is available. In fact, W. D. Brownawell shows that in the Nullstellensatz case (i.e. $r = 0$) we have the upper bound

$$D(Q, 0) \leqslant \max_i \deg A_i \leqslant \min\{n, m\}(nd^{\min\{n,m\}} + d).$$

Let I be the ideal of $C[x_1, \ldots, x_n]$ generated by the polynomials Q_i; let $\bar{I}$ be the homogeneous ideal of $C[x_0, \ldots, x_n]$ generated by the homogenizations ${}^hQ_i$ of the polynomials Q_i. We denote by V the affine variety $\{Q_1 = \ldots = Q_m = 0\}$ and by $\bar{V}$ the projective variety $\{{}^hQ_1 = \ldots = {}^hQ_m = 0\}$.

In the present paper, using some ideas of W. D. Brownawell and a powerful result from several complex variables developed by H. Skoda, we prove the following result:

THEOREM 1. *If V is a smooth affine variety of dimension $n-m$ and I is the ideal of V, then*

$$D(Q, r) \leqslant (m-1)d + (n-1)m^{n-m}(d-1)^{n-m}d^m + r.$$