Spaces of retractions which are homeomorphic to Hilbert space

by

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Abstract. Let $R(X)$ denote the space of retractions of a compact metric space X with sup-metric. In this paper, we prove that if X is a dendrite (= one-dimensional compact AR) then $R(X)$ is homeomorphic to Hilbert space l_2. In case X is a non-degenerate compact convex set in a locally convex linear topological space, let $Rc(X)$ be the subspace of $R(X)$ consisting of retractions with convex images. We also prove that $Rc(X)$ is homeomorphic to l_2.

0. Introduction. Let $X = (X, d)$ be a compact metric space. A *retraction* of X is a map $r\colon X \to X$ such that $r^2 = r$, i.e., $r|r(X) = \mathrm{id}$. By $R(X)$ we denote the space of retractions of X equipped with the sup-metric

$$d(r, r') = \sup\{d(r(x), r'(x))|\ x \in X\}.$$

Then $R(X)$ is a separable complete metric space. The Hilbert cube and Hilbert space are denoted by Q $(= I^{\omega})$ and l_2 respectively. A *Q-manifold* or an *l_2-manifold* is a separable (topological) manifold modeled on Q or l_2 respectively. In case X is a compact Q-manifold, using the result of Chapman [Ch], Sakai [Sa] has shown that $R(X)$ is an l_2-manifold. In finite-dimensional case, Basmanov and Savchenko [BS] has shown that $R(I)$ is homeomorphic ($\cong$) to l_2, where $I = [0, 1]$. Using this result, it can be shown that $R(S^1)\setminus\{\mathrm{id}\}$ is an l_2-manifold. One should remark that id is an isolated point of $R(X)$ in case X is a closed n-manifold. Recently, Cauty [Ca] has obtained the same result for case X is a closed n-manifold. Recently, Cauty [Ca] has obtained the same result for a compact surface (= connected 2-manifold), that is, $R(X)$ is an l_2-manifold if X is a bordered surface (i.e. $\partial X \neq \emptyset$) and $R(X)\setminus\{\mathrm{id}\}$ is an l_2-manifold if X is closed surface. Similar to the Homeomorphism Group Problem, the following problem [AK, HS 9] is still open for $n \geqslant 3$:

PROBLEM. For X a compact connected n-manifold $(n \geqslant 3)$ with $\partial X \neq \emptyset$, is $R(X)$ (or $R(X)\setminus\{\mathrm{id}\}$ when $\partial X = \emptyset$) an l_2-manifold? Particularly, is $R(I^n)$ $(n \geqslant 3)$ homeomorphic to l_2?

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In this paper we generalize the result of Basmanov and Savchenko in two different directions. A *dendrite* is a non-degenerate locally connected continuum which contains no circle, or equivalently, it is a one-dimensional compact absolute retract (AR) [Bo, Ch. V, (13.5)].

THEOREM I. *For any dendrite* X, $R(X) \cong l_2$.

In case X is a convex set in a linear topological space, we denote by $Rc(X)$ the space of all retractions r of X such that $\mathrm{Im}(r) = r(X)$ is convex. (Note that $R(I) = Rc(I)$.)

THEOREM II. *If X is a non-degenerate compact convex set in a locally convex linear topological space then* $Rc(X) \cong l_2$.

1. A strongly convex metric compactum. To prove Theorems I and II simultaneously, we introduce some notions. Let $X = (X, d)$ be a strongly convex metric compactum, where X is said to be *strongly convex* if for each pair of points $x, y \in X$ there exists a unique point $z \in X$ such that $d(x, z) = d(y, z) = d(x, y)/2$ [Bo, Ch. IX, §10]. It is well known that there is a map $\lambda\colon X^2 \times I \to X$ such that

(1) $$d(x, \lambda(x, y, t)) = t\cdot d(x, y) \qquad d(y, \lambda(x, y, t)) = (1-t)\cdot d(x, y)$$

for each $x, y \in X$ and $t \in I$. We denote

$$\overline{xy} = \{\lambda(x, y, t) |\ t \in I\}.$$

Then $\overline{xy}$ is the unique path with end-points x and y such that $\overline{xy}$ is isometric to $[0, d(x, y)]$. A subset A of X said to be *convex* if $\overline{xy} \subset A$ for each $x, y \in A$. Let $C(X)$ denote the hyperspace of subcontinua of X with the Hausdorff metric ϱ induced by d:

$$\varrho(A, B) = \max\{\sup\{d(x, B)|\ x \in A\},\ \sup\{d(y, A)|\ y \in B\}\},$$

where

$$d(x, A) = \inf\{d(x, y)|\ y \in A\}.$$

Denote by $cc(X)$ the subspace of $C(X)$ consisting of all compact convex sets in X, and by $Rc(X)$ the space of all retractions r of X such that $\mathrm{Im}(r) \in cc(X)$. We assume the following two conditions:

($*$) if $y \neq z$ and $0 < t < 1$ then $d(x, \lambda(y, z, t)) < \max\{d(x, y), d(x, z)\}$,

($**$) if $x \in \overline{x_1 x_2}$ then $d(x, \overline{y_1 y_2}) \leqslant \max\{d(x_1, y_1), d(x_2, y_2)\}$.

As corollaries of the following theorem, we have Theorems I and II.

1.1. THEOREM. *Let $X = (X, d)$ be a strongly convex metric compactum satisfying* ($*$) *and* ($**$). *Then* $Rc(X) \cong l_2$.

In case X is a dendrite, X admits a *convex metric* d, that is, for each pair of points $x, y \in X$ there exists $z \in X$ such that $d(x, z) = d(y, z) = d(x, y)/2$. By unique arc-wise connectedness, $X = (X, d)$ is strongly convex and all subcontinua of X are convex. As is easily observed, ($*$) and ($**$) are satisfied. Therefore Theorem I follows from the above theorem.

In case X is a non-degenerate compact convex set in a locally convex linear topological space, X can be affinely embedded in l_2 by [BP, Ch. III, Theorem 2.1] (cf. [BP, Ch. III, Proposition 3.1]). Then we may assume without loss of generality that X is a convex set in l_2. Let d be the metric for X induced by the norm of l_2. Then "convexity" with respect to the metric d is identical with "convexity" in the usual sense. As easily observed, ($*$) and ($**$) are satisfied. Therefore Theorem II follows from the above theorem.

Before the proof of Theorem 1.1, we consider basic properties which are induced by ($*$) and ($**$). First note that the above map λ is an equi-connecting map of X. The condition ($*$) implies that for each $x \in X$ and $\varepsilon > 0$,

(2) $\{y \in X|\ d(x, y) \leqslant \varepsilon\} \in cc(X)$ and

(3) $\{y \in X|\ d(x, y) = \varepsilon\}$ contains no non-degenerate convex set.

By the result of [Hi] or [Du], (2) implies the following

1.2. LEMMA. *X is an* AR.

By ($**$), X satisfies the following condition which is stronger than (2):

(4) $\{x \in X|\ d(x, A) \leqslant \varepsilon\} \in cc(X)$ for each $A \in cc(X)$ and $\varepsilon > 0$.

For each $A \in C(X)$, define $A[n] = \lambda(A[n-1]^2 \times I)$, $n \in N$, where $A[0] = A$, and denote by $\mathrm{co}(A)$ the closed convex hull of A in X, namely the smallest closed convex set in X containing A. Then we have the following:

1.3. LEMMA, $\mathrm{co}(A) = \mathrm{cl}(\bigcup_{n \in N} A[n])$.

Proof. Each $x, y \in \bigcup_{n \in N} A[n]$ are contained in same $A[n]$ since $A[1] \subset A[2] \subset \ldots$ Then $\overline{xy} \subset A[n+1] \subset \bigcup_{n \in N} A[n]$. Hence $\bigcup_{n \in N} A[n]$ is convex. From continuity of λ, it follows that $\mathrm{cl}(\bigcup_{n \in N} A[n])$ is also convex. Therefore $\mathrm{co}(A) \subset \mathrm{cl}(\bigcup_{n \in N} A[n])$. Conversely, it can be shown by induction that each $A[n]$ is contained in $\mathrm{co}(A)$, whence $\bigcup_{n \in N} A[n] \subset \mathrm{co}(A)$. Since $\mathrm{co}(A)$ is closed, $\mathrm{cl}(\bigcup_{n \in N} A[n]) \subset \mathrm{co}(A)$. ∎

1.4. LEMMA. co: $C(X) \to cc(X)$ *is a retraction.*

Proof. Since $\mathrm{co}|cc(X) = \mathrm{id}$, it suffices to show the continuity of co. Let $A, B \in C(X)$. First we show that $\varrho(A[1], B[1]) \leqslant \varrho(A, B)$. Let $x_1, x_2 \in A$ and $x \in \overline{x_1 x_2}$. For any $t > \varrho(A, B)$, we have $y_1, y_2 \in B$ such that $d(x_i, y_i) < t$, $i = 1, 2$, which implies $d(x, B[1]) \leqslant d(x, \overline{y_1 y_2}) < t$ by ($**$). Hence $d(x, B[1]) \leqslant \varrho(A, B)$ for each $x \in A[1]$. Similarly, $d(y, A[1]) \leqslant \varrho(A, B)$ for each $y \in B[1]$. Therefore $\varrho(A[1], B[1]) \leqslant \varrho(A, B)$. Then by induction, $\varrho(A[n], B[n]) \leqslant \varrho(A, B)$ for all $n \in N$. By Lemma 1.3, it follows that $\varrho(\mathrm{co}(A), \mathrm{co}(B)) \leqslant \varrho(A, B)$, which implies co is continuous. ∎

Let

$$\Xi = \{(A, x, t) \in cc(X) \times X \times [0, \infty)|\ d(x, A) \leqslant t\}$$

and define $\xi\colon \Xi \to cc(X)$ by

$$\xi(A, x, t) = A \cap \{y \in X|\ d(x, y) \leqslant t\}.$$

Then we have

1.5. Lemma. *ξ: $\Xi \to cc(X)$ is continuous.*

Proof. Assume that ξ is not continuous at $(A, x, t) \in \Xi$, that is, there are $\varepsilon > 0$ and $(A_n, x_n, t_n) \in \Xi$, $n \in N$, such that $(A_n, x_n, t_n) \to (A, x, t)$ as $n \to \infty$ but $\varrho(\xi(A_n, x_n, t_n), \xi(A, x, t)) > \varepsilon$. Then for each $n \in N$, (i) there is $a_n \in \xi(A_n, x_n, t_n)$ such that $d(a_n, \xi(A, x, t)) > \varepsilon$ or (ii) there is $b_n \in \xi(A, x, t)$ such that $d(b_n, \xi(A_n, x_n, t_n)) > \varepsilon$. Without loss of generality, we may assume that (i) holds for all $n \in N$ and $a_n \to a$ as $n \to \infty$ or (ii) holds for all $n \in N$ and $b_n \to b$ as $n \to \infty$. In the former case, $a \in \xi(A, x, t)$ since $A_n \to A$, $d(x_n, a_n) \to d(x, a)$ and $t_n \to t$ as $n \to \infty$. This is contrary to $d(a_n, \xi(A, x, t)) > \varepsilon$. In the latter case, $b \in \xi(A, x, t)$ and $d(b, \xi(A_n, x_n, t_n)) > \varepsilon$ for sufficiently large $n \in N$. Choose $a_n \in A_n$ so that $a_n \to b$ as $n \to \infty$. For each $n \in N$, take $c_n \in \xi(A_n, x_n, t_n)$. We may furthermore assume that $c_n \to c$ as $n \to \infty$. Observe that $c \in \xi(A, x, t)$. For sufficiently small $s > 0$, $d(b, \lambda(b, c, s)) < \varepsilon/2$. Then, from convexity, $\lambda(a_n, c_n, s) \in A_n$ for all $n \in N$. By (*),

$$d(x, \lambda(b, c, s)) < \max\{d(x, b), d(x, c)\} \leqslant t,$$

that is,

$$\delta = t - d(x, \lambda(b, c, s)) > 0.$$

For sufficiently large $n \in N$, $|t_n - t| < \delta/3$, $d(x_n, x) < \delta/3$ and $d(\lambda(b, c, s), \lambda(a_n, c_n, s)) < \min\{\delta/3, \varepsilon/2\}$. Then

$$\begin{aligned} d(x_n, \lambda(a_n, c_n, s)) &\leqslant d(x_n, x) + d(x, \lambda(b, c, s)) + d(\lambda(b, c, s), \lambda(a_n, c_n, s)) \\ &< \delta/3 + (t - \delta) + \delta/3 = t - \delta/3 < t_n, \end{aligned}$$

whence $\lambda(a_n, c_n, s) \in \xi(A_n, x_n, t_n)$. And

$$d(b, \lambda(a_n, c_n, s)) \leqslant d(b, \lambda(b, c, s)) + d(\lambda(b, c, s), \lambda(a_n, c_n, s)) < \varepsilon.$$

This is contrary to $d(b, \xi(A_n, x_n, t_n)) > \varepsilon$. Therefore ξ is continuous. ∎

1.6. Lemma. *There exists a map γ: $cc(X) \to Rc(X)$ such that $\operatorname{Im}(\gamma_A) = A$ and $d(x, \gamma_A(x)) = d(x, A)$ for all $A \in cc(X)$, where $\gamma_A = \gamma(A)$.*

Proof. For each $A \in cc(X)$ and each $x \in X$, $\gamma_A(x) \in A$ is uniquely determined so that $d(x, \gamma_A(x)) = d(x, A)$, that is, $\gamma_A(x)$ is the nearest point of A from x. In fact, there is an $a \in A$ such that $d(x, a) = d(x, A)$, and if $d(x, a_1) = d(x, a_2) = d(x, A)$ for $a_1, a_2 \in A$, then $\overline{a_1 a_2} \subset \{y \in X|\ d(x, y) = d(x, A)\}$ since $\overline{a_1 a_2}$ is contained in both A and $\{y \in X|\ d(x, y) \leqslant d(x, A)\}$ from convexity. Hence $a_1 = a_2$ from (3). Since $\xi(A, x, d(x, A)) = \{\gamma_A(x)\}$, $\gamma_A(x)$ is continuous with respect to both $A \in cc(X)$ and $x \in X$. Thus we have the desired map γ: $cc(X) \to Rc(X)$. ∎

2. Proof of Theorem 1.1. First of all, we show that $Rc(X)$ is an AR.

2.1. Lemma. *$Rc(X)$ is an* AR.

Proof. Let $M(X)$ we denote the space of (continuous) maps of X into itself equipped with the sup-metric. Since $M(X)$ is an AR, it suffices to show that $Rc(X)$ is

a retract of $M(X)$. As is easily observed, Im: $M(X) \to C(X)$ is continuous. By Lemma 1.4, we have a map η: $M(X) \to cc(X)$ such that $\eta|Rc(X) = \operatorname{Im}$. Let $\varphi = \gamma \circ \eta$: $M(X) \to Rc(X)$, where γ is a map obtained by Lemma 1.5. We denote

$$F = \{(f, x) \in M(X) \times X|\ x \in \operatorname{Im}(\varphi(f))\}.$$

Clearly F is closed in $M(X) \times X$. Since X is an AR by Lemma 1.2, we can define a map Ψ: $M(X) \times X \to X$ with the property:

$$\Psi(f, x) = \begin{cases} x & \text{if } (f, x) \in F, \\ f(x) & \text{if } (f, x) \in R(X) \times X. \end{cases}$$

Let ψ: $M(X) \to M(X)$ be the map induced by Ψ. Then $\operatorname{Im}(\varphi(f)) \subset \operatorname{Im}(\varphi(f))$ and $\psi(f)|\operatorname{Im}(\varphi(f)) = \mathrm{id}$ for each $f \in M(X)$. Since each $\varphi(f)$ is a retraction, we can define a map θ: $M(X) \to Rc(X)$ by $\theta(f) = \varphi(f) \circ \psi(f)$. Since $\operatorname{Im}(\varphi(r)) = \operatorname{Im}(r)$ and $\psi(r) = r$ for each $r \in Rc(X)$, $\theta|Rc(X) = \mathrm{id}$. Hence $Rc(X)$ is a retract of $M(X)$. ∎

A closed set A in a metric space $M = (M, d)$ is called a *strong Z-set* if for any map ε: $M \to (0, \infty)$ there is a map f: $M \to M$ such that $d(f(x), x) < \varepsilon(x)$ and $A \cap \operatorname{cl} f(M) = \emptyset$. Let X^* denote the subset of $R(X)$ consisting of constant maps. Obviously X^* is closed in $R(X)$.

2.2. Lemma. *$X^* \cup \{\mathrm{id}\}$ is a strong Z-set in $Rc(X)$.*

Proof. For each map ε: $Rc(X) \to (0, 1)$, choose a map δ: $Rc(X) \to (0, 1)$ so that $\delta(r) \leqslant \varepsilon(r)/4$ and $d(x, \operatorname{Im}(r)) < 2 \cdot \delta(r)$ implies $d(x, r(x)) < \varepsilon(r)/4$. By [Na, Corollary 3.4] and (4), we have a map α: $Rc(X) \to cc(X)$ defined by

$$\alpha(r) = \{x \in X|\ d(x, \operatorname{Im}(r)) \leqslant \delta(r)\}.$$

Let γ be the map obtained by Lemma 1.6. Since $\gamma_{\alpha(r)}(x), r(x) \in \alpha(r)$ for all $r \in Rc(X)$ and $x \in X$, we can define a map φ: $Rc(X) \to Rc(X)$ by

$$\varphi(r)(x) = \lambda(\gamma_{\alpha(r)}(x), r(x), u(x, r)),$$

where λ is the map satisfying (1) and u: $X \times Rc(X) \to I$ is a Urysohn map such that

$$u(x, r) = 0 \quad \text{if } x \in \alpha(r) \quad \text{and}$$

$$u(x, r) = 1 \quad \text{if } d(x, \alpha(r)) \geqslant \delta(r).$$

Then as easily observed, $\operatorname{Im}(\varphi(r)) = \alpha(r)$ and $d(\varphi(r), r) < \varepsilon(r)/2$ for each $r \in Rc(X)$.

Next choose $a, b \in X$ so that

$$d(a, b) = \operatorname{diam} X = \sup\{d(x, y)|\ x, y \in X\}.$$

And choose a map δ': $Rc(X) \to (0, \infty)$ such that $\delta'(r) < \delta(r)/2$ and $\varrho(A, \alpha(r)) \leqslant \delta'(r)$ implies $d(\gamma_A, \gamma_{\alpha(r)}) < \varepsilon(r)/2$. Let ξ be the map obtained in Lemma 1.6. Then we can define a map β: $Rc(X) \to cc(X)$ by

$$\begin{aligned} \beta(r) &= \xi(\alpha(r), a, d(a, b) - \delta'(r)) \\ &= \{x \in \operatorname{Im}(\varphi(r))|\ d(a, x) \leqslant d(a, b) - \delta'(r)\}. \end{aligned}$$

In fact, $\alpha(r)$ contains a closed ball with center $x \in \mathrm{Im}(r)$ and radius $\delta(r)$. From (1), we have $y \in X$ such that $d(x, y) = 3\delta(r)/4$ and $d(a, y) = d(a, x) - 3\delta(r)/4$. Then $d(y, z) \leqslant \delta(r)/4$ implies $d(x, z) \leqslant \delta(r)$ and $d(a, z) \leqslant d(a, x) - \delta'(r)$, that is, $z \in \beta(r)$. Thus $\beta(r)$ contains a closed ball with center y and radius $\delta(r)/4$, hence $\beta(r) \neq \emptyset$. Since $\beta(r) \subset \alpha(r) = \mathrm{Im}(\varphi(r))$ for each $r \in Rc(X)$, we can define a map ψ: $Rc(X) \to Rc(X)$ by $\psi(r) = \gamma_{\beta(r)} \circ \varphi(r)$. Since $\varrho(\beta(r), \alpha(r)) < \delta'(r)$,

$$\begin{aligned} d(\psi(r), \varphi(r)) &= d(\gamma_{\beta(r)}|\mathrm{Im}(\varphi(r)), \mathrm{id}) \\ &= d(\gamma_{\beta(r)}|\alpha(r), \gamma_{\alpha(r)}|\alpha(r)) \leqslant d(\gamma_{\beta(r)}, \gamma_{\alpha(r)}) < \varepsilon(r)/2, \end{aligned}$$

hence $d(\psi(r), r) < \varepsilon(r)$ for each $r \in Rc(X)$.

Now we will show that $(X^* \cup \{\mathrm{id}\}) \cap \mathrm{cl}\, \psi(Rc(X)) = \emptyset$. If there are $r_n \in Rc(X)$, $n \in N$, such that $\psi(r_n) \to \mathrm{id}$, i.e., $\mathrm{Im}(\psi(r_n)) = \beta(r_n) \to X$ as $n \to \infty$, so $\delta'(r_n) \to 0$, which implies $\delta'(\mathrm{id}) = 0$. This is a contradiction. Suppose there are $r_n \in Rc(X)$, $n \in N$, such that $\psi(r_n) \to r_0 \in X^*$ as $n \to \infty$. Then $\delta(r_n) \to 0$ because $\mathrm{Im}(\psi(r_n)) = \beta(r_n)$ contains a closed ball with radius $\delta(r_n)/4$. Hence we have also $r_n \to r_0$. Then $\delta(r_0) = 0$, which is a contradiction. ∎

In the proof of Theorem 1.1, we use the following version of [DT, Remark 2] (cf. [To$_{1,2}$]):

2.3. Lemma. *A separable complete metrizable* ANR M *is an l_2-manifold if M satisfies the following*:

($\#$) *For each $\varepsilon > 0$ there is $\delta > 0$ such that for each compactum $A \subset M$ there is a map f: $A \to M$ which is ε-homotopic to* id *and satisfies*

$$\mathrm{dist}(f(A), A) = \inf\{d(f(x), y) |\ x, y \in A\} > \delta.$$

Proof of Theorem 1.1. For each $a \neq b \in X$ and $t > 0$, let

$$R_{a,b,t} = \{r \in Rc(X) |\ d(r(a), r(b)) > t,\ b \notin \mathrm{Im}(r)\}.$$

Then each $R_{a,b,t}$ is open in $Rc(X)$ and

$$Rc(X) \setminus (X^* \cup \{\mathrm{id}\}) = \bigcup \{R_{a,b,t} |\ a \neq b \in X,\ t > 0\}.$$

If each $R_{a,b,t}$ is an l_2-manifold, $Rc(X)$ is also an l_2-manifold by [To$_2$, Theorem B1] and Lemma 2.2, hence $Rc(R) \cong l_2$ because a contractible l_2-manifold is homeomorphic to l_2 [He, Corollary 3]. Since $R_{a,b,t}$ is a separable complete metrizable ANR by Lemma 2.1, it suffices to show that $R_{a,b,t}$ has ($\#$) by Lemma 2.3.

Let $\varepsilon > 0$. For each compactum $A \subset R_{a,b,t}$, we have $\delta > 0$ such that $d(x, b) < \delta$ implies $x \notin \mathrm{Im}(r)$ and $d(r(x), r(b)) < \varepsilon/4$ for all $r \in A$. We can define a map φ: $A \to R_{a,b,t}$ as follows:

$$\varphi(r)(x) = \begin{cases} \lambda\left(r(b), r(a), \dfrac{3\varepsilon u(x)}{4d(r(a), r(b))}\right) & \text{if } d(x, b) \leqslant 2\delta/3, \\ \lambda(r(b), r(x), u(x)) & \text{if } d(x, b) \geqslant 2\delta/3, \end{cases}$$

where u: $X \to I$ is a Urysohn map such that

$$u(x) = 0 \quad \text{if } x = b \text{ or } d(x, b) = 2\delta/3 \quad \text{and}$$
$$u(x) = 1 \quad \text{if } d(x, b) = \delta/3 \text{ or } d(x, b) \geqslant \delta.$$

Then by using (1), we can see $d(r, \varphi(r)) < \varepsilon$ for each $r \in A$. In fact, if $d(x, b) < 2\delta/3$ then

$$\begin{aligned} d(r(x), \varphi(r)(x)) &\leqslant d(r(x), r(b)) + d\left(r(b), \lambda\left(r(b), r(a), \frac{3\varepsilon u(x)}{4d(r(a), r(b))}\right)\right) \\ &< \varepsilon/4 + 3\varepsilon/4 = \varepsilon. \end{aligned}$$

If $2\delta/3 \leqslant d(x, b) < \delta$ then

$$\begin{aligned} d(r(x), \varphi(r)(x)) &\leqslant d(r(x), r(b)) + d(r(b), \lambda(r(b), r(x), u(x))) \\ &< \varepsilon/4 + d(r(b), r(x)) < \varepsilon/2. \end{aligned}$$

If $d(x, b) \geqslant \delta$ then $\varphi(r)(x) = r(x)$. Since $\mathrm{Im}(\varphi(r)) = \mathrm{Im}(r)$ for all $r \in A$, φ is ε-homotopic to id by the homotopy Φ: $A \times I \to R_{a,t}$ defined by

$$\Phi(r, s)(x) = \lambda(\varphi(r), r(x), s).$$

We will show that $d(\varphi(r), r') > \varepsilon/4$ for all $r, r' \in A$. From convexity of d, we have $x_0 \in X$ such that $d(x_0, b) = \delta/3$. Then $d(r'(x_0), r'(b)) < \varepsilon/4$ and

$$\begin{aligned} d(\varphi(r)(x_0), \varphi(r)(b)) &= d\left(r(b), \lambda\left(r(b), r(a), \frac{3\varepsilon}{4d(r(a), r(b))}\right)\right) \\ &= 3\varepsilon/4 \end{aligned}$$

by (1). It follows that $d(\varphi(r)(b), r'(b)) > \varepsilon/4$ or $d(\varphi(r)(x_0), r'(x_0)) > \varepsilon/4$. Therefore $d(\varphi(r), r') > \varepsilon/4$. ∎

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On decomposition of 3-polyhedra into a Cartesian product

by

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Abstract. It is well known that the uniqueness of decomposition of 3-polyhedra into a Cartesian product in general does not hold. In this paper we prove that if there is nonuniqueness then one of the factors is an arc. We also answer the question when $K\times I\approx L\times I$, where K and L are compact 2-polyhedra.

1. Introduction. In 1938 K. Borsuk [1] proved that decomposition of a compact polyhedron into a Cartesian product of 1-dimensional factors is unique. However, if one of the factors is a 2-polyhedron, or 2-manifold with boundary, the uniqueness of decomposition does not hold. We give some examples.

Example 1.1 (R. H. Fox (1947) [2]). The sets K and L are unions of an annulus and two intervals as in Fig. 1.

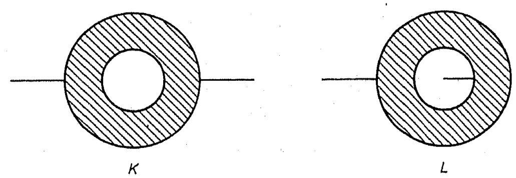

Fig. 1

Then $K\times I\approx L\times I$. ■

Example 1.2. The sets K and L are unions of a disc and six intervals as in Fig. 2.

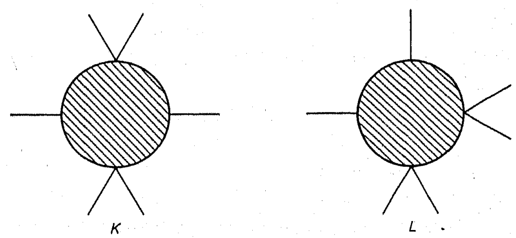

Fig. 2

Then $K\times I\approx L\times I$. ■