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DEPARTMENT OF MATHEMATICS
RUTGERS UNIVERSITY
Hill Center for the Mathematical Sciences
Bush Campus
New Brunswick, New Jersey 08903, USA

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A matrix paraphrase of Kloosterman sums

by

D. H. Lehmer (Berkeley, Calif.)

1. Introduction. In 1967 Lehmer and Lehmer [3] showed that there was a strong connection between the cyclotomic periods and the ordinary Kloosterman sums

$$s_h = \sum_{x=1}^{p-1} e^{2\pi i(x+h\bar{x})/p} \quad (h = 0, 1, \ldots, p-1) \tag{1}$$

where $\bar{x} \equiv 1/x \pmod p$ and the Gaussian periods

$$\sum_{y=0}^{f-1} e^{2\pi i g^{ey}+h/p}$$

where $p = ef+1$ and g is a primitive root of the odd prime p. In this paper we exploit this connection to give a matrix paraphrase of the Kloosterman sum and its periods.

2. Notation. Throughout the paper capital letters are reserved for matrices. The matrices will be of special kind known as circulants. A *circulant* is an n by n matrix of the form

$$M = \begin{bmatrix} a_0 & a_1 & a_2 \cdots & a_{n-1} \\ a_{n-1} & a_0 & a_1 \cdots & a_{n-2} \\ a_{n-2} & a_{n-1} & a_0 \cdots & a_{n-3} \\ \cdots & \cdots & \cdots & \cdots \\ a_1 & a_2 & a_3 \cdots & a_0 \end{bmatrix}.$$

The matrix M depends only on its first row. To save space we will write M as follows:

$$M = \operatorname{cir}(a_0, a_1, \ldots, a_{n-1}). \tag{2}$$

We number the rows and columns of M from 0 to $n-1$ to allow the use of residue classes modulo n. If we denote the element in the ith row and jth column by α_{ij} we have

$$\alpha_{ij} = a_{j-i} \tag{3}$$

where the subscript $j-i$ is taken modulo n. It is well known that

(4) $$\det M = \prod_{v=0}^{n-1} \sum_{s=0}^{n-1} a_s \zeta^{sv} \qquad (\zeta = e^{2\pi i/n}).$$

The characteristic polynomial of M is therefore

(5) $$(-1)^n \prod_{v=0}^{n-1} \{\lambda - (a_0 + a_1\zeta^v + a_2\zeta^{2v} + \dots + a_{n-1}\zeta^{(n-1)v})\}$$

and the eigenvalues of M are

(6) $$x_v = \sum_{s=0}^{n-1} a_s \zeta^{sv}.$$

THEOREM 1. *Let $\Phi(x_1, \dots, x_r)$ be a polynomial of degree m in its r variables. If $M_1, \dots, M_r$ are any of the n by n circulant matrices, then $\Phi(M_1, \dots, M_r)$ is an n by n circulant matrix.*

Proof. It is sufficient to show that the set of n by n circulant matrices is closed under addition, subtraction and multiplication. This is obvious for the first two operations. For multiplication we can write

$$A = \mathrm{cir}(a_0, a_1, \dots, a_{n-1}) = \{\alpha_{ij}\},$$
$$B = \mathrm{cir}(b_0, b_1, \dots, b_{n-1}) = \{\beta_{ij}\}.$$

Then by (3) we have $C = AB = \{\gamma_{ij}\}$ where

$$\gamma_{ij} = \sum_{k=0}^{n-1} \alpha_{ik}\beta_{kj} = \sum_{k=0}^{n-1} a_{k-i} b_{j-k}.$$

If we replace i by $i+m$ and j by $j+m$ we find $\gamma_{i+m,j+m} = \gamma_{ij}$. Hence C is a circulant.

We now suppose that n is an odd prime p and that the elements of our matrices are integers.

3. Cyclotomy. Let e be any positive divisor of $p-1 = ef$. Let $\zeta = e^{2\pi i/p}$ and let g be a primitive root mod p. Classic cyclotomy is based on the following exponential sums, called *Gaussian periods* of p with respect to g:

(7) $$\eta_i = \sum_{k=0}^{f-1} \zeta^{g^{ek+i}} \qquad (i = 0, 1, \dots, e-1).$$

The *cyclotomic class* $c(k)$ $(k = 0, 1, \dots, e-1)$ is that set $(x_0, x_1, \dots, x_{f-1})$ for which the index of each x with respect to g is congruent to $k \pmod e$. We can then write (7) as

(8) $$\eta_i = \sum_{s \in c(i)} \zeta^s \qquad (i = 0, 1, \dots, e-1).$$

4. The matrices Z_r. These are p circulant matrices $Z_0, Z_1, \dots, Z_{p-1}$ (Z is capital zeta) defined by

$$Z_r = \mathrm{cir}(a_0, a_1, \dots, a_{p-1}) \quad \text{where}$$
$$a_i = \delta_i^r \quad (\text{Kronecker symbol} \pmod p).$$

They constitute a paraphrase of the pth roots of unity. In fact

$$Z_r \cdot Z_k = Z_{r+k},$$

where the subscripts are taken $(\mathrm{mod}\, p)$ and $Z_r = Z_1^r$, $Z_0 = Z_p = I$, the unit matrix.

5. The matrices H_r. Following Whiteman [4] we introduced in [2] the matrix H_r defined by

$$H_r = \mathrm{cir}(a_0, a_1, \dots, a_{p-1}) \quad (H \text{ is capital eta})$$

where

$$a_j = \begin{cases} 1 & \text{if } j \in c(r), \\ 0 & \text{otherwise.} \end{cases}$$

These matrices constitute a paraphrase of the Gaussian periods. We can also write a counterpart of (8), namely

(9) $$H_r = \sum_{k \in c(r)} Z_1^k.$$

6. The Kloosterman sum and its paraphrase. The ordinary Kloosterman sum is defined by

$$s_h = \sum_{x=1}^{p-1} \zeta^{x+h\bar{x}} \qquad (x\bar{x} \equiv 1 \pmod p).$$

It has its matrix paraphrase

(10) $$S_h = \sum_{x=1}^{p-1} Z_1^{x+h\bar{x}} \qquad (h = 0, 1, \dots, p-1).$$

We have the trivial case $h = 0$ in which

(11) $$S_0 = \sum_{x=1}^{p-1} Z_x = \mathrm{cir}(0, 1, 1, \dots, 1) = J - I,$$

where J is the matrix all of whose elements are equal to 1.

THEOREM 2. *If $h \not\equiv 0 \pmod p$,*

$$S_h = \mathrm{cir}(a_0, a_1, \dots, a_{p-1}),$$

where

(12) $$a_r = \chi(r^2 - 4h) + 1$$

and where $\chi(a) = \left(\frac{a}{p}\right)$ *is the Legendre symbol.*

Proof. Since $h \not\equiv 0 \pmod p$ we have in view of (10)

$$S_h = \sum_{k=0}^{p-1} n_k Z_k,$$

where n_k is the number of solutions of the congruence

$$x + h\bar{x} \equiv k \pmod p.$$

But $n_k = \chi(k^2 - 4h) + 1$.

COROLLARY 3. *If* $h \neq 0$, *the matrix* S_h *is a* (0, 1, 2) *matrix.*

Proof. Obvious.

Theorem 2 paraphrases the well-known fact that (see for example [3], p. 386)

$$s_h = \sum_{s=0}^{p-1} \left(1 + \chi(s^2 - 4h)\right)\zeta^s. \tag{13}$$

EXAMPLE. Let $p = 5$, $e = f = g = 2$. We have

$$S_0 = \mathrm{cir}(0, 1, 1, 1, 1), \quad S_3 = \mathrm{cir}(0, 2, 0, 0, 2),$$
$$S_1 = \mathrm{cir}(2, 0, 1, 1, 0), \quad S_4 = \mathrm{cir}(2, 1, 0, 0, 1).$$
$$S_2 = \mathrm{cir}(0, 0, 2, 2, 0),$$

THEOREM 4. *The matrix* S_h *is symmetric.*

Proof. We write $S_h = \{\alpha_{ij}\}$ $(i, j = 0, 1, \ldots, p-1)$. Then, since S_h is a circulant matrix,

$$\alpha_{ij} = a_{j-i} = \chi\left((j-i)^2 - 4h\right) + 1$$

by Theorem 2. This is symmetric in i and j.

7. The Kloosterman periods θ_i. In 1967 [3] we introduced the notion of a Kloosterman period as a sum of Kloosterman sums over the members of a cyclotomic class. That is, θ_i was defined as

$$\theta_i = \sum_{h \in c(i)} s_h \quad (i = 0, 1, \ldots, e-1)$$

and we showed that

$$\theta_i = \sum_{j=0}^{e-1} \eta_j \eta_{i-j}. \tag{14}$$

8. The paraphrase of the Kloosterman periods. This paraphrase is defined as the matrix

$$\Theta_i = \sum_{h \in c(i)} S_h \quad (i = 0, 1, \ldots, e-1) \quad (\Theta \text{ is capital theta}). \tag{15}$$

The counterpart of (14) is

THEOREM 5. $\Theta_i = \sum_{j=0}^{e-1} H_j H_{i-j}$.

Proof. By (15) and (10) we have

$$\Theta_i = \sum_{h \in c(i)} \sum_{r=1}^{p-1} Z_{r+h\bar{r}} = \sum_{h \in c(i)} \sum_{j=0}^{e-1} \sum_{k \in c(j)} Z_{k+h\bar{k}}$$
$$= \sum_{j=0}^{e-1} \sum_{r \in c(j)} Z_r \sum_{q \in c(j-i)} Z_q = \sum_{j=0}^{e-1} H_j H_{i-j}.$$

9. The eigenvalues of S_h. During the next two sections we shall use the following lemma of Jacobsthal [1].

LEMMA. *Let* δ_a^b *be the Kronecker symbol modulo* p. *Then*

$$\sum_{s=0}^{p-1} \chi(s-a)\chi(s-b) = -1 + p\delta_a^b.$$

Proof. The Lemma is true for $a \equiv b \pmod p$. Suppose that $\Delta = b - a \not\equiv 0 \pmod p$. Then the sum

$$T(a, b) = \sum_{s=0}^{p-1} \chi(s-a)\chi(s-b) = \sum_{u=0}^{p-1} \chi(u)\chi(u-\Delta)$$

on substituting u for $s-a$. Therefore $T(a, b)$ depends only on the difference Δ between b and a. Setting $u = v\Delta$ we see that

$$T(\Delta) = \sum_{u=1}^{p-1} \chi(u)\chi(u-\Delta) = \sum_{v=1}^{p-1} \chi(v)\chi(v-1),$$

so that $T(\Delta)$ does not depend on Δ. To determine the constant T we compute the sum

$$\sum_{\Delta=1}^{p-1} T(\Delta) = \sum_{\Delta=1}^{p-1} \sum_{u=1}^{p-1} \chi(u)\chi(u-\Delta) = \sum_{u=1}^{p-1} \chi(u) \sum_{\Delta=1}^{p-1} \chi(u-\Delta)$$
$$= -\sum_{u=1}^{p-1} \chi^2(u) = -(p-1).$$

Thus the average value of $T(\Delta)$ is -1. Hence $T = -1$. This proves the lemma.

Now we consider the eigenvalues of S_h. First we take up the trivial case

$h=0$. By (5) we can write the characteristic polynomial of S_0 as

$$F(\lambda) = \det(\mathrm{cir}(-\lambda, 1, 1, 1, \dots, 1))$$
$$= \prod_{v=0}^{p-1} (-\lambda+\zeta^v+\zeta^{2v}+ \dots +\zeta^{(p-1)v})$$
$$= -(\lambda-(p-1)) \prod_{v=1}^{p-1} (\lambda+1).$$

Thus the eigenvalues of S_0 consist of $p-1$ with multiplicity one and -1 with multiplicity $p-1$.

We now consider the case in which $h \not\equiv 0 \pmod p$.

THEOREM 6. *Let $h \not\equiv 0 \pmod p$. The set of eigenvalues of S_h depends only on the quadratic character of h with respect to p. The set consists of $p-1$ with multiplicity one and of the $(p-1)/2$ ordinary Kloosterman sums s_k, where k has the same quadratic character as h, each with multiplicity two.*

Proof. Since $h \neq 0$ we have

$$S_h = \mathrm{cir}\big(1+\chi(0^2-4h),\ 1+\chi(1^2-4h), \dots, 1+\chi((p-1)^2-4h)\big).$$

By (5) the characteristic polynomial of S_h is

$$(16)\qquad -\prod_{v=0}^{p-1}\Big(\lambda-\sum_{s=0}^{p-1}(1+\chi(s^2-4h))\zeta^{sv}\Big).$$

The factor corresponding to $v=0$ is

$$\lambda-\sum_{s=0}^{p-1}(1+\chi(s^2-4h)) = \lambda-p-\sum_{s=0}^{p-1}\chi(s^2-4h).$$

As for the character sum

$$\sum_{s=0}^{p-1}\chi(s^2-4h) = \sum_{r=0}^{p-1}(1+\chi(r))\chi(r-4h)$$
$$= \chi(-h)+\sum_{r=1}^{p-1}\chi(r-4h)+\sum_{r=1}^{p-1}\chi(r)\chi(r-4h) = -1$$

by Jacobsthal's lemma. So when $v=0$ we get the eigenvalue $p-1$ with multiplicity one.

The factors of (16) for $v \neq 0$ are

$$\lambda-\sum_{s=0}^{p-1}(1+\chi(s^2-4h))\zeta^{sv}.$$

This leads to the eigenvalues

$$\sum_{s=0}^{p-1}(1+\chi(s^2-4h))\zeta^{sv} = \sum_{w=0}^{p-1}(1+\chi(w^2-4hv^2))\zeta^{w} = s_{hv^2}.$$

As v runs over the set $1, \dots, p-1$, hv^2 runs twice over the set of numbers of the same quadratic character as h. Hence the theorem.

10. The eigenvalues of Θ_h. By (6) the eigenvalues of Θ_h are

$$x_v = \sum_{s=0}^{p-1}\sum_{k\in c(h)}(1+\chi(s^2-4k))\zeta^{sv}.$$

If $v=0$

$$x_0 = \sum_{k\in c(h)}\sum_{s=0}^{p-1}(1+\chi(s^2-4k))$$
$$= \sum_{k\in c(h)} p+\sum_{k\in c(h)}\sum_{s=0}^{p-1}\chi(s^2-4k)$$
$$= pf+\sum_{k\in c(h)}\sum_{t=0}^{p-1}\chi(t-4k)(1+\chi(t))$$
$$= pf+\sum_{k\in c(h)}\sum_{t=0}^{p-1}\chi(t)\chi(t-4k).$$

By Jacobsthal's lemma the inner sum is -1. Hence

$$x_0 = pf-f = f(p-1).$$

Next suppose that $v \neq 0$. Let v belong to $c(j)$ and let $sv \equiv w \pmod p$ in (16). This gives

$$(17)\qquad x_v = \sum_{w=0}^{p-1}\sum_{k\in c(h)}(1+\chi(w^2-4v^2k))\zeta^w = \sum_{k\in c(h)} s_{v^2k} = \sum_{r\in c(h+2j)} s_r = \theta_{h+2j}.$$

As v runs from 1 to $p-1$, j runs from 0 to $e-1$, f times over. The eigenvalues of Θ_h consist of $(p-1)f$ with multiplicity one and the set of e ordinary Kloosterman periods $\theta_h, \theta_{h+2}, \dots, \theta_{h+2(e-1)}$ each with multiplicity f. The subscripts are taken modulo e.

11. The sum and the sum of squares of the S_h. We begin with

THEOREM 7. *The sum of all the matrices S_h is $(p-1)J$.*

Proof.

$$\sum_{h=0}^{p-1} S_h = \sum_{h=0}^{p-1}\sum_{x=1}^{p-1} Z_1^{x+hx} = \sum_{x=1}^{p-1} Z_1^{x}\sum_{h=0}^{p-1} Z_1^{xh}$$
$$= \sum_{x=1}^{p-1} Z_1^{x}\sum_{r=0}^{p-1} Z_1^{r} = (J-I)J = J^2-J = pJ-J = (p-1)J.$$

THEOREM 8. $\sum_{h=0}^{p-1} S_h^2 = (p-1)((p-2)J+pI)$.

Proof. If $h=0$ we have from (11)

$$S_0^2 = (J-I)^2 = pJ-2J+I = (p-2)J+I \tag{18}$$
$$= \mathrm{cir}(p-1,\, p-2,\, p-2,\, \ldots,\, p-2).$$

Let $h \neq 0$ and let $S_h = \{\alpha_{ij}^{(h)}\}$. Then $S_h^2 = \beta_{ij}^{(h)}$ with

$$\beta_{ij}^{(h)} = \sum_{k=0}^{p-1} \alpha_{ik}^{(h)} \alpha_{kj}^{(h)} = \sum_{k=0}^{p-1} a_{k-i}^{(h)} a_{j-k}^{(h)}.$$

Since S_h^2 is a circulant we need only to compute the top row elements of S_h^2. First consider

$$\beta_{00}^{(h)} = \sum_{k=0}^{p-1} a_k^{(h)} a_{p-k}^{(h)} = \sum_{k=0}^{p-1} \left(1+\chi(k^2-4h)\right)^2$$
$$= \sum_{k=0}^{p-1} 1 + 2\sum_{k=0}^{p-1} \chi(k^2-4h) + \sum_{k=0}^{p-1} \chi^2(k^2-4h).$$

Hence

$$\beta_{00}^{(h)} = p + 2\sum_{k=0}^{p-1} \chi(k^2-4h) + p - (1+\chi(h)). \tag{19}$$

Summing both sides over h we have

$$\sum_{h=1}^{p-1} \beta_{00}^{(h)} = 2p(p-1) + 2\sum_{h=1}^{p-1}\sum_{k=0}^{p-1} \chi(k^2-4h) - (p-1) - \sum_{h=1}^{p-1} \chi(h)$$
$$= 2p^2-3p+1-2\sum_{k=0}^{p-1} \chi(k^2)$$
$$= 2p^2-3p+1-2(p-1) = (p-1)(2p-3).$$

Using (18)

$$\sum_{r=0}^{p-1} \beta_{00}^{(r)} = \beta_{00}^{(0)} + \sum_{h=1}^{p-1} \beta_{00}^{(h)} = p-1+(p-1)(2p-3) = 2(p-1)^2.$$

We now evaluate $\beta_{0j}^{(h)}$ for a fixed $h \not\equiv 0 \pmod p$ and $j \neq 0$. By (19)

$$\beta_{0j}^{(h)} = \sum_{k=0}^{p-1} a_k a_{j-k} = \sum_{k=0}^{p-1} \left(1+\chi(k^2-4h)\right)\left(1+\chi\left((j-k)^2-4h\right)\right)$$
$$= \sum_{k=0}^{p-1} 1 + \sum_{k=0}^{p-1} \chi(k^2-4h) + \sum_{k=0}^{p-1} \chi\left((j-k)^2-4h\right) + \sum_{k=0}^{p-1} \chi(k^2-4h)\,\chi\left((j-k)^2-4h\right).$$

Next we sum over h:

$$\sum_{h=1}^{p-1} \beta_{0j}^{(h)} = p(p-1) - \sum_{k=0}^{p-1} \chi(k^2) - \sum_{k=0}^{p-1} \chi^2(j-k)$$
$$+ \sum_{k=0}^{p-1}\sum_{h=0}^{p-1} \chi\left(h-(k/2)^2\right)\chi\left(h-\left(\frac{j-k}{2}\right)^2\right).$$

We use the Jacobsthal lemma in the form

$$\sum_{h=1}^{p-1} \chi(h-a)\,\chi(h-b) = -1+p\delta_b^a - \chi(ab)$$

and get

$$\sum_{h=1}^{p-1} \beta_{0j}^{(h)} = p^2-p-2(p-1)+p\sum_{k=0}^{p-1} \delta_{(j-k)^2}^{k^2} - \sum_{k=0}^{p-1} \chi(k^2)\,\chi(j-k)^2.$$

The congruence $k^2 \equiv (j-k)^2 \pmod p$ has the single solution $k \equiv j/2 \pmod p$. Hence

$$\sum_{r=0}^{p-1} \beta_{0j}^{(r)} = p^2-4p+4 = (p-2)^2.$$

Since $\beta_{0j}^{(0)} = p-2$ by (18), we can write

$$\sum_{r=0}^{p-1} \beta_{0j}^{(r)} = (p-2)^2+p-2 = (p-1)(p-2)$$

and

$$\sum_{h=0}^{p-1} S_h^2 = (p-1)(p-2)J + \left[2(p-1)^2-(p-1)(p-2)\right]I$$
$$= (p-1)\left((p-2)J+pI\right).$$

This proves Theorem 8.

12. The sum and the sum of squares of the Θ matrices. The following theorem follows easily from Theorem 7.

THEOREM 9. *The sum of all the matrices Θ is $(p-2)J+I$.*

Proof.
$$\sum_{k=0}^{e-1} \Theta_k = \sum_{k=0}^{e-1}\sum_{h\in c(k)} S_h = \sum_{h=1}^{p-1} S_h = \sum_{h=0}^{p-1} S_h - S_0$$
$$= (p-1)J-(J-I) = (p-2)J+I.$$

THEOREM 10. *If $e=2$ the sum of the squares of the Θ's is*

$$\tfrac{1}{2}\left[(p^3-4p^2+5p-4)J+(p^2+1)I\right].$$

Proof. Since $e=2$ there are only two of the matrices Θ. One of them is

$$\frac{p-3}{2}J+\frac{p-1}{2}I$$

and the other is

$$\frac{p-1}{2}J-\frac{p-1}{2}I.$$

Summing their squares and replacing J^2 by pJ we get

$$\tfrac{1}{2}[(p^3-4p^2+5p-4)J+(p^2+1)I].$$

The period polynomial of the Θ_h is defined by

$$\prod_{h=0}^{e-1}(X-\Theta_h) = X^e + B_1X^{e-1} + \ldots + B_e.$$

From Theorem 9 and Theorem 10 we easily deduce the next theorem.

THEOREM 11. *If $e = 2$ the matrix coefficients are*

$$B_1 = (p-2)J+I, \quad B_2 = \tfrac{1}{4}\{p^3-4p^2+7p-4\}J - \tfrac{1}{4}(p^2-1)I.$$

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DEPARTMENT OF MATHEMATICS
UNIVERSITY OF CALIFORNIA
Berkeley, California 94720, USA

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