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If x and y are independent realizations of q, then $x-y$ realizes the type $p_{\eta_A}|a$, where η_A is the defining function of A. Thus q and p_{η_A} are non-orthogonal. Since q is an arbitrary type, T is non-multidimensional. It is clear that T is unsuperstable.

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A subclass of the class MOBI *

by

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Abstract. Necessary and sufficient conditions are given for a regular space to be an open and compact image of a σ-discrete metacompact Moore space. The class of regular spaces satisfying these conditions is invariant under open mappings with compact metric fibers. This gives a characterization of the minimal class of regular spaces containing all σ-discrete metric spaces and invariant under open and compact mappings.

For a class $\mathscr{K}$ of topological spaces, let $\mathrm{MOBI}_i(\mathscr{K})$ be the minimal class of T_i-spaces containing all metric spaces from $\mathscr{K}$ and invariant under open and compact mappings (see [BCh1]).

It is easy to observe that a T_i-space is in $\mathrm{MOBI}_i(\mathscr{K})$ if and only if it can be obtained as an image of a metric space from $\mathscr{K}$ under a mapping which is a composition of a finite number of open and compact mappings with T_i-domains [B].

If the class $\mathscr{K}$ contains the class of all metric spaces, we write MOBI_i instead of $\mathrm{MOBI}_i(\mathscr{K})$.

The purpose of this note is to prove a characterization of the class MOBI_3 (σ-discrete). This gives a partial solution to the problem of characterizing MOBI_3 (see [A] and [Ch2]) and generalizes the characterization of MOBI_3(scattered) from [BCh2].

There seems to be a pattern that the solutions of problems concerning MOBI_3 are similar to the solutions of the corresponding problems in MOBI_2. The main difference is that the techniques needed in the regular case are more complicated than those required in MOBI_2.

In the present paper we follow this pattern. We prove that a regular space is in $\mathrm{MOBI}_3(\sigma\text{-discrete})$ if and only if it is in $\mathrm{MOBI}_2(\sigma\text{-discrete})$ (see [Ch3]) and has a base of countable order.

It turns out that in characterizing $\mathrm{MOBI}_i(\sigma\text{-discrete})$, the regular case ($i = 3$) is much more complicated than the Hausdorff case ($i = 2$). In fact, the techniques used in this paper have been distilled from [BCh2] rather than from [Ch3].

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1. Introduction. Unless stated otherwise, all spaces are assumed to be regular. All mappings are continuous and onto. Open and compact mappings are the open mappings with compact fibers.

A sequence $\langle G_n: n \geq 1\rangle$ of subsets of a space Y is called *decreasing* (*strictly decreasing*) if $G_{n+1} \subset G_n$ ($\bar{G}_{n+1} \subset G_n$) for $n \geq 1$.

Recall that a space Y is said to *have a base of countable order* if there exists a sequence $\langle \mathcal{G}_n: n \geq 1\rangle$ of bases of Y such that each decreasing (or, equivalently, strictly decreasing) sequence $\langle G_n: n \geq 1\rangle$, where $G_n \in \mathcal{G}_n$ for $n \geq 1$, satisfies

(d) if $y \in \bigcap \{G_n: n \geq 1\}$, then $\{G_n: n \geq 1\}$ is a base for y in Y.

If, in addition,

(λ) there exists a $y \in \bigcap \{\bar{G}_n: n \geq 1\}$

is satisfied, then Y is said to *have a λ-base* (see [WW2] and [ChČN]).

It is well known (see [WW1]), that all spaces in MOBI_3 have a base of countable order and all spaces in MOBI_3 (complete) have a λ-base.

The two known subclasses of the class MOBI_3 are the classes of spaces with a point-countable base which are either locally nice (see [BCh1, 1.8]) or scattered ([BCh2]). In both cases the base of countable order is hidden behind a stronger property. In the present paper we actually use the base of countable order.

In order to show how this will be done, we start with a proposition which is a version of our main result for complete spaces.

Proposition 1.1. *For a regular space Y the following conditions are equivalent.*

(a) *Y is in* MOBI_3 *(scattered),*

(b) *Y is a scattered space in* MOBI_3,

(c) *Y is a scattered space with a point-countable base,*

(c′) *Y has a λ-base and is in* MOBI_2 *(σ-discrete).*

The equivalence of the conditions (a), (b) and (c) has been proved in [BCh2]. Since a metric space is scattered if and only if it is σ-discrete and complete ([N, 3.17, 20], [WW2]) and, in fact, first-countable scattered spaces have a λ-base [WW2], the proof of 1.1 reduces to showing that in the class MOBI_2 (σ-discrete) being scattered is a consequence of having a λ-base. Before proving this fact (see 1.2 and 3 below), we have to recall a characterization of MOBI_2 (σ-discrete) from [Ch3] and the related definitions.

A *neighbornet* for a space Y is a relation $V \subset Y \times Y$ such that for each $y \in Y$, $y \in \operatorname{Int} V(y)$, where $R(y) = \{z \in Y: \langle y, z\rangle \in R\}$ for a relation $R \subset Y \times Y$ [J1].

A neighbornet V is called *co-countable* (*co-finite*) if $V^{-1}(y)$ is countable (finite) for $y \in Y$ [J2]. A neighbornet V in Y is called *transitive* or *antisymmetric* if the relation V is transitive ($V \circ V = V$) or, respectively, antisymmetric ($V \cap V^{-1} = \{\langle y, y\rangle: y \in Y\}$) [J1]. Since all neighbornets are reflexive, a neighbornet V which is transitive is a quasiorder while a transitive and antisymmetric neighbornet is a partial order. In both these cases, the fact that the relation is a (co-countable) neighbornet means that (the initial segments are countable and) the final segments are open.

Lemma 1.2 (see [Ch3, 4.2]). *If a T_1-space Y has a co-countable neighbornet, then it has a co-countable neighbornet which is both transitive and antisymmetric.*

Proof. Suppose that V is a co-countable neighbornet in Y. It is easy to check that $Q = \bigcup \{V^n: n \geq 1\}$ is a transitive co-countable neighbornet in Y [J2]. Thus Q is a quasi-order $\leq$ with countable initial segments and open final segments. The relation $R = Q \cap Q^{-1} = \{\langle y, z\rangle: y \leq z \text{ and } z \leq y\}$ is an equivalence relation with countable equivalence classes.

For each equivalence class of R, fix an ordering of this class of order type less than or equal to ω. Define $P \subset Y \times Y$ by putting $P(y)$ to be $Q(y)$ without the finite set of predecessors of y in $R(y)$.

Clearly, P is a co-countable neighbornet in Y and our construction assures that P is a partial order.

In Section 2 of [Ch3], it was shown that a Hausdorff space Y is in the class MOBI_2 (σ-discrete) if an only if it is a first-countable space with a co-countable neighbornet.

We are now ready to finish the proof of (c′)$\Rightarrow$(c) in 1.1 and to show how bases of countable order work with neighbornets.

Proposition 1.3. *If a space Y has an antisymmetric neighbornet and a λ-base, then Y is a scattered space.*

Proof. Let V be an antisymmetric neighbornet in Y and $\langle \mathcal{G}_n: n \geq 1\rangle$ be a sequence of bases of Y witnessing the fact that Y has a λ-base.

If Y is not a scattered space, then there exists a nonempty, closed and dense in itself subset F of Y.

Since no point of F is isolated in F, one can construct, by induction, sequences $\langle y_n: n \geq 0\rangle$ and $\langle G_n: n \geq 0\rangle$ such that

(0) $G_0 = Y$

and, for $n \geq 0$,

(1) $y_n \in F \cap G_n \setminus \{y_0, \ldots, y_{n-1}\}$,

(2) $y_n \in G_{n+1} \in \mathcal{G}_{n+1}$ and $\bar{G}_{n+1} \subset G_n$,

(3) $G_{n+1} \subset V(y_n)$.

By (2), the sequence $\langle G_n: n \geq 1\rangle$ satisfies (d) and (λ). Thus there exists a $y \in \bigcap \{G_n: n \geq 1\}$ and $\{G_n: n \geq 1\}$ is a base for y in Y. In particular, there exists and $n \geq 0$ such that $G_{n+1} \subset V(y)$. Since $y_n \in G_{n+1}$. We obtain $y_n \in V(y)$. On the other hand, by (3), $y \in G_{n+1} \subset V(y_n)$ and this contradicts the assumption that V is antisymmetric.

2. The main result.

Theorem 2.1. *For a regular space Y the following conditions are equivalent.*

(a) *Y is an open and compact image of a σ-discrete metacompact Moore space,*

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(a′) *Y is in* MOBI_3 *(σ-discrete),*

(b) *Y has a co-countable neighbornet and is in* MOBI_3,

(b′) *Y has an antisymmetric neighbornet and is in* MOBI_3,

(c) *Y has an antisymmetric neighbornet, point-countable base and a base of countable order,*

(c′) *Y has a co-countable neighbornet and a base of countable order.*

Proof. In [J2] it is shown that σ-discrete metacompact Moore spaces are open finite-to-one images of σ-discrete metric spaces. This gives (a)⇒(a′).

The implication (a′)⇒(b) follows from the fact that σ-discrete metric spaces have a co-countable neighbornet and the latter property is invariant under open mappings with separable fibers [J2].

Lemma 1.2 gives (b)⇒(b′) and (b′)⇒(c) is well known.

In order to prove (c)⇒(c′), assume that V is an antisymmetric neighbornet and $\mathscr{W}$ is a point countable base in Y. For each $y \in Y$ fix a $W(y) \in \mathscr{W}$ such that $y \in W(y) \subset V(y)$. Since V is antisymmetric, the indexing of $\mathscr{W}' = \{W(y):\ y \in Y\}$ is one-to-one and, consequently, $\mathscr{W}'$ generates a co-countable neighbornet.

It remains to prove (c′)⇒(a). In the next section we show that for any regular space Y with a co-countable neighbornet and a base of countable order there exists a regular first-countable space X with a co-finite neighbornet and an open and compact mapping f of X onto Y. Since regular first-countable spaces with a co-finite neighbornet are, precisely, σ-discrete metacompact Moore spaces [J2], this will prove (c′)⇒(a).

3. The construction. Let Y be a regular space with a co-countable neighbornet and let $\langle \mathscr{G}_n:\ n \geqslant 1\rangle$ be a sequence of bases of Y witnessing the fact that Y has a base of countable order.

By 1.2, we can assume that Y has a co-countable neighbornet V which is a partial order $\leqslant$ (such that the final segments $V(y) = \{z \in Y:\ z \geqslant y\}$ are open and the initial segments $V^{-1}(y) = \{z \in Y:\ z \leqslant y\}$ are countable).

As in [Ch3], the space X will consist of (increasing with respect to V) finite sequences of elements of Y and the points compactifying the fibers of the projection e which maps each sequence onto its last term. However, the topology on the set of sequences is different than the topology from [J2] which was used in [Ch3] and not all the (increasing) sequences in Y are considered.

We begin with the inductive definition of the sequences which are to become elements of X. The construction resembles the induction from the proof of 1.3.

We define, by induction, for each $n \geqslant 0$, a subset S_n of the set of all the increasing n-element sequences in Y and, for each $s \in S_{n+1}$, an open subset $G(s)$ of Y containing the last term of s.

We start with

(0) $$S_0 = \{\emptyset\} \quad \text{and} \quad G(\emptyset) = Y$$

and proceed according to the following three conditions.

(1) $$S_{n+1} = \{p^\frown y:\ p \in S_n \text{ and } y \in G(p)\setminus\{c(p)\}\},$$

where $p^\frown y$ denotes the extension of p by y and $e(p)$ is the last term of p ($\{e(\emptyset)\} = \emptyset$).

Moreover, for $s = p^\frown y \in S_{n+1}$

(2) $$y \in G(p^\frown y) \in \mathscr{G}_{n+1}, \quad G(p^\frown y) \subset G(p)$$

and

(3) $$\overline{G(p^\frown y)} \subset V(y).$$

Given the sets S_n and $G(s)$ for $s \in S_n$, define

$$S = \bigcup \{S_n:\ n \geqslant 1\}$$

and

$$P = S \cup S_0.$$

Observe that (1) and (2) imply

(4) $$\text{if } p, q \in P, p \subset q \text{ and } q^\frown y \in S, \text{ then } p^\frown y \in S.$$

For $p \in P$ define $A(p) = \{s \in S:\ p \subset s\}$ and recall that $e(s) = y$ for $s = p^\frown y \in S$.

Consider $e\colon S \to Y$ defined above. Conditions (1) and (3) in the definition of S assure that all the sequences in S are strictly increasing with respect to the partial order V. Thus, for each $y \in Y$, $s \in e^{-1}(y)$ means that s is a sequence in $V^{-1}(y)$. Since $V^{-1}(y)$ is countable, it follows that e is countable-to-one (see [J2]).

We shall extend e by adding compactification points to each infinite fiber of e.

Let $*$ denote a point not in Y and define S^* to be the set of all sequences $p^\frown *^\frown y$ such that $p^\frown y \in S$ and

(5) $$e^{-1}(y) \cap A(p) \text{ cannot be covered by any finite subcollection of } \{A(p^\frown z):\ p^\frown z \in P\}.$$

Put

$$X = S \cup S^*.$$

Extend the function $e\colon S \to Y$ to a function $f\colon X \to Y$ by putting $f(p^\frown *^\frown y) = y$ for $p^\frown *^\frown y \in S^*$.

Before defining the topology of X we shall prove a fact which explains the definition of X and can be used to show that the fibers of f are compact.

3.1. *If D is an infinite subset of $e^{-1}(y)$, then there exists a $p \in P$ such that the set $e^{-1}(y) \cap A(p) \cap D$ cannot be covered by any finite subcollection of $\{A(p^\frown z): p^\frown z \in P\}$.*

Proof. Suppose that such a p does not exist. Put $p_0 = \emptyset$ and note that $e^{-1}(y) \cap A(p_0) \cap D = D$ is infinite. By induction on $n \geqslant 0$ construct a sequence $\langle y_n:\ n \geqslant 0\rangle$ such that for each $n \geqslant 0$, $p_n = \langle y_0, \ldots, y_{n-1}\rangle \in P$ and

(6) $$e^{-1}(y) \cap A(p_n) \cap D \text{ is infinite}.$$

The fact that for a $p_n \in P$ satisfying (6) we can find a $y_n \in Y$ such that $p_{n+1} = p_n\hat{}y_n \in P$ satisfies (6) is an easy consequence of our assumption that the infinite set in (6) can be covered by finitely many sets $A(p_n\hat{}z)$.

From (2), it follows that $G(p_n) \in \mathscr{G}_n$ and that $\langle G(p_n): n \geq 1\rangle$ is a decreasing sequence.

Since (6) implies that $e^{-1}(y) \cap A(p_n)$ is nonempty, condition (4) assures that $p_n\hat{}y \in S$ and (1) gives $y \in G(p_n)$ for $n \geq 1$.

Thus $\{G(p_n): n \geq 1\}$ is a base for y in Y and, in particular, there exists an $n \geq 0$ such that $G(p_{n+1}) \subset V(y)$. Since $y_n = e(p_{n+1}) \in G(p_{n+1})$, this contradicts the fact that the sequence $p_{n+1}\hat{}y$ is increasing in the partial order V (see the proof of 1.3).

Observe that if p is given by 3.1 for an infinite set $D \subset e^{-1}(y)$, then $s^* = p\hat{}*\hat{}y \in S^*$. The topology of X will be defined in such a way that s^* will be an accumulation point for D.

To define the topology of X we need some more notation.

For $p \in P$ put

$$B(p) = \{x \in X: p \subset x\},$$

where $p \subset x$ means that the sequence p is an initial segment of the sequence x. Note that

$$A(p) = B(p) \cap S \tag{7}$$

and, by (0), (1), (4) and the definition of S^*,

$$f(B(p)) \subset G(p). \tag{8}$$

For each $y \in Y$, fix a strictly decreasing base $\langle U_j(y): j \geq 0\rangle$ of neighborhoods of y in Y such that $U_0(y) = Y$.

For $s = p\hat{}y \in S$ and $j \geq 0$, define

$$B(p\hat{}y, j) = B(p\hat{}y) \cap f^{-1}(U_j(y)). \tag{9}$$

Fix a $y \in Y$. From (5) and 3.1, it follows that $f^{-1}(y)\setminus S \neq \emptyset$ if and only if $e^{-1}(y)$ is infinite. If $e^{-1}(y)$ is infinite, then $V^{-1}(y)$ is infinite and we can choose a one-to-one enumeration k_y: $V^{-1}(y) \mapsto \omega$ of $V^{-1}(y)$. For a $t \in e^{-1}(y)$ define

$$k_y(t) = \sum \{k_y(z): z \text{ is a term of } t\}.$$

Let $s^* = p\hat{}*\hat{}y \in S^*$. The traces on $f^{-1}(y)$ of the basic neighborhoods of s^* will be of the form

$$F(s^*, j) = f^{-1}(y) \cap B(p)\setminus \bigcup \{B(p\hat{}z): z \in V^{-1}(y) \text{ and } k_y(z) < j\}. \tag{10}$$

Note that in view of (5) and (7), for each $j \geq 0$,

$$F(p\hat{}*\hat{}y, j) \cap e^{-1}(y) \neq \emptyset. \tag{11}$$

This will assure that $e^{-1}(y)$ is dense in $f^{-1}(y)$ and 3.1 will imply that every infinite subset of $e^{-1}(y)$ has an accumulation point in $f^{-1}(y)$.

We complete the definition of neighborhoods of $s^* = p\hat{}*\hat{}y$ in X by putting

$$\begin{aligned} B(s^*, j) &= F(s^*, j) \cup \bigcup \mathscr{B}(s^*, j), \text{ where} \\ \mathscr{B}(s^*, j) &= \{B(t, j + k_y(t)): t \in F(s^*, j) \cap e^{-1}(y)\}. \end{aligned} \tag{12}$$

Note that if $p\hat{}z \in S$ and $j \geq 0$, then

$$B(p\hat{}*\hat{}y, j) \cap B(p\hat{}z) = \emptyset \text{ if either } z \notin V^{-1}(y) \text{ or } k_y(z) < j. \tag{13}$$

Moreover, for each $x \in X$, $j \geq 0$ and $p \in P$, the definition of $B(x, j)$ implies

$$\text{if } x \in B(p), \text{ then } B(x, 0) \subset B(p) \tag{14}$$

and

$$f(B(x, j)) \subset U_j(f(x)). \tag{15}$$

Define the topology in X by using the sets $B(x, j)$ as weak bases at x; that is, a set $B \subset X$ is open in X if and only if, for each $x \in B$ there exists a $j \geq 0$ such that $B(x, j) \subset B$.

3.2. *The function f: $X \to Y$ is continuous.*

Proof. The continuity of f follows from (15).

3.3. *The sets $B(x, j)$ are open in X.*

Proof. First observe that (14) implies that, for each $s \in S$, the set $B(s)$ is open in X. Thus, from 3.2 and (9), it follows that, for each $s \in S$ and $j \geq 0$, the set $B(s, j)$ is open in X.

Consider $B(s^*, j)$, where $s^* = p\hat{}*\hat{}y \in S^*$ and $j \geq 0$. In view of (12) and the first part of the proof, in order to prove that $B(s^*, j)$ is open in X, it suffices to show that $t^* \in F(s^*, j)\setminus\{s^*\}$ implies $F(t^*, j) \subset F(s^*, j)$. Suppose that $t^* = q\hat{}*\hat{}y' \neq s^*$ is in $F(s^*, j)$. From (10), it follows that $y = y'$ and $p \subset q$. Since $t^* \neq s^*$, there exists a $z \in Y$ such that $p\hat{}z \subset q$. Hence, $t^* \in F(s^*, j)$ implies, by (13), that $z \in V^{-1}(y)$ and $k_y(z) \geq j$. Condition (10) gives $F(t^*, j) \subset F(t^*, 0) \subset F(s^*, j)$.

From 3.3, it follows that X is a first-countable space.

In order to show that X is regular, we shall need the next two facts.

3.4. *For each $q \in P$, the set $B(q)$ is clopen in X.*

Proof. If $q = \emptyset$, the $B(q) = X$. For $q \in S \subset X$ the set $B(q) = B(q, 0)$ is open in X. We shall show that it is also closed in X.

If $s = p\hat{}y \notin B(q)$ and $B(s) \cap B(q) \neq \emptyset$, then q is a strict extension of s, which implies $f(q) \in V(y)$ and, since V is a partial order, $y = f(s) \notin V(f(q))$. On the other hand, 3.2, (8) and (3) give $f(\overline{B(q)}) \subset \overline{G(q)} \subset V(f(q))$ and, consequently, we obtain $s \notin \overline{B(q)}$.

If $s^* = p\hat{}*\hat{}y \notin B(q)$ and $B(p) \cap B(q) \neq \emptyset$, then q is a strict extension of p and there exists a $z \in Y$ such that $p\hat{}z \subset q$. Take a $j > k_y(z)$ if $z \in V^{-1}(y)$ or $j = 0$ if $z \notin V^{-1}(y)$. By (13), $B(s^*, j) \cap B(p\hat{}z) = \emptyset$ and, since $B(q) \subset B(p\hat{}z)$, we obtain $s^* \notin \overline{B(q)}$.

3.5. *For each $s^* \in S^*$ the sequence $\langle F(s^*, j)\colon j \geqslant 0\rangle$ is a decreasing sequence of closed subsets of X and its intersection is $\{s^*\}$.*

Proof. The least obvious fact, that the sets $F(s^*, j)$ are closed in X follows from 3.2, 3.4 and (10).

3.6. *The space X is a T_1-space.*

Proof. The points of S^* are closed by 3.5. If s is a point of S, then $\{s\} = B(s) \cap f^{-1}(s)$ is closed by 3.2 and 3.4.

We are now ready to prove that X is a regular space.

3.7. *For each $x \in X$, the sequence $\langle B(x, j)\colon j \geqslant 0\rangle$ is strictly decreasing.*

Proof. If $x = s = p^\frown y \in S$ and $j \geqslant 0$, then by 3.4, 3.2 and (9), $\overline{B(s, j+1)} \subset B(s) \cap f^{-1}(\overline{U_{j+1}(y)}) \subset B(s) \cap f^{-1}(U_j(y)) = B(s, j)$.

If $x = s^* = p^\frown * ^\frown y \in S^*$ and $j \geqslant 0$, then, by 3.4 and (13), $\overline{B(s^*, j+1)} \subset B(p) \setminus \bigcup \{B(p^\frown z)\colon z \in V^{-1}(y) \text{ and } k_y(z) < j+1\}$. Thus (10) and (12) imply that $\overline{B(s^*, j+1)} \cap f^{-1}(y) \subset F(s^*, j+1) = B(s^*, j+1) \cap f^{-1}(y)$. Hence, by 3.5 and (12), our task is reduced to showing that $\overline{B(s^*, j+1)} \setminus f^{-1}(y) \subset B(s^*, j)$.

Suppose that $y' \neq y$ and $x' \in f^{-1}(y')$. If x' is in the closure of an element of $\mathscr{B}(s^*, j+1)$, then x' is in the corresponding element of $\mathscr{B}(s^*, j)$, by the first part of our proof. Thus, in order to finish the proof, it suffices to show that $\mathscr{B}(s^*, j+1)$ is locally finite at x'.

Choose an $i \geqslant 0$ such that $U_i(y') \cap U_i(y) = \emptyset$. By (15), if $B(x', i)$ intersects an element $B(t, j+1+k_y(t))$ of $\mathscr{B}(s^*, j+1)$, then $j+1+k_y(t) < i$. In view of the definition of $k_y(t)$, the number of such $t \in e^{-1}(y)$ is finite. Thus $B(x', i)$ intersects finitely many elements of $\mathscr{B}(s^*, j+1)$ and the proof is finished.

3.8. *The space X has a co-finite neighbornet.*

Proof. We shall show that the collection $\{B(s, 0)\colon s \in S\} \cup \{B(s^*, 0)\colon s^* \in S^*\}$ defines a co-finite neighbornet in X.

Let x be a point in X. If $x \in B(s, 0) = B(s)$, then $s \subset x$ and so the number of such $s \in S$ is finite. If $x \in (p^\frown * ^\frown y, 0)$, then $p \subset x$ and y is a term of x. Again, the number of such $p^\frown * ^\frown y$ in S^* is finite.

We have shown that the space X has the required properties and the function $f\colon X \to Y$ is continuous. It remains to prove that f is an open and compact mapping.

3.9. *The mapping $f\colon X \to Y$ is open.*

Proof. If $s \in S$, then, by (1), $e(A(s)) = G(s)$. Thus (7) and (8) imply $f(B(s)) = G(s)$, which, in view of (9) gives $f(B(s, j)) = G(s) \cap U_j(f(s))$ for $j \geqslant 0$.

If $s^* \in S^*$ and $j \geqslant 0$, then (10), (11), (12) and the first part of the proof assure that $f(B(s^*, j))$ is open in Y.

The fact that the fibers of f are compact can be proved by observing that if $f^{-1}(y)$ is infinite, then, by (11), $e^{-1}(y)$ is dense in $f^{-1}(y)$ and so 3.1 implies that every locally finite collection of open subsets of $f^{-1}(y)$ is finite. Since the definition of S^* assures that $f^{-1}(y)$ is countable, this can be used to show that $f^{-1}(y)$ is compact.

We shall modify the proof of 3.1 in order to give a direct reasoning showing that the fibers of f are compact.

3.10. *The fibers of f are compact.*

Proof. Fix a $y \in Y$ and put $E = e^{-1}(y) \subset S$, $F = f^{-1}(y) \subset X$. If $s \in E$, then $B(s) \cap F = \{s\}$, thus all points of E are isolated in F. If $s^* \in F \setminus E$, and $j \geqslant 0$, then $B(s^*, j) \cap F = F(s^*, j)$, thus 3.5 implies that F is a zero-dimensional space.

Suppose that $\mathscr{U}$ is a cover of F consisting of clopen subsets of F and assume that no finite subcollection of $\mathscr{U}$ covers F. Since, by (11), E is dense in F and the elements of $\mathscr{U}$ are closed, it follows that no finite subcollection of $\mathscr{U}$ covers E.

We shall construct, by induction, a sequence $\langle y_n\colon n \geqslant 0\rangle$ in Y such that, for each $n \geqslant 0$, $p_n = \langle y_0, \ldots, y_{n-1}\rangle$ will satisfy

(16) $E \cap A(p_n)$ cannot be covered by any finite subcollection of $\mathscr{U}$.

Since $A(\emptyset) = S$, condition (16) is satisfied by $p_0 = \emptyset$. Suppose that we have $p_n = \langle y_0, \ldots, y_{n-1}\rangle$ satisfying (16).

Assume first that $p_n^\frown * ^\frown y$ is not in S^*. Since $E \cap A(p_n) \neq \emptyset$ and (4) imply that $p_n^\frown y \in S$, this means that (5) does not hold and, consequently, we can find a $y_n \in Y$ such that $p_{n+1} = p_n^\frown y_n$ satisfies (16).

If $p_n^\frown * ^\frown y = s^* \in S^*$, then there exists a $j \geqslant 0$ such that $F(s^*, j)$ is contained in an element U of $\mathscr{U}$. Thus (16) implies that $E \cap A(p_n) \setminus F(s^*, j)$ cannot be covered by any finite subcollection of $\mathscr{U}$. Since, by (7) and (10), the set $E \cap A(p_n) \setminus F(s^*, j)$ is covered by the finite collection $\{A(p_n^\frown z)\colon z \in V^{-1}(y) \text{ and } k_y(z) < j\}$, we can find an element $A(p_{n+1})$ in this collection such that (16) will be satisfied.

This concludes the inductive construction and one can obtain a contradiction as in the proof of 3.1 (see 1.3).

4. Remarks. We start with some remarks concerning the construction of X.

Remark 4.1. *If the space Y is zero-dimensional, then the space X can be constructed to be dimensional.*

Proof. If Y is zero-dimensional, then we can assume that the bases $\langle U_j(y)\colon j \geqslant 0\rangle$ consist of sets clopen in Y. The proof of 3.7 can be modified (by replacing $j+1$ with j) to show that $\overline{B(x, j)} \subset B(x, j)$ for $x \in X$ and $j \geqslant 0$.

Remark 4.2. *If the space Y is completely regular, then the space X can be constructed to be completely regular.*

Proof. Assume that Y is completely regular and fix a $y \in Y$. By induction on $j \geqslant 0$, construct sequences $\langle U_j\colon j \geqslant 0\rangle$ of neighborhoods of y in Y and $\langle \varphi_j\colon j \geqslant 0\rangle$ of mappings of Y into the unit interval $I = [0, 1]$ satisfying, for $j \geqslant 0$,

(17) $$U_0 = Y,$$

(18) $$\varphi_j(y) = 0 \quad \text{and} \quad \varphi_j(z) = 1 \quad \text{for } z \notin U_j,$$

$$\overline{U}_{j+1} \subset U_j \cap U_{j+1}(y)\,, \tag{19}$$

$$\varphi_j(z) < 1/j \quad \text{for } z \in U_{j+1}\,. \tag{20}$$

By (19), the sequence $\langle U_j\colon j \geqslant 0\rangle$ is a strictly decreasing base for y in Y. Use the sequence $\langle U_j\colon j \geqslant 0\rangle$ in place of $\langle U_j(y)\colon j \geqslant 0\rangle$ to define the sets $B(s,j)$ for $s \in e^{-1}(y)$ and $j \geqslant 0$ (this affects (9) and, indirectly, (12)).

We shall show that for every $x \in f^{-1}(y)$ and $j \geqslant 1$, there exists a mapping $\psi\colon X \to I$ separating x from $X \setminus B(x,j)$. Since the same modification can be applied to any $y \in Y$ (we fix y in order to simplify the notation), it will follow that X can be made completely regular.

If $x = p\hat{\ }y = s \in S$, then by 3.4, it is sufficient to define the separation on $B(s)$. In view of (18), the mapping $\psi = \varphi_j \circ f$ separates x from $B(s) \setminus f^{-1}(U_j) = B(s) \setminus B(x,j)$.

If $x = p\hat{\ }*\hat{\ }y = s^* \in S^*$, then, by 3.4, it is sufficient to define the separation on $X' = B(p) \setminus \bigcup \{B(p\hat{\ }z)\colon z \in V^{-1}(y) \text{ and } k_y(z) < j\}$. Define ψ to be 0 on $F(s^*,j)$, the combination of $\varphi_{j+k_y(t)} \circ f$ on pairwise disjoint clopen sets $B(t)$ for $t \in e^{-1}(y) \cap F(s^*,j)$ and 1 in the remaining points of X'.

Clearly, ψ separates x from $X' \setminus B(x,j)$. We shall show that ψ is continuous on X'.

Since $\psi \equiv 1$ on $X' \setminus \overline{B(x,j)}$ and, by 3.7, $\overline{B(x,j)} \subset B(x,j-1)$, we only have to check the continuity of ψ on

$$X' \cap B(x,j-1) \subset F(s^*,j) \cup \bigcup \{B(t)\colon t \in e^{-1}(y) \cap F(s^*,j)\}\,.$$

This, by the definition of ψ, reduces to showing the continuity of ψ at the points of $F(s^*,j) \setminus e^{-1}(y)$.

If $t^* \in F(s^*,j) \setminus e^{-1}(y)$, then $\psi(t^*) = 0$. For an arbitrary $k > 0$ find a $i > j$ such that $k_y(t) > k$ for $t \in e^{-1}(y) \cap F(t^*,i)$. Condition (20) assures that $\psi(B(t^*,i)) \subset [0, 1/k)$, which shows that ψ is continuous at t^*.

If the space Y is scattered, then, since the fibers of f are scattered, it follows that the space X is also scattered. Thus, our construction is more general than the construction from [BCh2]. However, if the space Y is scattered and the parameters in both constructions are chosen in a suitable way, then the resulting spaces X are very similar. If the space Y is a scattered space of height 2 and the neighbornet V is chosen in a natural way (each $V(y)$ either consists of one point or contains exactly one non-isolated point), then our construction is identical with the construction described in [BCh2, 3] (see [Ch1, 1.2]). In particular, this shows that our construction may give a non-normal space X even if the space Y is normal.

Since having a co-countable neighbornet is equivalent to having a closure-preserving cover by countable closed sets (see [J2] and spaces having a base of countable order are invariant under perfect mappings (see [Wo] or [ChČN]), our characterization gives

COROLLARY 4.3. *The class* MOBI_3 *(σ-discrete) is invariant under perfect mappings.*

Finally, let us note that, since each scattered space is right separated (see [GJ]), Proposition 1.3 can be sharpened to

PROPOSITION 4.4. *A regular space Y is a first-countable scattered space if and only if Y has a λ-base and an antisymmetric neighbornet.*

Moreover, the proof of 1.3 can be modified to show (see [GJ])

PROPOSITION 4.5. *A regular space Y is a scattered space of point-countable type if and only if Y is monotonically Čech complete and has an antisymmetric neighbornet.*

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