Caliber (ω_1, ω) is not productive

by

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Abstract. Caliber (ω_1, ω) is a property between separability and the countable chain condition which was introduced by Comfort and Negrepontis in their book *Chain Conditions in Topological Spaces*. Separability is productive while the productivity of the countable chain condition is consistent but independent of the axioms of set theory. In this paper, we construct, in ZFC, a space of caliber (ω_1, ω) whose square does not have caliber (ω_1, ω). The combinatorics used in the construction are of independent interest.

1. Introduction. In *Chain Conditions in Topological Spaces*, Comfort and Negrepontis [1] introduced the property of caliber (ω_1, ω): A topological space has caliber (ω_1, ω) if point-finite families of open sets are countable. This property lies between separability and the countable chain condition. Separability is productive while the productivity of the countable chain condition is consistent but independent of the axioms of set theory. In this paper, we construct, in ZFC, a space X of caliber (ω_1, ω) whose square does not have caliber (ω_1, ω).

In compact Hausdorff spaces (or even Baire spaces), caliber (ω_1, ω) coincides with precaliber (ω_1, ω) (any uncountable family of open sets has an infinite centred subfamily) which is equivalent to the countable chain condition (construct inductively maximal centred subfamilies — either some subfamily is infinite or the intersections of these finite subfamilies are an uncountable disjoint family of open sets).

Under $MA_{\aleph_1}$, the product of two spaces with the countable chain condition has the countable chain condition (this was shown by Kunen in 1968 (see [3])) and so, under $MA_{\aleph_1}$, the product of two compact Hausdorff spaces with caliber (ω_1, ω) also has caliber (ω_1, ω). The example in this paper is completely regular but cannot, therefore, be compact (or even Baire).

A useful discussion of calibers and precalibers can be found in [2] while a discussion of MA and productivity can be found in [3].

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2. Getting X^2 to have an uncountable point-finite open family. Let A_0 and A_1 be disjoint sets of cardinality $\aleph_1$. Let $\Pi_i\colon \omega_1 \to A_i$ $(i \in 2)$ be bijections which witness this fact. We shall define a family F of countable subsets of ω_1 which are increasing ω-sequences. We shall also define a partition of F into subfamilies F_0 and F_1. The counterexample X is the subspace of $2^{A_0 \cup A_1}$ defined by $p \in X$ if and only if

1 $\alpha \in \omega_1 \Rightarrow (p(\Pi_0(\alpha)) = 0 \vee p(\Pi_1(\alpha)) = 0)$;

2. $i \in 2 \Rightarrow (\exists f \in F_i)\colon A_i \cap p^{-1}(1) \overset{*}{\subseteq} \Pi_i'' f$

($\overset{*}{\subseteq}$ denotes inclusion mod finite).

Lemma 1. *If $f_i \in F_i$ $(i \in 2)$ implies $|f_0 \cap f_1| < \omega$ then X^2 has an uncountable point-finite family.*

Proof. Let $U_\alpha = \prod_{i \in 2}\{p \in X\colon p(\Pi_i(\alpha)) = 1\}$. Each U_α is a nonempty open subset of X^2 (nonempty because U_α contains the function $(f_0, f_1) \in X^2$ defined by $f_i^{-1}(1) = \{\Pi_i(\alpha)\}$). In fact, $\{U_\alpha\colon \alpha \in \omega_1\}$ is an uncountable point-finite open family. To demonstrate this, suppose that $(p_0, p_1) \in \bigcap \{U_{\alpha(n)}\colon n \in \omega\}$. Consider the first coordinate: $p_0\big(\Pi_0(\alpha(n))\big) = 1$ for each $n \in \omega$ while, by (2) above, there is $f_0 \in F_0$ such that $A_0 \cap p_0^{-1} \overset{*}{\subseteq} \Pi_0'' f_0$. Thus $\{\Pi_0(\alpha(n))\colon n \in \omega\} \overset{*}{\subseteq} \Pi_0'' f_0$ and so $\{\alpha_n\colon n \in \omega\} \overset{*}{\subseteq} f_0$. A similar argument shows that there is $f_1 \in F_1$ such that $\{\alpha_n\colon n \in \omega\} \overset{*}{\subseteq} f_1$ and so $|f_0 \cap f_1| = \omega$ which is impossible.

Remark: A standard Δ-system argument shows that X has the countable chain condition. In fact, X has precaliber $\aleph_1$. Since precaliber $\aleph_1$ is productive, X^2 has the countable chain condition as well.

3. Getting X to have caliber (ω_1, ω). The precise definition of F is a bit difficult. What we need is to be able to recapture a finite subset of ω_1 from its maximum and an integer: if $A \subset \omega_1$ has order-type ω and supremum α then A_n is defined to be $\{a(n)\colon a \in A\}$ if this set also has order-type ω and supremum α (where each ordinal $a \in \omega_1$ is interpreted as a bijection $a\colon \omega \to a$). This definition can be extended to finite strings of integers in a natural way: if $\sigma = \tau * n$ where $*$ denotes concatenation then $A(\sigma) = (A(\tau))(n)$ if this set also has order-type ω and supremum α. If A_σ exists, then we say σ *codes* A.

A technical problem arises here. How do we distinguish between a set A and another set $A(n)$? We need to well-found the $A(\cdot)$ process and rigorize the notion that A and $A(n)$ have different behavior. This is manifested, for example, in the length of a maximal string of n's which code A and $A(n)$. The definition of *urelement* is only a little more difficult than this: the *behavior* of A on σ where σ codes A is a function $f\colon [\sigma] \times \operatorname{dom}\sigma \to \omega$ where $[\sigma]$ is the set of all nonempty substrings of σ.

If τ is a nonempty substring of σ and $n \in \operatorname{dom}\sigma$ then $f(\tau, n)$ is the maximal integer m such that $\sigma n * \tau^m$ codes A (τ^m denotes the concatenation of m copies of τ). Of course, some sets may not have a defined behavior but this is not a problem. Unfortunately, we need behavior to be hereditary to ensure that the hypothesis of Lemma 2 is satisfied. We say that A has behavior f *hereditarily* if, for each $\tau \in [\sigma]$ and $n \in \operatorname{dom}\sigma$, $A(\sigma n * \tau^{f(\tau,n)+1})$ is bounded below $\sup A$. Note that if B is an

infinite subset of a set A with behavior f hereditarily, then B also has behavior f.

The definition of F (and F_0 and F_1) can now be made. Let $\{S(\sigma, f, x)\colon \sigma$ is a string of integers, $f\colon [\sigma] \times \operatorname{dom}\sigma \to \omega$ and $x\colon 1 + \operatorname{dom}\sigma \to 2\}$ partition the limit ordinals into stationary sets. For each $\alpha \in S(\sigma, f, x)$, let $F^*(\alpha)$ be the family of all subsets A of ω_1 of order-type ω and supremum α such that σ codes A and A has behavior f hereditarily. Let $F(\alpha) = \{A(\sigma \restriction n)\colon A \in F^*(\alpha), n \leqslant \operatorname{dom}\sigma\}$ and let $F = \bigcup \{F(\alpha)\colon \alpha$ is a countable limit ordinal$\}$. To complete the definition, let $A(\sigma \restriction n) \in F_i$ if $x(n) = i$. Note that each F_i is closed under finite union, closed under adding a finite set and closed under taking an infinite subset. What is not so evident is that F_0 and F_1 are disjoint:

Lemma 2. *F_0 and F_1 are disjoint (and so the hypothesis of Lemma 1 is satisfied by the closure properties of each F_i).*

Proof. Suppose that $A, B \in F_*(\alpha)$, $\alpha \in S(\sigma, f, x)$ and $m < n$ but that $A(\sigma \restriction m) = B(\sigma \restriction n)$. Let $\tau \in [\sigma]$ be such that $\sigma \restriction m * \tau = \sigma \restriction n$. Let f_A, f_B be the behavior of A and B respectively. We can calculate $f_A(\tau, m) = f_B(\tau, m) + 1$, which contradicts $f_A = f = f_B$.

The basic thinning-out lemma we need is:

Lemma 3. *If $\{\{a_i^\alpha\colon i \in n\}\colon \alpha \in \omega_1\}$ is any disjoint family of subsets of ω_1 (where we assume $a_0^\alpha > a_1^\alpha > a_2^\alpha > \ldots$ for simplicity), then there is a closed unbounded set C, a string σ, a behavior f and an uncountable $A \subset \omega_1$ such that for each $\alpha \in C$ and each countable $Y \subset A$ such that $\{a_0^\beta\colon \beta \in Y\}$ is an increasing sequence converging to α:*

1. *$\{\{a_i^\beta\colon \beta \in X\}\colon i < n\} = \{\{a_0^\beta\colon \beta \in X\}(\sigma \restriction i)\colon i < n\}$;*
2. *$\{a_0^\beta\colon \beta \in X\}$ has code σ and hereditary behavior f on σ.*

Proof. Define a string σ_α for each $\alpha \in \omega_1$ by $\{a_i^\alpha\colon i < n\} = \{a_0^\alpha(\sigma(\alpha) \restriction i)\colon i < n\}$. Now $\sigma(\alpha) = \sigma$ is fixed on a stationary $A \subset \omega_1$. For each $\tau \in [\sigma]$, $n \in \operatorname{dom}\sigma$, $\alpha \in A$, let $f_\alpha(\tau, n)$ be the maximal integer m such that $\alpha(\sigma \restriction n * \tau^m) \geqslant \alpha$. Assume $f_\alpha = f$ is fixed. Furthermore, press down (that is apply Ulam's theorem which states that if S is a stationary subset of ω_1 and $f\colon S \to \omega_1$ satisfies $(\forall \alpha \in S) f(\alpha) < \alpha$ then $(\exists$ stationary $T \subset S)(\exists \eta \in \omega_1)\colon (\forall \alpha \in T) f(\alpha) = \eta)$ finitely many times to find $\eta \in \omega_1$ such that for each $\tau \in [\sigma]$, $n \in \operatorname{dom}\sigma$, $\alpha \in A$ we have $\alpha(\sigma \restriction n * \tau^{m+1}) < \eta$ where $m = f_\alpha(\tau, n)$. Thin out A so that $\alpha < \beta \in A$ implies $a_0^\alpha < a_{n-1}^\beta$ and then find $C \subset \omega_1 - \eta$ such that $c < d \in C$ and $c < a_0^\alpha \leqslant d$ implies $c < a_{n-1}^\alpha$.

We can now prove the result:

Lemma 4. *X has caliber (ω_1, ω).*

Proof. For each $\alpha \in \omega_1$, let $p_\alpha \in Fn(A_0 \cup A_1, 2)$ be a finite partial function which indicates a nonempty basic open subset $\{f \in X\colon f \supset p_\alpha\}$ of X which will be denoted $[p_\alpha]$. By assumption (1) in the definition of X, for each $\alpha \in \omega_1$, we can find $n \in \omega$, $\{A_i^\alpha\colon i < n\} \subset \omega_1$ and $x\colon n \to 2$ such that

$$\operatorname{supp} p_\alpha = \{\Pi_j(a_i^\alpha)\colon i < n, x(i) = j\}.$$

Assume the $\{a_i^\alpha: i<n\}$ are disjoint outside a root (that is, assume that there is a finite set Δ such that $\{a_i^\alpha-\Delta: i<n\}$ is a disjoint family) be applying the delta-system lemma for finite sets and reindexing. We can ignore the root since it is finite. We can assume that n is fixed and x is fixed and then we can apply Lemma 3 to find a closed unbounded set C, a string σ, a behavior f and an uncountable $A\subset\omega_1$. Let $\alpha\in C\cap S(\sigma,f,x)$ be such that there is an infinite $Y\subset A$ such that $\{a_0^\alpha: \alpha\in Y\}$ increases to α (this requires intersection with another closed unbounded set). Now $\{a_0^\alpha: \alpha\in Y\}$ has code σ and behavior f hereditarily which implies that $\{a_0^\alpha: \alpha\in Y\}\in F^*(\alpha)$. This means that $\{a_i^\alpha: \alpha\in Y\}=\{a_0^\alpha: \alpha\in Y\}(\sigma\restriction i)\in F_{x(i)}$. By closure under finite unions $\{a_i^\alpha: \alpha\in Y, i\in x^{-1}(j)\}\in F_j$ and so $p^{-1}(1)\cap A_j\in \Pi_j(f)$ for some $f\in F_j$. By the definition of X, $\{[p_\alpha]: \alpha\in Y\}$ has nonempty intersection with X.

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Slender modules, endo-slender abelian groups and large cardinals

by

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Dedicated to Professor Hiroyuki Tachikawa on his 60-th birthday

Abstract. We prove the following theorems introducing some new notions.

THEOREM A. *The following* (1)–(3) *are equivalent*:

(1) *For an arbirary infinite cardinal* μ, *there exists an* $L_{\mu\omega}$-*compact cardinal*;

(2) *For an arbitrary ring* R *and module* M_R, *there exists a cardinal* $\varkappa$ *such that* $R_M=R_M^{[\varkappa]}$, *where* $R_M A=\bigcap\{\mathrm{Ker}(h): h\in\mathrm{Hom}_R(A,M)\}$ *and* $R_M^{[\varkappa]}A=\sum\{R_M X_R: X_R\leqslant A_R \text{ and } |X|<\varkappa\}$;

(3) *For an arbitrary ring* R *and module* M_R, *the torsion class* ${}^\perp M_R$ *is singly generated, i.e.*, ${}^\perp M_R={}^\perp(A_R^\perp)$ *for some* A_R.

An abelian group A is endo-slender, if A is a slender module over its endomorphism ring.

THEOREM B. *Let* $A(=\Pi_{i\in I}A_i)$ *be a direct product of reduced torsion free groups* A_i *of rank* 1. *Then*, A *is endo-slender iff for any infinite subset* X *of* I *there exists a* $j\in I$ *such that*

$$\{i\in X: t(A_i)\leqslant t(A_j)\}$$

is infinite.

THEOREM C. *Let* B *be an* (ω,∞)-*distributive complete Boolean algebra and* A *a countable reduced torsionfree abelian group. Then, the Boolean power* $A^{(B)}$ *is endo-slender iff* B *satisfies* $(*\omega_1)$, *i.e. For any family of nonzero elements* $\{b_n: n<\omega\}$ *there exist a nonzero* b, *an infinite subset* I *of* ω *and* h_n $(n\in I)$ *such that* $h_n: [0,b_n]\to[0,b]$ *is a countably complete homomorphism and* $h_n(b_n)=b$ *for each* $n\in I$.

0. Introduction. There had been several studies about slender modules and rings as generalizations of slender abelian groups even before the works of Huber [23] and Mader [26]. However, they found a somewhat new situation, where slenderness occurs through consideration of abelian groups and modules as modules over their endomorphism rings. On the other hand, the fundamental theorem about slender groups due to J. Łoś was generalized to arbitrary cardinalities by the author [8, 10]. This clarified why a measurable cardinal appears concerning abelian groups. Though