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Continued fractions of period five and real quadratic fields of class number one

by

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1. Introduction. Although several results in the literature give necessary and sufficient conditions for a real quadratic field to have class number one (see [3], [5], and [6]), we find that none of these gives *specific* enough information, not only to be useful as a tool for testing class number one *in general*, but also to qualify as a true "Rabinowitsch-like" result for real quadratic fields (see [14]–[15]). Hence in [7]–[12] we found necessary and sufficient conditions for a real quadratic field to have class number one in terms of *exactly* specifying the prime factorization (over the integers) of certain quadratic polynomials. The aforementioned criteria in [3], [5], and [6] do not *explicitly* do this *in general*. We looked at $\boldsymbol{Q}(\sqrt{d})$ for a positive square-free integer d, and examined the continued fraction expansion of $w = (1+\sqrt{d})/2$ if $d \equiv 1 \pmod 4$ (respectively $w = \sqrt{d}$ if $d \equiv 2, 3 \pmod 4$). We were able to find exact criteria which we were seeking, when the period k of the continued fraction expansion of $w = \langle a, \overline{a_1, \ldots, a_k}\rangle$ satisfies $k \leqslant 4$. As an offshoot we were actually able to list *all* real quadratic fields of class number one (with possibly only one more value remaining) for $k \leqslant 4$. A subset of such d are the so-called *Richaud–Degert* (R-D) type real quadratic fields (see [2] and [16]); i.e., those of the form $d = l^2+r$ with $4l \equiv 0 \pmod r$ and $-l < r \leqslant l$. Since we found the condition $-l < r \leqslant l$ to be somewhat artificial we removed it and we call such fields *extended* R-D types. The virtual solution of the class number one problem for extended R-D types in [13] shows that five of the six conjectures in the literature involving such types are true with the remaining one failing for at most one value. The conjectures are by S. Chowla [1], R. A. Mollin [8], R. A. Mollin and H. C. Williams [11] and H. Yokoi [17].

It is the purpose of this paper to solve the problem when $k = 5$. For $k = 5$ we find that if $d \not\equiv 5 \pmod 8$ then $h(d) = 1$ if and only if $d = 41$. When $d \equiv 5 \pmod 8$ we pose a conjecture which we claim lists all such d with $h(d) = 1$.

The main result is to give a specific Rabinowitsch condition for $k = 5$ in terms of the polynomial $f_d(x) = -x^2-x+(d-1)/4$ when $d \equiv 5 \pmod 8$.

In §2 the above notation will be in force.

2. Results. It is well known that if $h(d) = 1$ then $d = p$, $2p'$ or $p'q$ where p is prime and $p' \equiv q \equiv 3 \pmod 4$ are primes. Moreover, it is well known that an odd period k for w necessarily implies that d is a sum of two squares. Thus:

Remark 1. If the period $k > 1$ of w is odd, then $h(d) = 1$ implies that $d \equiv 1 \pmod 4$ is prime.

THEOREM. *If $k = 5$ for $Q(\sqrt{d})$ with $d \equiv 1 \pmod 4$ then $w = \langle a, \overline{b, c, c, b, 2a-1}\rangle$ where $2a-1 = b(c^2+1)^2 + c(c^2+1) - f((bc+1)^2+b^2)$, and $d = (2a-1)^2 + 4r$ where $r = (c^2+1)^2 - (bc^2+c+b)f$, for some positive integers b and c, and a non-zero integer f. Set $s = c(c^2+1) - f(bc+1)$. Thus:*

(I) $d \equiv 1 \pmod 8 \Rightarrow h(d) = 1$ *if and only if* $d = 41$.

(II) $d \equiv 5 \pmod 8 \Rightarrow h(d) = 1$ *if and only if all of the following conditions are satisfied*:

(1) *r is prime or $r = (bs+1)^2$.*

(2) *$bs+1$ is prime.*

(3) *If $r = (bs+1)^2$ then $f_d(x)/(bs+1)$ is prime whenever $0 \leqslant x \leqslant a-1$, $x \equiv 2^{-1}(\pm s-1) \pmod{bs+1}$, and $x \not\equiv 2^{-1}(\pm s-1) \pmod{(bs+1)^2}$. Also $f_d(x)/(bs+1)^2$ is 1 or prime whenever $0 \leqslant x \leqslant a-1$ and $x \equiv 2^{-1}(\pm s-1) \pmod{(bs+1)^2}$.*

(4) *If $r \neq (bs+1)^2$ then $f_d(x)/(bs+1)$ is prime or r^2 or $(bs+1)^2$ whenever $0 \leqslant x \leqslant a-1$ and $x \equiv 2^{-1}(\pm(br-s)-1)^{-1} \pmod{bs+1}$.*

(5) *If $r \neq (bs+1)^2$ then $f_d(x)/r$ is 1 or prime or r^2 or $r(bs+1)$ whenever $0 \leqslant x \leqslant a-1$ and $x \equiv 2^{-1}(\pm s-1) \pmod r$.*

(6) *$f_d(x)$ is prime whenever $0 \leqslant x \leqslant a-1$ and x does not satisfy any of the congruences in (3)–(5).*

Proof. The first statement of the theorem is a straighforward verification using the methods of Kraïtchik [3]. Throughout the proof we will be using the fact that $2a-1 = br+s$ where $0 < s < r$, which follows from examination of the continued fraction of w. Observe that $s \neq 0$ since $s = 0$ implies that d is of extended R-D type, and extended R-D types have $k \leqslant 4$.

Now, by a result of Lu [6] (see also [12]), $h(d) = 1$ if and only if

$$2a+2b+2c = \lambda_1(d)+1 \tag{2.1}$$

where $\lambda_1(d)$ denotes the number of solutions of $u^2+4vy = d$ in non-negative integers u, v and y. Since u must be odd we set $u = 2x+1$ and so $f_d(x) = -x^2-x+(d-1)/4 = vy$ with $0 \leqslant x \leqslant a-1$. Now we examine the number of divisors of $f_d(x)$; i.e., $\lambda_1(d)$.

Case (I): $d \equiv 1 \pmod 8$, in which case r is even. Also $r > 2$ since a tedious check shows that $d = (2a-1)^2+8$ does not yield $k = 5$. Now we determine the number of values of x for which $f_d(x) \equiv 0 \pmod{r/2}$. If $2x+1 \equiv \pm(2a-1) \pmod{2r}$ then we do have $f_d(x) \equiv 0 \pmod{r/2}$. If $2x+1 = 2a-1+2lr$ for some integer l then, since $0 \leqslant x \leqslant a-1$, we have $(1-a)/r \leqslant l \leqslant 0$.

CLAIM 1. *b is odd.*

If b is even and c is odd then by the first statement of the theorem f must be even. Thus $2a-1$ is forced to be even, a contradiction. If b is even and c is even then r is odd, again a contradiction, which secures the claim.

Now let $\lceil x \rceil$ denote the least integer greater than or equal to x.

CLAIM 2. $\lceil (1-a)/r \rceil = (1-b)/2$.

Recall from the statement at the outset of the proof that $2a-1 = br+s$, with $0 < s < r$. Therefore, $\lceil (1-a)/r \rceil = \lceil (1-br-s)/2r \rceil$ and

$$-b/2 \geqslant (1-br-s)/2r > -b/2-1.$$

Since b is odd by Claim 1, Claim 2 is now secured. Hence l takes on $(b+1)/2$ such values.

If $2x+1 = -2a+1+2lr$ then, as above, $a/r \leqslant l \leqslant (2a-1)/r$. Clearly, $\lfloor (2a-1)/r \rfloor = b$, where $\lfloor x \rfloor$ denotes the greatest integer less than or equal to x; whence $l \leqslant b$. Also

$$\lceil a/r \rceil = \lceil (br+s+1)/2r \rceil = (b+1)/2.$$

Thus $(b+1)/2 \leqslant l \leqslant b$; whence l takes on $(b+1)/2$ such values. Consequently there are at least $b+1$ values of x for which $f_d(x) \equiv 0 \pmod{r/2}$. If $r/2$ is divisible by 4 or an odd prime, then for all but possibly the two values of x yielding $f_d(x) = pr$ where $p = 2$ or $p > 2$ and $p \mid r$ (in which case these two values of x yield at least 9 distinct divisors of $f_d(x)$), $f_d(x)$ must be divisible by at least 6 distinct divisors. Hence there are at least $6(b-1)+9 = 6b+3$ distinct divisors of $f_d(x)$ for the above $b+1$ values of x. For the remaining $a-(b+1)$ values of x where $0 \leqslant x \leqslant a-1$ we must have at least four distinct divisors since $f_d(x)$ is even. (Note that we have excluded the possibility that $f_d(x) = 4$ by insisting that 4 divides $r/2$.) Thus, for these remaining values of x there are at least $4(a-b-1) = 4a-4b-4$ distinct divisors of $f_d(x)$. In total then we have $\lambda_1(d) \geqslant 4a+2b-1$.

We now invoke (2.1) to get $2a+2b+2c \geqslant 4+2b$; whence $c \geqslant a$. This is a contradiction since it is a general fact from the theory of continued fractions that, for example, if

$$w = \langle a, \overline{a_1, a_2, \ldots, a_{k-1}, 2a-1}\rangle$$

when $d \equiv 1 \pmod 4$ then $a_i < a$ for each i. This contradiction forces $r = 4$. A straightforward check using the first statement of the theorem shows that $r = 4$ implies $b = 1$, $c = 2$ and $f = 3$; i.e., $d = 41$.

Case (IIa): $d \equiv 5 \pmod 8$; whence r is odd. $2x+1 \equiv \pm(br-s) \pmod{bs+1}$.

CLAIM 3. $f_d(x) \equiv 0 \pmod{bs+1}$.

From the first statement of the theorem we get $2a-1 = br+s$. Thus,

$$4f_d(x) = -(2x+1)^2+d = -(2x+1)^2+(br+s)^2+4r.$$

Therefore,

$$4f_d(x) \equiv -(br-s)^2+(br+s)^2+4r \pmod{bs+1}.$$

Hence $4f_d(x) \equiv 4brs+4r \equiv 0 \pmod{bs+1}$. This proves Claim 3.

Hence

$$2x+1 = br-s+l(bs+1) \quad \text{when } 2x+1 \equiv br-s \pmod{bs+1},$$

$$2x+1 = s-br+l(bs+1) \quad \text{when } 2x+1 \equiv -br+s \pmod{bs+1},$$

for some integer l, where l is even (and possibly $l=0$ when $2x+1 \equiv br-s \pmod{bs+1}$). Since $0 \leqslant x \leqslant a-1$, if $2x+1 \equiv br-s \pmod{bs+1}$ then

$$(1-br+s)/(bs+1) \leqslant l \leqslant 2s/(bs+1).$$

Since l must be 0 or even and $2s/(bs+1)$ is clearly less than 2, we have $l \leqslant 0$.

To simplify the writing, set

$$u = \lceil (1-br+s)/(bs+1) \rceil .$$

Since it is straightforward to check that

$$-c-1 < (1-br+s)/(bs+1) \leqslant 1-c$$

we have $u = -c$ or $1-c$. If $u=-c$ then, when c is odd, l takes on $(c+1)/2$ values. When c is even then l takes on $c/2+1$ values. If $u = 1-c$ then, when c is odd, l takes on $(c+1)/2$ values. If c is even then l takes on $c/2$ values.

As above, if $2x+1 \equiv s-br \pmod{bs+1}$, then

$$(1-s+br)/(bs+1) \leqslant l \leqslant (2br)/(bs+1).$$

Since l is even we easily get $l \leqslant 2c$. Moreover, it is straightforward to check that

$$c+1 > (br+1-s)/(bs+1) \geqslant c-1.$$

Set

$$v = \lceil (br+1-s)/(bs+1) \rceil .$$

Therefore, $v=c$ or $c+1$. If $v=c$ and c is odd then l takes on $(c+1)/2$ values. If c is even then l takes on $c/2+1$ values. If $v=c+1$ and c is odd then l takes on $(c+1)/2$ values. When c is even l takes on $c/2$ values. Hence in total we have $c+1$ values taken on by l. To see this we observe that when c is even then $u=-c$ if and only if $v=c+1$ (and so $u=1-c$ if and only if $v=c$). The case where c is odd is obvious.

Case (IIb): $2x+1 \equiv 2a-1 \pmod r$; whence $f_d(x) \equiv 0 \pmod r$. Thus, $2x+1 = 2a-1+lr$. (Note that by a substitution of the latter into $f_d(x)$ and using the parameterization given in the first statement of the theorem one may check that $f_d(x) \neq r^2(bs+1)$.) Since $0 \leqslant x \leqslant a-1$ we have $1 \leqslant 2a-1+lr \leqslant 2a-1$; whence $(2-2a)/r \leqslant l \leqslant 0$. Since

$$\lceil (2-2a)/r \rceil = \lceil ((1-s)/r)-b \rceil = -b$$

we have $-b \leqslant l \leqslant 0$ with $l=0$ or l even. Therefore, if b is odd then there are $(b+1)/2$ such values of l and if b is even then there are $b/2+1$ such values of l.

Case (IIc): $2x+1 \equiv -2a+1 \pmod r$; whence $f_d(x) \equiv 0 \pmod r$. Thus, $2x+1 = -2a+1+lr$, and so, as above, $1 \leqslant -2a+1+lr \leqslant 2a-1$. Therefore, $2a/r \leqslant l \leqslant 2(2a-1)/r$; whence

$$b+(s+1)/r \leqslant l \leqslant 2b+2s/r.$$

Since $\lceil b+(s+1)/r \rceil = b+1$ and $\lfloor 2b+2s/r \rfloor = 2b+1$ we get $b+1 \leqslant l \leqslant 2b+1$ with l even. If b is odd then there are $(b+1)/2$ values and if b is even there are $b/2$ values.

Now we analyze cases (IIa–c) for the number of divisors of $f_d(x)$. First we assume $g=1$ where $g = \text{g.c.d.}(r, bs+1)$. We handle the case $g>1$ at the end of the proof.

In case (IIa) when $2x+1 \equiv br-s \pmod{bs+1}$ there are two possibilities. Either $f_d(x)$ equals $r^2(bs+1)$ or it does not. If $f_d(x_0) = r^2(bs+1)$ then there are at least 6 divisors of $f_d(x)$ for $x = x_0$. If c is odd then there are at least $4(c-1)/2 = 2c-2$ divisors for the remaining values of x. If c is even then there are at least $4(c/2-1) = 2c-4$ divisors for the remaining values of x when $u = 1-c$; and $4(c/2) = 2c$ values when $u=-c$. This yields a total of at least $2c+4$ distinct divisors of $f_d(x)$ when c is odd, and at least $2c+2$ divisors when $u=1-c$, c even. Finally there are at least $2c+6$ divisors when c is even and $u=-c$.

If $f_d(x) \neq r^2(bs+1)$ for any x with $0 \leqslant x \leqslant a-1$ then, when c is odd, there are at least $4(c+1)/2 = 2c+2$ divisors; and when c is even there are at least $4(c/2) = 2c$ divisors if $u = 1-c$, c even. There are at least $4(c/2+1) = 2c+4$ divisors if c is even and $u=-c$.

In case (IIa), when $2x+1 \equiv -br+s \pmod{bs+1}$, a tedious check shows that

$$f_d\big((cs-br+2c(bs+1)-1)/2\big) = (bs+1)^2$$

and $f_d(x) \neq bs+1$ for any x with $0 \leqslant x \leqslant a-1$. If $f_d(x) = r^2(bs+1)$ and c is odd then for the remaining values of x there are at least $4((c+1)/2-2) = 2c-6$ divisors. If c is even and $v=c$ then there are at least $4(c/2-1) = 2c-4$ divisors, and if $v=c+1$ there are at least $4(c/2-2) = 2c-8$ divisors. In total then there are at least $2c+3$ distinct divisors when c is odd. When c is even and $v=c$ there are at least $2c+5$ divisors, and at least $2c+1$ divisors when $v=c+1$. Finally we observe that $f_d(x) = r^2(bs+1)$ cannot happen in *both* of the cases $2x+1 \equiv -br+s \pmod{bs+1}$ *and* $2x+1 \equiv br-s \pmod{bs+1}$. (The reason for this is that $bs+1$ does not divide $br-s$.) We will need to keep this observation in mind when analyzing the minimum of $\lambda_1(d)$ later.

If $f_d(x) \neq r^2(bs+1)$ then there are at least $2c+1$ divisors when c is odd. When c is even and $v=c$ there are at least $2c+3$ divisors; and when $v=c+1$ there are at least $2c-1$ of them. In case (IIb), $f_d(a-1) = r$ yielding 2 divisors.

Also $f_d((br-s-1)/2) = r(bs+1)$ which we already counted in case (IIa). Therefore we revise down by one the number of values found in case (IIb). Hence in total we have the following for case (IIb). If b is odd then there are at least $4((b-3)/2)+2 = 2b-4$ divisors, and if b is even there are at least $4(b/2-1)+2 = 2b-2$ divisors.

In case (IIc), if $f_d(x) = r^2(bs+1)$ then we already counted this in case (IIa) so we revise down the number of values found in case (IIc) by one. Thus in total for this instance of case (IIc) we have the following. If b is odd then there are at least $4((b-1)/2) = 2b-2$ divisors, and if b is even there are at least $4(b/2-1) = 2b-4$ divisors. On the other hand, if $f_d(x) \neq r^2(bs+1)$ then the case (IIc) count remains as is and we have the following. If b is odd then there are at least $4(b+1)/2 = 2b+2$ divisors and if b even there are at least $4(b/2) = 2b$ divisors.

Case (IId): For the remaining values of x not counted in cases (IIa–c) we have the following. If $f_d(x) = r^2(bs+1)$ then there are $a-(c+1+b-1)$ values of x remaining which yield at least $2(a-(c+1+b-1)) = 2a-2c-2b$ divisors of $f_d(x)$. If $f_d(x) \neq r^2(bs+1)$ then there are $a-(c+b+1)$ values of x remaining, which yield at least $2a-2b-2c-2$ divisors of $f_d(x)$.

Now we total all the divisors. If $f_d(x) = r^2(bs+1)$ for some x with $0 \leqslant x \leqslant a-1$ then $\lambda_1(d) \geqslant 2a+2b+2c-1$. Hence from (2.1) we get

$$2a+2b+2c = \lambda_1(d)+1 \geqslant 2a+2b+2c;$$

i.e., the minimum is achieved and conditions (II)(1)–(6) of the theorem hold. On the other hand, if $f_d(x) \neq r^2(bs+1)$ then

$$\lambda_1(d) \geqslant 2a-2b-2c-2+4c+3+4b-2 = 2a+2b+2c-1.$$

Again invoking (2.1) we see that the minimum is achieved. This completes the proof when $g = 1$. Now we assume that $g > 1$.

If both $(bs+1)/g > 1$ and $r/g > 1$ then a tedious check of the above cases shows that too many divisors of $f_d(x)$ occur, whence (2.1) is violated. Thus $bs+1 = g$ or $r = g$. However, $r > bs+1$ so $bs+1 = g$. To complete the proof we show that $bs+1$ is prime and $r = (bs+1)^2$.

Case (IIe): $2x+1 \equiv s \pmod{(bs+1)^2}$; whence $f_d(x) \equiv 0 \pmod{(bs+1)^2}$. If $2x+1 = s+l(bs+1)^2$ then $0 \leqslant l \leqslant b$. If s is odd then l and b are even, and there are $b/2+1$ values. If s is even then l and b are odd, and there are $(b+1)/2$ values.

Case (IIf): $2x+1 \equiv -s \pmod{(bs+1)^2}$; whence $f_d(x) \equiv 0 \pmod{(bs+1)^2}$. If $2x+1 = -st+l(bs+1)^2$ then $1 \leqslant l \leqslant b$. If s is odd then l and b are even, so there are $b/2$ values. If s is even then l and b are odd, and there are $(b+1)/2$ values.

Hence from cases (IIe–f) a total of $b+1$ values emerge. Therefore with the exception of those values of x where $f_d(x) = (bs+1)^3$ or $(bs+1)^2$ (yielding a total of at least 7 divisors), each value of x yields at least 6 divisors. Hence cases (IIe–f) yield a total of at least $6(b-1)+7 = 6b+1$ divisors.

Now we count those values of x for which $bs+1$ *properly* divides $f_d(x)$. From case (IIa) above we get $c+1$ values. However, we already counted $b+1$ of them as values where $bs+1$ does *not* properly divide $f_d(x)$. Thus, there are $c-b$ values for which $bs+1$ *does* properly divide $f_d(x)$. For these values there are at least 4 divisors each yielding at least $4(c-b)$ divisors. In total then we have at least $4c+2b+1$ divisors. However, we have not yet counted those remaining x between 0 and $a-1$. There are $a-(b+1+c-b)$ of them yielding at least $2a-2c-2$ divisors. Thus the $g > 1$ case yields a minimum number of divisors of: $2a+2b+2c-1$. Invoking (2.1) shows that the minimum is achieved. This completes the proof of the Theorem. ■

The table below illustrates the Theorem.

Table

$d \equiv 5 \pmod 8$, $h(d) = 1$ and $k = 5$

d	a	b	c	f	r	$bs+1$	$f_d(x)$ for $0 \leqslant x \leqslant a-1$
149	6	1	1	−1	7	5	37, 35, 31, 25, 17, 7
157	6	1	3	7	9	3	39, 37, 33, 27, 19, 9
181	7	4	2	1	3	5	45, 43, 39, 33, 25, 15, 3
269	8	1	2	·2	11	5	67, 65, 61, 55, 47, 37, 25, 11
397	10	2	6	17	9	3	99, 97, 93, 87, 79, 69, 57, 43, 27, 9
941	15	1	5	21	25	5	235, 233, 229, 223, 215, 205, 193, 179, 163, 145, 125, 103, 79, 53, 25
1013	16	2	2	1	13	11	253, 251, 247, 241, 233, 223, 211, 197, 181, 163, 143, 121, 97, 71, 43, 13
2477	25	2	1	−3	19	23	619, 617, 613, 607, 599, 589, 577, 563, 547, 529, 509, 487, 463, 437, 409, 379, 347, 313, 277, 239, 199, 157, 113, 67, 19
2693	26	2	4	7	23	11	673, 671, 667, 661, 653, 643, 631, 617, 601, 583, 563, 541, 517, 491, 463, 433, 401, 367, 331, 293, 253, 211, 167, 121, 69, 23
3533	30	4	1	1	13	29	883, 881, 877, 871, 863, 853, 841, 827, 811, 793, 773, 751, 727, 701, 673, 643, 611, 577, 541, 503, 463, 421, 377, 331, 283, 233, 181, 127, 71, 13
4253	33	9	3	1	7	19	1063, 1061, 1057, 1051, 1043, 1033, 1021, 1007, 991, 973, 953, 931, 907, 881, 853, 823, 791, 757, 721, 683, 643, 601, 557, 511, 463, 413, 361, 307, 251, 193, 133, 71, 7

We have checked up to $2 \cdot 10^4$ on a computer and the values in the table (together with 41) are all those of period 5 with class number one under that bound. We pose the following:

CONJECTURE 1. If $k = 5$ then $h(d) = 1$ if and only if $d \in \{41, 149, 157, 181, 269, 397, 941, 1013, 2477, 2693, 3533, 4253\}$.

Our computational evidence also gives us the confidence to pose the following:

CONJECTURE 2. If $k = 7$ then $h(d) = 1$ if and only if $d \in \{89, 109, 113, 137, 373, 389, 509, 653, 797, 853, 997, 1493, 1997, 2309, 2621, 3797, 4973\}$.

CONJECTURE 3. If $k = 9$ then $h(d) = 1$ if and only if $d \in \{73, 97, 233, 277, 349, 353, 613, 821, 877, 1181, 1277, 1613, 1637, 1693, 2357, 3557, 3989, 4157, 4517, 7213\}$.

Remark 2. The $k = 5$ case appears, from our data, to be the last period where at most 2 primes (r and $bs+1$) come into play. Given this fact and the intricacies of the proof of the Theorem it seems clear that this approach is exhausted as a general technique for finding the *exact* prescription for the factorization of $f_d(x)$ tantamount to $h(d) = 1$.

We hope to be able to prove these conjectures in the near future and advance what is known for larger periods. There is much work yet to be done.

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