On pairs of compacta with $\dim(X\times Y)<\dim X+\dim Y$

by

Stanisław Spież (Warszawa)

Abstract. In this note we give a necessary and sufficient condition for maps $f_1\colon X_1\to D^2\times\{0\}\subset D^4$, $f_2\colon X_2\to\{0\}\times D^2\subset D^4$ of 2-dimensional compacta into discs to admit mappings $F_i\colon X_i\to D^4$ which have disjoint images and coincide with f_i on $f_i^{-1}(\partial D^4)$, for $i=1,2$. As a corollary we obtain that $\dim(X_1\times X_2)<4$ iff any two mappings $X_1\to R^4$, $X_2\to R^4$ can be approximated by mappings with disjoint images. We also characterize pairs X_1, X_2 of finite dimensional compacta with $\dim(X_1\times X_2)<n=\dim X_1+\dim X_2$ in terms of extensions of mappings from closed subsets of X into certain CW-complexes. In the Appendix we also give an alternative proof of the latter result and of a result of [Sp2] by applying Eilenberg–MacLane spaces.

Introduction. Recall that a mapping $h\colon Z\to I^p$, here $I=[-1,1]$, is said to be *inessential* (in the sense of Alexandrov–Hopf) if there exists a mapping $h'\colon Z\to\partial I^p$ such that $h'(x)=h(x)$ for each $x\in h^{-1}(\partial I^p)$. In this note we prove the "only if" part in the case $m=2=n$ of the following

THEOREM 1. *Let $f\colon X\to I^m$, $g\colon Y\to I^n$ be mappings of finite dimensional compacta, so that $m=\dim X$ and $n=\dim Y$. Then, the product mapping $f\times g\colon X\times Y\to I^{m+n}$ is inessential if and only if there exist two mappings*

$$F\colon X\to I^{m+n}\quad \textit{and}\quad G\colon Y\to I^{m+n}$$

with disjoint images and such that

$$F(x)=\big(f(z),0\big)\quad \textit{and}\quad G(y=\big(0,g(y)\big)$$

for each $x\in f^{-1}(\partial I^m)$ and each $y\in g^{-1}(\partial I^n)$ (i.e. the mappings f and g are transversely trivial in the sense of [Kr]).

The "if" part of the above theorem was proved in [K–L]. The case $m=1$ of the "only if" part was proved also in [K–L] and the case $m\geqslant 3$, $n\geqslant 2$ of this part in [Sp1]. The proof of the missing case ($m=2=n$) of the theorem is the same as that given in [Sp1] except that one has to apply a homotopy version of the Whitney trick for 4-dimensional manifolds (which is described in § 1) instead of the isotopy Whitney trick for higher dimensional manifolds.

A consequence of Theorem 1 and of (2.1) in [Kr] (cf. [M–R]) is the "only if" part of the following characterizations of compacta with the properties which occur in the famous constructions of L. S. Pontryagin [Po] and V. Boltyanskii [Bo] (the "if" part was proved in [Kr]).

Corollary 1. *Any two mappings* $X\to R^k$ *and* $Y\to R^k$ *of finite dimensional compacta, where* $k=\dim X+\dim Y$, *can be approximated arbitrarily closely by mappings with disjoint images if and only if* $\dim(X\times Y)<\dim X+\dim Y$.

Corollary 2 ([Sp1], [Sp2]). *Let* X *be an* m*-dimensional compactum. Then* $\dim(X\times X)<2m$ *if and only if the set of all imbeddings* $X\to R^{2m}$ *is dense in the space of all mappings* $X\to R^{2m}$.

We are also interested in characterizing pairs X, Y of compacta with the property $\dim(X\times Y)<\dim X+\dim Y$ in terms of extensions of mappings of closed subsets of X, Y into certain CW-complexes. In the statement of the following result, which in a weaker form was announced in [Sp2], we need the following notation. Let $S_0^{m-1}\vee S_1^{m-1}$ be the one point union, with the base point $*$, of $(m-1)$-spheres S_0^{m-1} and S_1^{m-1}. Let a_j be a generator of the group $\pi_{m-1}(S_j^{m-1},*)$ for $j\in\{0,1\}$. By $P_{k,l}^m$, where $m\geqslant 2$, we denote the CW-complex obtained by attaching to $S_0^{m-1}\vee S_1^{m-1}$ two m-cells by mappings corresponding to $a_0a_1^l$ and a_0^k, and by $R_{k,l}^m$ the CW-complex obtained by attaching to $S_0^{m-1}\vee S_1^{m-1}$ one m-cell by a mapping corresponding to $(a_0a_1^k)^l$. (We use multiplicative notation for the higher homotopy groups also.)

Theorem 2. *Let* X *and* Y *be compacta such that* $\dim X=m$ *and* $\dim Y=n$. *Then,* $\dim(X\times Y)<m+n$ *if and only if there exists a permutation* (U,V) *of the letters* P *and* R *such that for arbitrary closed subsets* A, B *of* X, Y, *respectively, any mappings*

$$A\to S_0^{m-1}\quad \text{and}\quad B\to S_0^{n-1}$$

admit extensions

$$X\to U_{k,l}^m\quad \text{and}\quad Y\to V_{k,l}^n$$

for some integers k *and* l.

In this note we prove the "only if" part in the case $m=2=n$, see § 3 (the proof in the higher dimensions is the same). In the proof, Lemma (3.9) in [Sp2] plays an essential role. The inverse implication can be proved similarly to (1.5) in [Kr] (compare also the proof of the "if" part of Theorem 3 in [Sp2], § 4).

In the Appendix, we give an alternative proof of Theorem 3 in [Sp2] that is independent of the results of [Sp2] and we sketch an alternative proof of Theorem 2.

I would like to remark that recently I have learned by letters from D. Repovš that he, A. N. Dranishnikov, and E. V. Shchepin have obtained results along the lines of Corollary 1, case $m=n=2$ (see [D–R–S1] and that some results of [Kr] and [Sp1] have been very recently recovered in [D–S] in an alternative way. (Added in proof: After this paper was accepted the author received a preprint [D–R–S2] containing an alternative proof of Corollary 1 in the case $m=2=n$.)

1. A version of the Whitney lemma. The proof of the case $m=2=n$ of the "only if" part of Theorem 1 is the same as that of Theorem (3.2) in [Sp1] except that one has to apply Lemma (1.1) below instead of Lemma (2.1) in [Sp1]. The proof of Lemma (1.1) is a modification of Lemma (3.1) in [Fr] (compare also Lemma

(immersed Whitney move), section 1 in [F–Q]). In this section we will use notations of [R–S] and [Sp1].

(1.1) Lemma. *Let* $X=X_1\cup X_2$ *be a disjoint union of compact 2-polyhedra* X_1 *and* X_2 *with triangulations* T_1 *and* T_2, *respectively, and let* M *be a* 1*-connected oriented* 4*-dimensional* PL*-manifold. Suppose* $f\colon X\to M$ *is a* PL*-map such that*

(a) *the singularities of* f *are transverse double points;*

(b) $f(X_1^{(1)})\cap f(X_2)=\emptyset=f(X_1)\cap f(X_2^{(1)})$, *here* $X_1^{(1)}$ *and* $X_2^{(1)}$ *are the* 1*-skeletons corresponding to the triangulations* T_1 *and* T_2;

(c) $f(X)\cap\partial M=f(X_1^{(1)}\cup X_2^{(1)})\cap\partial M$;

(d) *for each pair of (oriented)* 2*-simplexes* $\sigma_1\in T_1$, $\sigma_2\in T_2$ *the intersection number* $f(\sigma_1)\wedge f(\sigma_2)$ *is* 0.

Then there exists a PL*-mapping* $g\colon X\to M^4$ *such that*

$$g(X_1)\cap g(X_2)=\emptyset\quad \text{and}\quad g(x)=f(x)\quad \text{for each } x\in X_1^{(1)}\cup X_2^{(1)}.$$

Remark. Observe that, by general position theorem, the assumptions (a), (b) and (c) are not too restrictive.

In the proof of (1.1) we will use the following homotopy version of the Whitney move (see [Fr], the proof of (3.1), and [F–Q] section 1).

(1.2) Lemma. *Let* M *be an oriented* 4*-dimensional* PL*-manifold and let* $P_1,P_2\subset M$ *be two transverse locally flat oriented* 2*-dimensional* PL*-submanifolds which intersect in exactly two points* a *and* b *such that* $\varepsilon(a)=-\varepsilon(b)$. *Suppose* α_i *is an arc in* P_i, *for* $i=1,2$, *joining* a *and* b *and* $h\colon D\to M$ *is a* PL*-map of a* PL*-disc such that*

(1.3) $h|\partial D$ *is a* PL*-homeomorphism onto the Whitney circle* $\alpha_1\cup\alpha_2$;

(1.4) *the singularities of* h *are transverse double points contained in* $h(\operatorname{Int}D)$;

(1.5) $h(\operatorname{Int}D)\cap(P_1\cup P_2)=\emptyset$.

Then for any neighborhood U *of* $h(D)$ *in* M *there exist a* PL*-disc* B, *a regular neighborhood of* α_1 *in* $P_1\cap U$ *and a* PL*-homotopy* $H\colon B\times[0,1]\to U$ *such that*

$$H(x,t)=x\quad \text{for each}\quad (x,t)\in B\times\{0\}\cup(\partial B\times[0,1])\quad \text{and}\quad H(x,1)\in M\setminus P_2$$
for each $x\in B$.

Proof of (1.1). Suppose $\sigma_1\in T_1$, $\sigma_2\in T_2$ are two simplexes with nonempty intersection $f(|\sigma_1|)\cap f(|\sigma_2|)$. The condition $f(\sigma_1)\wedge f(\sigma_2)=0$ implies that $f(|\sigma_1|)\cap\cap f(|\sigma_2|)$ is a set of an even number of points, say

$$\{a_1,\ldots,a_s;\ b_1,\ldots,b_s\}$$

such that $\varepsilon(a_i)=-\varepsilon(b_i)$. Let α_i, $i=1,2$, be a PL-arc joining a_s and b_s in $f(\operatorname{Int}|\sigma_i|)$ missing all singularities of f and all points at which $f(\operatorname{Int}|\sigma_i|)$ is not locally flat in M (the number of such points is finite). Since $\pi_1(M)=0$ there exists a PL-mapping $h\colon D\to M$ of a PL-disc which satisfies the conditions (1.3), (1.4) and such that $h(\operatorname{Int}D)$ meets $f(X)$ transversely and is disjoint from $f(X_1^{(1)}\cup X_2^{(1)})$.

Without loss of generality we may assume that

$$h(\operatorname{Int}D)\cap f(X_2)=\emptyset.$$

This can be achieved by a PL-homotopy of $f(X_2)$ which removes points of intersection of $h(\operatorname{Int} D)$ and $f(X_2)$ by pushing these points along disjointly imbedded arcs in $h(D)$ to the edge α_2. These are "Casson moves" (or finger moves), see the proof of (3.1) in [Fr] and section 1 in [F–Q]. Moreover, we may assume that this procedure does not introduce new points of intersection of $f(X_2)$ and $f(X_1)$.

Observe that there are open neighborhoods P_1 and P_2 of α_1 and α_2 in $f(\operatorname{Int}|\sigma_1|)$ and $f(\operatorname{Int}|\sigma_2|)$, respectively, which satisfy the assumptions of Lemma (1.2). Applying (1.2), we can modify the map f on the inverse image of a small neighborhood of α_1 in P_1 in order to get a map $f'\colon X \to M$, which satisfies the conditions (a), (b), (c) and (d) of (1.1) and such that

$$f'(X_1) \cap f'(X_2) = f(X_1) \cap f(X_2) \setminus \{a, b\}.$$

Thus the lemma follows by applying successively the above argument.

2. Some lemmas concerning presentations of groups. First we introduce some conventions. By $\operatorname{cdim}_G X$ we denote the cohomological dimension of X with respect to G; Q the additive group of all rationals, Q_p the additive group of all rationals whose denominators are coprime with a prime p and $Z(k) = Z/kZ$. By (k, l) we denote the greatest common divisor of integers k and l. If $(k, l) = 1$ for each l belonging to a set $\mathscr{F}$ of integers then we write $(k, \mathscr{F}) = 1$. In the sequel of the paper, we fix a copy of the unit circle and denote it by S_0 or by S. Let $\bigvee \{S_j\colon j \in J\}$, where $0 \in J$, denote the one point union, with the base point $*$, of circles $\{S_j\colon j \in J\}$ and let a_j be a generator of $\pi_1\{S_j, *\}$. If $\{v_i\colon i \in I\}$ is any collection of words in symbols $\{a_j\colon j \in J\}$ then one can form a 2-dimensional CW-complex whose 1-skeleton is $\bigvee \{S_j\colon j \in J\}$ and whose 2-cells $\{\sigma_i\colon i \in I\}$ are such that the attaching map of σ_i is given by $v_i \in \pi_1(\bigvee \{S_j\colon j \in J\}, *)$ for each $i \in I$. We call

(2.1) $$P = \{\{a_j\colon j \in J\}; \{v_i\colon i \in I\}\}$$

the *presentation* of the arising complex $K(P)$. The elements v_j are called *relators* of P.

We say (see [Sp2]) that a mapping $f\colon (L, L_0) \to (K, S)$ is *admissible* if for any homomorphism h of abelian groups the following condition is satisfied

$$H^2(\varphi) \otimes h = 0 \quad \text{iff} \quad H^2(\psi) \otimes h = 0,$$

where $\varphi\colon (L, L_0) \to (D^2, S)$ and $\psi\colon (K, S) \to (D^2, S)$ are extensions of $f|L_0$ and the identity map on S, respectively. (By D^2 we denote the unit 2-ball.)

The following lemma is essential in the proof of Theorem 2.

(2.2) Lemma (see [Sp2], (3.9) and (2.5)). *Let (L, L_0) be a pair of compact polyhedra such that* $\dim L = 2$. *Then any mapping $L_0 \to S$ can be extended to an admissible mapping $(L, L_0) \to (K(P), S)$ where P has the following form*

$$P = \{a_0, \ldots, a_k;\ a_0^{n(j)} a_j^{m(j)} \text{ for } 1 \leqslant j \leqslant k,\ [a_i, a_j] \text{ for } 0 \leqslant i < j \leqslant k\},$$

such that $n(j) \neq 0$ and $n(j)$ divides $m(j)$ for each $j \in \{1, \ldots, k\}$.

We will also need the following lemmas:

(2.3) Lemma. *Let P be given by* (2.1) *and let P' be defined as P except that, for some $i \in I$ and some positive integer n, the relator v_i is replaced by $(v_i)^n$. If (X, A) is a pair of compacta such that $\dim X = 2$ and $\operatorname{cdim}_{Z(n)} X \leqslant 1$, then a map $f\colon A \to S$ admits an extension $X \to K(P')$ provided f admits an extension $X \to K(P)$.*

The proof is essentially the same as that of Lemma (4.4) in [Sp2], since the assumption $\dim(X \times X) < 4$ can be relaxed to $\operatorname{cdim}_{Z(n)} X \leqslant 1$ (see Remark (4.3) in [Sp2]).

(2.4) Lemma. *Let P and P' be as assumed in* (2.3). *Then there exists a mapping*

$$K(P') \to K(P)$$

which is the identity on S.

(2.5) Lemma. *Let us consider two presentations*

$$P = \{a_0, a_1;\ a_0^n a_1^{k \cdot l}, [a_0, a_1]\}$$

and

$$P' = \{a_0, a_1, a_2;\ a_0^n a_1^k, a_0^n a_2^l, [a_i, a_j] \text{ for } 0 \leqslant i < j \leqslant 2\},$$

where n, k and l are positive integers. If $(k, l) = 1$, then there exists a mapping $K(P) \to K(P')$ which is the identity on S.

Proof. Let t and s be integers such that $t \cdot k + s \cdot l = 1$. Then the required map is induced by the substitutions $a_0 \to a_0$ and $a_1 \to a_1^s a_2^t$.

3. Proof of Theorem 2. As discussed in the Introduction, we will restrict ourselves to the case $m = 2 = n$ and the "only if" part. Let X_1, X_2 be 2-dimensional compacta such that $\dim(X_1 \times X_2) < 4$ and let

$$g_s\colon (X_s, A_s) \to (D^2, S), \quad s = 1, 2,$$

be mappings, where A_s is a closed subset of X_s for $s \in \{1, 2\}$. We need the following

(3.1) Lemma. *There exist mappings*

$$h_s\colon (X_s, A_s) \to (K(P_s), S), \quad s = 1, 2,$$

such that $h_s(x) = g_s(x)$ for $x \in A_s$ and P_s is a presentation of the following form

$$P_s = \{a_0, \ldots, a_{k(s)};\ a_0^{n(s,j)} a_j^{m(s,j)} \text{ for } 1 \leqslant j \leqslant k(s),\ [a_i, a_j] \text{ for } 0 \leqslant i < j \leqslant k(s)\},$$

where $n(s,j)$ divides $m(s,j)$, $n(s,j) \neq 0$, $m(s,j)$ is either 0 or a power of a prime, and the greatest common divisor of the integers $m(1, i)$, $m(2,j)$ divides the product $n(1, i) \cdot n(2,j)$ for each pair $(i,j) \in \{1, \ldots, k(1)\} \times \{1, \ldots, k(2)\}$.

Proof. By (2.5), we may additionally require that in (2.2) each $m(j)$ is either 0 or a power of a prime. The proof follows from (2.2) by the argument used to prove (4.2) in [Sp2].

Observe that under assertions of (3.1) either all $m(1,j)$'s are nonzero or all $m(2,j)$'s are nonzero. Without loss of generality we will assume that the latter condition is satisfied. Let k denote the lowest common multiple of the integers $m'(2,j) = m(2,j)/n(2,j)$ for $1 \leqslant j \leqslant k(2)$. Let us observe that

(3.2) Remark. If, in (3.1), we have $m(1,j) = 0$ then k divides $n(1,j)$.

For $s \in \{1, 2\}$, let $\mathscr{F}_s$ denote the set of all primes p such that $\operatorname{cdim}_{Z(p)} X_s = 2$. Let us note (see [Ko]) that $\dim(X_1 \times X_2) < 4$ implies that $\mathscr{F}_1 \cap \mathscr{F}_2 = \emptyset$. The proof of Theorem 2 is divided into the following two lemmas:

(3.3) Lemma. *There exist an integer l such that $(l, \mathscr{F}_2) = 1$ and a mapping*

$$f_1\colon X_1 \to K_1 = K(\{a_0, a_1;\ a_0^k, a_0 a_1^l\})$$

such that $f_1(x) = g_1(x)$ for each $x \in A_1$.

Proof. By (2.5) in [Sp2], and (3.1), (3.2) and (2.3), there exists a mapping $h_1\colon (X_1, A_1) \to (K(P_1), S)$ as asserted in (3.1) and (3.2) except that, for each j such that $(m(1,j), \mathscr{F}_1) = 1$, the condition $n(1,j)$ divides $m(1,j)$ is replaced by the condition k divides $n(1,j)$. By Lemma (2.4), we may additionally require that $n(1,j) = 1$ for each j such that $(m(1,j), \mathscr{F}_1) \neq 1$ and $m(1,j) \neq 0$. Let l be the lowest common multiple of all integers $m(1,j)$'s such that $(m(1,j), \mathscr{F}_1) \neq 1$ and $m(1,j)) \neq 0$. Observe that then $(l, \mathscr{F}_2) = 1$.

Now, there exists a mapping

$$K(P_1) \to K_1,$$

which is the identity on S; it is defined by the following substitutions

$$\begin{aligned} &a_j \to \textit{trivial element} && \text{if } (m(1,j), \mathscr{F}_1) = 1, \\ &a_j \to (a_1)^{l/m(1,j)} && \text{if } (m(1,j), \mathscr{F}_1) \neq 1 \text{ and } m(1,j) \neq 0, \end{aligned}$$

and by applying (2.4).

(3.4) Lemma. *Let l be an integer such that $(l, \mathscr{F}_2) = 1$. Then there exists a mapping*

$$f\colon X_2 \to K_2 = K(\{a_0, a_2; (a_0 a_1^k)^l\})$$

such that $f_2(x) = g_2(x)$ for each $x \in A_2$.

Proof. Let h_2 be as asserted in (3.1). By (2.4), there exists a mapping $K(P_2) \to K(P_2')$, which is the identity on S, where P_2' is defined as P_2 except that $n(2,j)$ is replaced by 1 and $m(2,j)$ is replaced by $m'(2,j) = m(2,j)/n(2,j)$ for each $1 \leqslant j \leqslant k(2)$. Then there exists a mapping

$$K(P_2') \to K(\{a_0, a_1; a_0 a_1^k\}),$$

which is the identity on S; it is induced by the substitutions $a_1 \to (a_j)^{k/m'(2,j)}$. The lemma follows by applying (2.3) successively.

4. Appendix. An alternative approach to a result of [Sp2] and Theorem 2. We will give an alternative proof of Theorems 3 and 3′ of [Sp2] and of the "only if" part of Theorem 2, which does not depend on Lemma (2.2) and uses Eilenberg–MacLane

spaces. We shall consider the case of 2-dimensional compacta only, as the proof in higher dimensions is the same.

First, we introduce some notations. For a nonempty set $\mathscr{F}$ of primes, by $Q_{\mathscr{F}}$ we denote the intersection $\bigcap \{Q_p\colon p \in \mathscr{F}\}$. If $\mathscr{F}$ is empty then we define $Q_{\mathscr{F}} = Q$. By $Q^{(p)}$ we denote the additive group of all rationals whose denominators are powers of a prime p, and by N the set of nonnegative integers. The following informations, stated in (4.1) and (4.5), on the Eilenberg–MacLane spaces $K(Q_{\mathscr{F}}, 1)$ and $K(Q^{(p)}/Z, 1)$ are necessary.

(4.1) Lemma. *Let $\mathscr{F}$ be a set of primes and let $\tilde{\mathscr{F}}$ denote the set of primes not belonging to $\mathscr{F}$. Suppose P is a presentation whose symbols form the set*

$$\mathscr{A} = \{a_0\} \cup \{a_{p,m}\colon p \in \tilde{\mathscr{F}}, m \in N\},$$

and whose relators form the set

$$\{(a_{p,n})^{p^{n-m}} a_{p,m}^{-1}\colon p \in \tilde{\mathscr{F}}, m, n \in N \text{ and } m < n\} \cup \{[c, d]\colon c, d \in \mathscr{A}\},$$

where each $a_{p,0} = a_0$ for each $p \in \tilde{\mathscr{F}}$. Then, the 2-dimensional complex $K(P)$ satisfies $\pi_1(K(P), *) \simeq Q_{\mathscr{F}}$. Therefore, $K(P)$ can be considered as a 2-skeleton of an Eilenberg–MacLane space $K(Q_{\mathscr{F}}, 1)$.

Proof. Let $h\colon \pi_1(K(P), *) \to Q_{\mathscr{F}}$ be a homomorphism such that $h(a_0) = 1$ and $h(a_{p,m}) = p^{-m}$. We will prove that h is an isomorphism, by using the following fact (see [Fu], Theorem 8.4, Chapter II):

$$Q_{\mathscr{F}}/Z = \bigoplus \{Q^{(p)}/Z\colon p \in \tilde{\mathscr{F}}\}. \tag{4.2}$$

Observe that any $a \in \pi_1(K(P), *)$ can be presented in the following form

$$a = a_0^n \cdot \prod_{p \in \mathscr{F}} a_{p,n(p)}^{k(p)}, \tag{4.3}$$

where $k(p)$ is 0 for almost all p and $0 \leqslant k(p) < p^{n(p)}$. Suppose that

$$h(a) = n + \sum_{p \in \mathscr{F}} k(p) \cdot p^{-n(p)} \tag{4.4}$$

is 0 (in $Q_{\mathscr{F}}$). Then $\sum_{p \in \mathscr{F}} k(p) \cdot p^{-n(p)}$, considered as an element of $Q_{\mathscr{F}}/Z$, is equal to 0. Thus, by (4.2), we have that each $k(p)$ is 0, and consequently $n = 0$. It follows that $a = 0$, which proves that h is a monomorphism.

Using (4.2), one can prove that any $b \in Q_{\mathscr{F}}$ can be presented in the form given by (4.4), where $k(p)$ is 0 for almost all p and $0 \leqslant k(p) < p^{n(p)}$. Thus $b = h(a)$, where a is given by (4.3). It follows that h is an epimorphism.

(4.5) Lemma. *Suppose $\tilde{P}$ is a presentation whose symbols form the set $\{a_n\colon n \in N\}$ and whose relators form the set*

$$\{a_0\} \cup \{a_n (a_{n+1})^{-p}\colon n \in N\}.$$

Then, the 2-dimensional complex $K(\tilde{P})$ satisfies $\pi_1(K(\tilde{P}), *) \simeq Q^{(p)}/Z$. Therefore, $K(\tilde{P})$ can be considered as a 2-skeleton of an Eilenberg–MacLane space $K(Q^{(p)}, 1)$.

Proof. The homomorphism $h\colon \pi_1(K(\tilde{P}), *) \to Q^{(p)}/Z$, given by $h(a_0) = 0$ and $h(a_m) = p^{-m}$ for $n > 0$, is an isomorphism.

As a consequence of (4.1), we obtain the following

(4.6) LEMMA. *Let $\mathscr{F}$ be a set of primes and let (X, A) be a pair of compacta such that $\dim X = 2$ and that $\operatorname{cdim}_{Q_{\mathscr{F}}} X \leqslant 1$. Then any mapping $A \to S$ can be extended to a mapping*

$$X \to K(\{a_0, a_1;\ a_0 a_1^n\})$$

for some integer n satisfying $(n, \mathscr{F}) = 1$.

Proof. Suppose a mapping $A \to S$ is given. Since $\operatorname{cdim}_{Q_{\mathscr{F}}} X \leqslant 1$ we can extend this mapping to a mapping $f\colon X \to K(Q_{\mathscr{F}}, 1)$, see [Ko] (let us recall that a_0 corresponds to S). Since X is a 2-dimensional compactum thus we may assume that $f(X)$ is contained in a compact subset of $K(P)$. Thus there exist a finite subset $\mathscr{F}' = \{p_1, \ldots, p_k\}$ of $\tilde{\mathscr{F}}$ and an integer r such that

$$f(X) \subset K(P') \subset K(P),$$

where P' is the presentation whose symbols form the set

$$\mathscr{A}' = \{a_0\} \cup \{a_{p,m}\colon p \in \mathscr{F}', m \in N \text{ and } m \leqslant r\},$$

and whose relators form the set

$$\{(a_{p,n})^{p^{n-m}} a_{p,m}^{-1}\colon p \in \mathscr{F}', m, n \in N \text{ and } m < n \leqslant r\} \cup \{[c, d]\colon c, d \in \mathscr{A}'\}.$$

The substitutions $a_{p,m} \to (a_{p,r})^{p^{r-m}}$, where $p \in \mathscr{F}'$, $0 < m \leqslant r$ induce a mapping

$$K(P') \to K(P''),$$

which is the identity on S, where P'' is the presentation whose symbols form the set

$$\mathscr{A}'' = \{a_0\} \cup \{a_{p,r}\colon p \in \mathscr{F}'\},$$

and whose relators form the set

$$\{a_0^{-1} a_{p,r}^{p^r}\colon p \in \mathscr{F}'\} \cup \{[c, d]\colon c, d \in \mathscr{A}''\}.$$

Finally, there exists a mapping

$$K(P'') \to K(\{a_0, a_1;\ a_0 a_1^n\}),$$

which is the identity on S, where $n = (p_1 \cdot \ldots \cdot p_k)^r$, and Lemma (4.6) follows.

Now, suppose X is a 2-dimensional compactum such that $\dim(X \times X) < 4$. By Theorem 41–4 in [Ko], it follows that $\operatorname{cdim}_Q X < 2$ and $\operatorname{cdim}_{Z(p)} X < 2$ for each prime p. By (2.3) and (4.6), any mapping $A \to S$ can be extended to a mapping

$$X \to K(\{a_0, a_1;\ (a_0 a^n)\})$$

and consequently to a mapping

$$X \to K(\{a_0, a_1;\ a_0^n (a_1)^{n^2}, [a_0, a_1]\})$$

for some integer n. Thus we obtain an alternative proof of the "only if" part of Theorem 3 in [Sp2] that characterizes compacta X such that $\dim(X \times X) < 2\dim X = 4$. Theorem 3′ of [Sp2], dealing with the case $\dim X > 2$, can be reproved similarly.

(4.7) LEMMA. *Let k be an integer and let (X, A) be a pair of compacta such that $\dim X = 2$ and that $\operatorname{cdim}_{Q^{(p)}/Z} X \leqslant 1$ for each prime p which divides k. Then any mapping $A \to S$ can be extended to a mapping*

$$X \to K(\{a_0, a_1;\ a_0^k, a_0 a_1^l\})$$

for some integer l which has the following property: if a prime p divides l then p divides k.

Outline of the proof. The case where k is a power of a prime p can be obtained as a consequence of (4.5) similarly to the proof (4.6). Then, the lemma can be proved by applying the proof of (4.4) in [Sp2] and (2.5) in [Sp2].

An alternative proof of Theorem 2 (the "only if" part in the case $m = 2 = n$): Let X and Y be 2-dimensional compacta such that $\dim(X \times Y) < 4$. By 41–4 in [Ko], we have $\operatorname{cdim}_Q X < 2$ or $\operatorname{cdim}_Q Y < 2$. Without loss of generality we may assume that $\operatorname{cdim}_Q X < 2$. By $\mathscr{F}$ we denote the set of all primes such that $\operatorname{cdim}_{Q_p} X < 2$. Then, by Bockstein's first theorem (see [Ko], p. 233), we have $\operatorname{cdim}_{Q_{\mathscr{F}}} X < 2$. By (4.6), any mapping $A \to S$, where A is a closed subset of X, admits an extension

$$X \to K(\{a_0, a_1;\ a_0 a_1^k\})$$

for some integer k satisfying $(k, \mathscr{F}) = 1$.

By $\mathscr{P}(k)$ we denote the set of all primes p which divide k. Since $\mathscr{P}(k) \cap \mathscr{F} = \emptyset$, thus by 41–4 in [Ko] it follows that $\operatorname{cdim}_{Q^{(p)}/Z} Y < 2$ for each $p \in \mathscr{P}(k)$. Let $\mathscr{P}'$ be the set of all primes $p \in \mathscr{P}(k)$ such that $\operatorname{cdim}_{Z(p)} Y < 2$. Let us consider k as the product $s \cdot t$ of integers such that $(s, \mathscr{P}(k) \setminus \mathscr{P}') = 1$ and $(t, \mathscr{P}') = 1$. By (4.7), any mapping $B \to S$, where B is a closed subset of Y, admits an extension

$$Y \to K(\{a_0, a_1;\ a_0^t, a_0 a_1^l\})$$

for some integer l which has the following property: if a prime p divides l then $p \in \mathscr{P}(k) \setminus \mathscr{P}'$.

Observe that if a prime p divides s then $\operatorname{cdim}_{Z(p)} Y < 2$, and if a prime q divides l then $\operatorname{cdim}_{Z(q)} Y = 2$ and consequently $\operatorname{cdim}_{Z(q)} X < 2$ by Theorem 41–4 in [Ko]. Thus by (2.3), there exist extensions

$$X \to K(\{a_0, a_1;\ (a_0 a_1^k)^l\}) \quad \text{and} \quad Y \to K(\{a_0, a_1;\ a_0^k, a_0 a_1^l\})$$

of the mappings $A \to S$ and $B \to S$ respectively, and Theorem 2 follows.

I am grateful to Henryk Toruńczyk for helpful conversations and suggesting the use of Eilenberg–MacLane spaces to get alternative proofs in the Appendix. I would also like to thank Zbigniew Karno for helpful discussions.

References

[D-R-S1] A. N. Dranishnikov, D. Repovš and E. V. Shchepin, *A criterion for approximation of maps of 2-dimensional compacta into R^4 by embeddings*, Abstr. Amer. Math. Soc. 10 (1989), No. 89T-54-132, 313.

[D-R-S2] — — — *On intersection of compacta of complementary dimensions in Euclidean space*, Topology Appl., to appear.

[D-S] A. N. Dranishnikov and E. V. Shchepin, preprint.

[Fr] M. H. Freedman, *The topology of four dimensional manifolds*, J. Differential Geometry 17 (1982), 357–453.

[F-Q] M. Freedman and F. Quinn, *Topology of 4-manifolds*, preprint.

[Fu] L. Fuchs, *Infinite abelian groups*, vol. I, New York 1970.

[Ko] Y. Kodama, *Cohomological dimension theory*, Appendix to: K. Nagami, *Dimension theory*, New York 1970.

[Kr] J. Krasinkiewicz, *Imbeddings into R^m and dimension of products*, Fund. Math. 133 (1989), 247–253.

[K-L] J. Krasinkiewicz and K. Lorentz, *Disjoint membranes in cubes*, to appear in Bull. Pol. Acad. of Sciences.

[M-R] D. McCullough and L. R. Rubin, *Some m-dimensional compacta admitting a dense set of imbeddings into R^{2m}*, Fund. Math. 133 (1989), 235–243.

[Po] L. S. Pontrjagin, *Sur une hypothèse fondamentale de la théorie de la dimension*, C. R. Acad. Paris 190 (1930), 1105–1107.

[R-S] C. P. Rourke and B. J. Sanderson, *Introduction to Piecewise-Linear Topology*, Springer Verlag 1982.

[Sp1] S. Spież, *Imbeddings in R^{2m} of m-dimensional compacta with* $\dim(X\times X) < 2m$, Fund. Math. 134 (1990), 103–113.

[Sp2] — *The structure of compacta satisfying* $\dim(X\times X) < 2\dim X$, Fund. Math. 135 (1990), 127–145.

INSTITUTE OF MATHEMATICS
POLISH ACADEMY OF SCIENCES
Śniadeckich 8
00-950 Warszawa, Poland

Received 3 April 1989;
in revised form 18 May 1989