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Extending holomorphic maps from compact sets in infinite dimensions

by

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Abstract. The aim of this paper is to study the extension of holomorphic maps from compact sets in metric vector spaces with values in some locally convex spaces and in complete C-manifolds. Moreover, the theorem of Siciak–Zakharyuta for continuous separately holomorphic functions with values in Banach–Lie groups is established.

Introduction. The extension of separately holomorphic functions defined on special subsets of C^n has been investigated by many authors, for example Siciak [8], Zakharyuta [11]. In [9] and [10] Siciak and Waelbroeck have considered this problem for compact sets in C^n. Moreover, Waelbroeck has also considered this problem for unique compact sets in a Banach space. Here a compact set K in a topological vector space is called *unique* if for every holomorphic function f on a neighbourhood of K such that $f|K = 0$ there exists a neigbourhood U of K such that $f|U = 0$. This paper is devoted to the study of the extension of continuous functions on compact sets K in topological vector spaces with values in locally convex complex manifolds to holomorphic functions on a neighbourhood of K.

In Section 1 we investigate the interrelation between the holomorphic extendability and weakly holomorphic extendability of continuous functions on a compact set K in a metric vector space E with values in a locally convex space F such that F^* is a Baire space. We prove that if either E or F is nuclear, then holomorphic extendability and weakly holomorphic extendability are equivalent. This has been established by Siciak in [9] and Waelbroeck in [10] in the case where $\dim E < \infty$. Our method in the case where E is a nuclear metric vector space is based on an idea of Waelbroeck [10]. We first prove the nuclearity of the DF-space $\operatorname{inj\,lim}\{H^\infty(U)/A_U\colon U \supset K\}$ where $H^\infty(U)$ denotes the Banach space of bounded holomorphic functions on U equipped with the sup norm and $A_U := \{f \in H^\infty(U)\colon f|K = 0\}$. (In the case where $K \subset C^n$ this proof is not difficult.) Next following Waelbroeck, using the closed graph theorem for maps of barrelled locally convex spaces into B-complete spaces, we obtain the above result.

In the case where E and F are Banach spaces we prove that there exists a Banach space $\tilde{F}$ containing F as a closed subspace such that every continuous function on a compact set in E with values in F having the weakly holomorphic extension property can be extended to a holomorphic function on a neighbourhood of K but with values in $\tilde{F}$.

Using the above result we prove the equivalence of the weakly holomorphic extendability and the holomorphic extendability of continuous functions on a compact set in a metric vector space with values in a complete C-manifold. Finally, in Section 3 we prove the Siciak–Zakharyuta theorem for separately holomorphic functions f with values in a Banach–Lie group under the additional assumption that f is continuous.

§ 1. Extending weakly holomorphic functions from compact sets with values in locally convex spaces. In this section we investigate the holomorphic extension of weakly holomorphic functions defined on compact subsets of metric vector spaces E with values in locally convex spaces. The problem has been investigated by Siciak [9] and Waelbroeck [10] in the finite-dimensional case and for unique compact sets in Banach spaces also by Waelbroeck [10].

1.1. THEOREM. *Let E be a metric vector space and F a sequentially complete locally convex space such that F^* is a Baire space. Let one of the following two conditions hold:*

(1) *E is a nuclear space.*

(2) *F is a complete nuclear space.*

Then every map $f\colon K \to F$, where K is a compact set in E, such that $l \circ f$ can be extended to a holomorphic function on a neighbourhood of K for every $l \in F^*$ can be extended to a holomorphic function $\hat{f}$ on a neighbourhood of K.

We need the following lemmas.

1.2. LEMMA. *Let $T\colon A \to B$ be a nuclear map between Banach spaces A and B. Then there exists a neighbourhood V of zero in A such that $T(V) \subset U$, where U is the unit ball in B, and the canonical map $\hat{T}\colon H^\infty(U) \to H^\infty(V)$ induced by T is nuclear.*

Proof. By hypothesis, we can choose $\{a_n\}_{n=1}^\infty \subset A^*$ and $\{y_n\}_{n=1}^\infty \subset B$ such that

$$Tx = \sum_{n=1}^\infty \langle a_n, x\rangle y_n \quad \forall x \in A,$$

$$M = \sum_{n=1}^\infty \|a_n\|\,\|y_n\| < \infty.$$

Without loss of generality we may assume that $M \leqslant 1/2$. Take a balanced neighbourhood V of zero in A such that $TV \subset U$ and $\sup\{\|x\|\colon x \in V\} \leqslant 1$. For each $k \in N$, define a continuous linear map $l_k\colon H^\infty(U) \to H^\infty(V)$ by $[l_k g](x) = (d^k g(0))\hat{}\,(Tx)/k!$, $\forall g \in H^\infty(U)$, $\forall x \in V$, where $(d^k g(0))\hat{}$ denotes the polynomial map associated to $d^k g(0)$. Then l_k is a nuclear map. Indeed,

$$[l_k g](x) = A_k(Tx, \ldots, Tx) = A_k\big(\sum \langle a_n, x\rangle y_n, \ldots, \sum \langle a_n, x\rangle y_n\big)$$

$$= \sum_{n_1+\ldots+n_k=0}^{\infty} A_k(y_{n_1}, \ldots, y_{n_k})\langle a_{n_1}, x\rangle \ldots \langle a_{n_k}, x\rangle,$$

where $A_k = (d^k g(0))\hat{}/k!$. Obviously the map $g \mapsto A_k(g; y_{n_1}, \ldots, y_{n_k})$ belongs to $[H^\infty(U)]^*$ for each fixed $(y_{n_1}, \ldots, y_{n_k}) \in F^k$. We have

$$\sum_{n_1+\ldots+n_k=0}^{\infty} \|A_k(\cdot\,; y_{n_1}, \ldots, y_{n_k})\|\, |\langle a_{n_1}, x\rangle| \ldots |\langle a_{n_k}, x\rangle|$$

$$\leqslant \sum \sup_{\|g\| \leqslant 1} \|A_k(g; y_{n_1}, \ldots, y_{n_k})\|\, \|a_{n_1}\| \ldots \|a_{n_k}\|$$

$$\leqslant \sum \sup_{\|g\| \leqslant 1} \|A_k(g; \ldots)\|\, \|y_{n_1}\|\, \|a_{n_1}\| \ldots \|y_{n_k}\|\, \|a_{n_k}\|$$

$$\leqslant \sum \sup_{\|g\| \leqslant 1} \frac{k^k}{k!} \left\|\frac{(d^k g(0))\hat{}}{k!}\right\| \|y_{n_1}\|\, \|a_{n_1}\| \ldots \|y_{n_k}\|\, \|a_{n_k}\|$$

$$\leqslant \frac{k^k}{k!} \sum \sup_{\substack{\|u\| \leqslant 1 \\ \|g\| \leqslant 1}} \left\| \frac{1}{2\pi i} \int_{|\lambda|=\alpha} \frac{g(\lambda u)}{\lambda^{k+1}}\, d\lambda \right\| \|y_{n_1}\|\, \|a_{n_1}\| \ldots \|y_{n_k}\|\, \|a_{n_k}\|$$

$$\leqslant \frac{k^k}{k!} \sum \frac{\|g\|_U}{\alpha^k} \|y_{n_1}\|\, \|a_{n_1}\| \ldots \|y_{n_k}\|\, \|a_{n_k}\|$$

$$\leqslant \frac{k^k}{k!} \frac{1}{(2\alpha)^k},$$

where α is a suitable constant, $\frac{1}{2} < \alpha < 1$. Put

$$\hat{T}_s := \sum_{k=0}^{s} l_k.$$

Then $\{\hat{T}_s\}$ is a Cauchy sequence in the nuclear norm. By Lemma 3.13 of [3], $\hat{T}(g) := \lim_{s\to\infty} \hat{T}_s(g)$ is nuclear.

1.3. LEMMA. *Let K be a compact subset of a nuclear metric vector space E. Then for every neighbourhood U of K, there exists a neighbourhood V of K contained in U such that the restriction map $R\colon H^\infty(U) \to H^\infty(V)$ is nuclear.*

Proof. (a) First we prove the lemma for $K = \{0\}$. For each balanced neighbourhood W of zero in E, put

$$E_W^* = \{l \in E^*\colon \|l\|_W := \sup_{x \in W} |\langle l, x\rangle| < \infty\}, \qquad W^{**} = \{l \in E_W^*\colon \|l\|_W \leqslant 1\}.$$

Consider the canonical map $e\colon E \to E^{**}$. Observe that $e(W) \subset W^{**}$ and

$$\sup_{x \in W} |\langle l, x\rangle| = \|l\|_W = \sup_{x \in W^{**}} |\langle l, x\rangle|$$

for every $l \in E_W^*$. This implies that

$$\sup_{(u_1,\ldots,u_n) \in W^n} |T(u_1, \ldots, u_n)| = \sup_{u_1 \in W} \ldots \sup_{u_n \in W} |T(u_1, \ldots, u_n)|$$

$$= \sup_{(u_1,\ldots,u_n) \in W^{**n}} |\tilde{T}(u_1, \ldots, u_n)|$$

for every $T \in \mathscr{L}_W(E^n)$, where $\mathscr{L}_W(E^n)$ denotes the space of continuous n-linear forms on E^n which are bounded on W^n and $\tilde{T}$ denotes the n-linear form on E_W^{**n} induced by T.

Let $f \in H^\infty(W)$. Then

$$f(x) = \sum_{n=0}^{\infty} (P_n f)(x) \qquad \forall x \in W,$$

where

$$P_n f(x) = \frac{1}{2\pi i} \int_{|\lambda|=r} \frac{f(\lambda x)}{\lambda^{n+1}} d\lambda \qquad (r > 0),$$

$x \in W/r$. Thus $\|P_n f\|_{W/r} \leqslant \|f\|_{W/r^n}$. Hence $\|P_n \tilde{f}\|_{W^{**}/r} \leqslant \|f\|_{W/r^n}$. This implies that the form $\tilde{f}(x) = \sum_{n=0}^{\infty} (P_n f)^{\sim}(x)$ defines a holomorphic function $\tilde{f}$ on W^{**} and $\tilde{f} \circ e = f$. We have

$$\|\tilde{f}\|_{W^{**}/r} \leqslant C_r \|f\|_W \qquad \forall r > 1,\ f \in H^\infty(W).$$

Thus we obtain a continuous linear extension map

$$S_W\colon H^\infty(W) \to H^\infty(W^{**}/r).$$

Take a balanced neighbourhood V of zero contained in U such that

$$\lim_{n\to\infty} n^5 \delta_n(V, U) = 0,$$

where

$$\delta_n(V, U) = \inf\{\delta(V, U, L)\colon L \subset E,\ \dim L \leqslant n\},$$

$$\delta(V, U, L) = \inf\{\varepsilon > 0\colon L + \varepsilon U \supset V\}.$$

Let $L \subset E$, $\dim L \leqslant n$, be such that $V \subseteq \delta_n(V, U)U + L$. Put $L^\perp = \{l \in E_U^*\colon l|L = 0\}$. Then $\dim(E_U^*/L^\perp) \leqslant n$. Hence we can write $E_U^* = L^\perp \oplus \tilde{L}$ with $\dim \tilde{L} \leqslant n$. Since $V \subseteq \delta_n(V, U)U + L$, we have $U^* \cap L^\perp \subseteq \delta_n(V, U)V^*$, where U^* and V^* are the unit balls in E_U^* and E_V^* respectively.

Let now $x \in U^*$. We write $x = x_1 + x_2$ with $x_1 \in L^\perp$, $x_2 \in \tilde{L}$ and $\|x_1^*\| \leqslant n+1$. This implies that $x \in (n+1)\delta_n(V, U)V^* + \tilde{L}$. Thus $\delta_n(U^*, V^*) \leqslant (n+1)\delta_n(V, U)$. Hence

$$n^4 \delta_n(U^*, V^*) \leqslant n^4 (n+1)\delta_n(V, U) \to 0.$$

By [5], it follows that the restriction map $\eta\colon E_U^* \to E_V^*$ is nuclear. This yields the nuclearity of $\eta^*\colon E_V^{**} \to E_U^{**}$. By Lemma 1.2, the map $\hat{\eta}^*\colon H^\infty(U^{**}) \to H^\infty(V^{**}/r')$ is nuclear for some $r' > 1$. Hence the composition map $H^\infty(U) \to H^\infty(U^{**}/r) \to H^\infty(V^{**}/rr') \to H^\infty(V/rr')$ is nuclear.

(b) Now we can prove Lemma 1.3 when K is an arbitrary compact set in E. By compactness of K, there exist $\{x_i\}_{i=1}^n$ and balanced neighbourhoods $\{U_i\}_{i=1}^n$ of zero in E such that

$$K \subseteq \bigcup_{i=1}^{n} (x_i + U_i) \subseteq U.$$

By (a) there exist $\{y_j\}_{j=1}^m$ and balanced neighbourhoods $\{V_j\}_{j=1}^m$ such that the restriction maps $\alpha_{ij}\colon H^\infty(x_i + U_i) \to H^\infty(y_j + V_j)$ are nuclear for all $y_j + V_j \subseteq x_i + U_i$.

Let $I_i = \{j\colon y_j + V_j \subseteq x_i + U_i\}$ and $V = \bigcup_{j=1}^m (y_j + V_j)$. Consider the commutative diagram

$$\begin{array}{ccc} H^\infty(U) & \xrightarrow{R} & H^\infty(V) \\ \downarrow{\scriptstyle\beta} & & \downarrow{\scriptstyle\gamma} \\ \prod_{i=1}^{n} H^\infty(x_i + U_i) & \xrightarrow{\alpha} & \prod_{i=1}^{n} \prod_{j\in I_i}^{m} H^\infty(y_j + V_j) \end{array}$$

in which α, β and γ are the canonical maps. Since α_{ij} is nuclear for all i and $j \in I_i$, the map α is also nuclear. Hence R is absolutely summing. Take now a neighbourhood $\tilde{V}$ of K such that the restriction map $R_0\colon H^\infty(V) \to H^\infty(\tilde{V})$ is absolutely summing. Then the composition map $R_0 \circ R\colon H^\infty(U) \to H^\infty(V) \to H(\tilde{V})$ is nuclear [3].

1.4. Lemma. *Let A, B, C be Banach spaces and let $\tilde{A}$, $\tilde{B}$, $\tilde{C}$ be their respective factor spaces. Let $S\colon A \to B$, $T\colon B \to C$ be nuclear maps which factor through $\tilde{A}$ and $\tilde{B}$, respectively. Then the canonical composition map $\tilde{T} \circ \tilde{S}\colon \tilde{A} \to \tilde{B} \to \tilde{C}$ is absolutely summing.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccc} A & \xrightarrow{S} & B & \xrightarrow{T} & C \\ {\scriptstyle\alpha}\downarrow & & {\scriptstyle\beta}\downarrow & & \downarrow{\scriptstyle\gamma} \\ \tilde{A} & \xrightarrow{\tilde{S}} & \tilde{B} & \xrightarrow{\tilde{T}} & \tilde{C} \end{array}$$

in which α, β, γ are the canonical maps. From the equality $\beta^* \circ \tilde{T}^* = T^* \circ \gamma^*$ and the nuclearity of T^* it follows that $\tilde{T}^*$ is absolutely summing. Similarly $\tilde{S}^*$ is absolutely summing. Hence $(\tilde{S}^* \circ \tilde{T}^*)^*$ is nuclear. Thus $(\tilde{T} \circ \tilde{S})^{**} = \tilde{T}^{**} \circ \tilde{S}^{**} = (\tilde{S}^* \circ \tilde{T}^*)^*$ is nuclear. It follows that $\tilde{T} \circ \tilde{S}$ is absolutely summing and the lemma is proved.

1.5. Lemma. *Let K be a compact subset of a metric vector space. Then $\tilde{H}^\infty(K) := \operatorname{inj\,lim} H^\infty(U)/A_U$ is regular.*

Proof. Since $S(H^\infty(U)/A_U) \subset \lambda_U S(H^\infty(U))$, where $S(H^\infty(U)/A_U)$ and $S(H^\infty(U))$ denote the unit balls in $H^\infty(U)/A_U$ and $H^\infty(U)$ respectively and $\lambda_U\colon H^\infty(U) \to H^\infty(U)/A_U$ is the canonical map, it follows that $S(H^\infty(U)/A_U)$ is compact in $H(U)/\tilde{A}_U$, where

$$\tilde{A}_U := \{f \in H(U)\colon f|K = 0\}.$$

Hence $S(H^\infty(U)/A_U)$ is closed in $\operatorname{inj\,lim} H(U)/\tilde{A}_U$. This implies that $S(H^\infty(U)/A_U)$ is closed in $\operatorname{inj\,lim} H(U)/A_U$. Hence $\operatorname{inj\,lim} H^\infty(U)/A_U$ is regular [2].

Proof of Theorem 1.1. (i) Let $\{U_n\}$ be a decreasing fundamental system of neighbourhoods of K. Put $A_n = A_{U_n}$ for every $n \in N$. By Lemma 1.3 we can assume that the restriction map R_n: $H^\infty(U_n) \to H^\infty(U_{n+1})$ is nuclear, and by Lemma 1.4 that the map $\hat{R}_n$: $H^\infty(U_n)/A_n \to H^\infty(U_{n+1})/A_{n+1}$ induced by R_n is also nuclear.

Let now f: $K \to F$ be a continuous map such that $l \circ f$ can be extended to a holomorphic function $(l \circ f)^\wedge$ on a neighbourhood of K for all $l \in F^*$. Define a linear map S: $F^* \to \tilde{H}^\infty(K) \cong \operatorname{inj\,lim} H^\infty(U_n)/A_n$ by $S(l) = (l \circ f)^\wedge \operatorname{mod} A_n$, where $n \in N$ is such that $(l \circ f)^\wedge \in H^\infty(U_n)$. By the closed graph theorem [6], S is continuous. Since F^* is a Baire space, F^* is continuously mapped by S into $H^\infty(U_{n_0})/A_{n_0}$ for some n_0. Since the map $\hat{R}_{n_0}$: $H^\infty(U_{n_0})/A_{n_0} \to H^\infty(U_{n_0+1})/A_{n_0+1}$ is compact, we can assume that S is compact. We shall prove that S is continuous, as a map of F_c^* into $H^\infty(U_{n_0})/A_{n_0}$, where F_c^* is F^* equipped with the topology of uniform convergence on compact subsets of F.

We first prove that for each $b^* \in (H^\infty(U_{n_0})/A_{n_0})^*$, $b^* \circ S$: $F_c^* \to C$ is continuous. By [6], it suffices to show that the restriction of $b^* \circ S$ to each W^0 is continuous, where W^0 denotes the polar of a neighbourhood W of zero in F. Let $\{x_\alpha^*\} \subset W^0$ and $x_\alpha^* \to 0$ in the $\sigma(F^*, F)$-topology. Then $\{S(x_\alpha^*)\}$ converges to $\bar{g} \in H^\infty(U_{n_0})/A_{n_0}$. Since $\lim(Sx_\alpha^*)(z) = \lim x_\alpha^* f(z) = 0$ $\forall z \in K$ it follows that $g(z) = 0$ for every $z \in K$, where $g \in H^\infty(U_{n_0})$ is such that $g \operatorname{mod} A_{n_0} = \bar{g}$. Thus $\bar{g} = 0$. On the other hand, by compactness of $\overline{\{S(x_\alpha^*)\}}$, it follows that $b^* \circ S(x_\alpha^*) \to 0$. Let now $x_\alpha^* \to 0$ in F_c^*. We then have

$$\sup_{\|b^*\| \leqslant 1} |\langle Sx_\alpha^*, b^* \rangle| = \sup_{\|b^*\| \leqslant 1} |\langle x_\alpha^*, S^* \circ b^* \rangle| \to 0.$$

Hence S: $F_c^* \to H^\infty(U_{n_0})/A_{n_0}$ is continuous.

From the nuclearity of $\hat{R}_{n_0}$, this map can be factorized through l^1. Thus there exists a continuous linear map $\tilde{S}$: $F_c^* \to H^\infty(U_{n_0+1})$ such that $\lambda_{n_0+1} \circ \tilde{S} = \hat{R}_{n_0} \circ S$, where λ_{n_0+1}: $H^\infty(U_{n_0+1}) \to H^\infty(U_{n_0+1})/A_{n_0+1}$ is the canonical map. Then the form given by $\hat{f}(z)(x^*) = [\tilde{S}(x^*)](z)$ for $z \in U_{n_0+1}$ and $x^* \in F^*$ defines a holomorphic map $\hat{f}$: $U_{n_0+1} \to F$ which is a holomorphic extension of f.

(ii) As in (i) we consider the map S: $F^* \to \operatorname{inj\,lim} H^\infty(U_n)/A_n$. Since F^* is complete bornological, F^* is a space of type (β) [4]. By the closed graph theorem of Grothendieck [4], S is continuous. Since F^* is Baire, as in (i) there exists n_0 such that S: $F^* \to H^\infty(U_{n_0})/A_{n_0}$ is continuous. On the other hand, since F^* is nuclear there exists a continuous linear map $\tilde{S}$: $F^* \to H^\infty(U_{n_0})$ such that $S = \lambda_{n_0} \circ \tilde{S}$. It remains to repeat the last part in the proof of (i).

1.6. THEOREM. *Let K be a compact subset of a metric vector space E and f a map from K into a Banach space B such that $l \circ f$ can be extended to a holomorphic function on a neighbourhood of K for every $l \in B^*$. Then there exists a Banach space $\tilde{B}$ containing B as a closed subspace such that f can be extended to a holomorphic map on a neighbourhood of K but with values in $\tilde{B}$.*

Proof. Let T be the unit ball in B. Put $\tilde{B} = l^\infty(T)$. Then $\tilde{B}$ contains B as a closed subspace. As in Theorem 1.1 consider the map S from $\tilde{B}^*$ into the regular inductive limit $\operatorname{inj\,lim} H^\infty(U_n)/A_n$. By [6], S is continuous, hence there exists n_0 such that $S(\tilde{B}^*) \subseteq H^\infty(U_{n_0})/A_{n_0}$ and S: $\tilde{B}^* \to H^\infty(U_{n_0})/A_{n_0}$ is continuous. Consider the map $P \circ S^*$: $(H^\infty(U_{n_0})/A_{n_0})^* \to \tilde{B}$, where P: $\tilde{B}^{**} \to \tilde{B}$ is a continuous linear projection. By the Hahn–Banach theorem there exists a continuous linear map $\tilde{S}$: $H^\infty(U_{n_0})^* \to \tilde{B}$ which is an extension of $P \circ S^*$. Hence $\tilde{S}^*$: $\tilde{B}^* \to H^\infty(U_{n_0})^{**} \to H(U_{n_0})$ is continuous. As in Theorem 1.1, $\tilde{S}^*$ induces a holomorphic extension $\hat{f}$: $U_{n_0} \to \tilde{B}$ of f.

§ 2. Extending weakly holomorphic functions with values in a C-manifold. Let X be a locally convex complex manifold, i.e. a complex manifold modelled by open sets in locally convex complex spaces. Let C_X denote the Carathéodory pseudometric on X defined by

$$C_X(x, y) = \sup\{\varrho(f(x), f(y)): \ f \in H(X, D)\}$$

where $D = \{z \in C: |z| < 1\}$, ϱ is the Bergman–Poincaré metric on D and $H(X, D)$ is the space of holomorphic maps from X into D.

As in [7] we say that X is a *C-manifold* if C_X is a metric on X. Moreover, if X is complete for C_X, then X is called a *complete C-manifold.* In the finite-dimensional case, it is known [7] that C_X induces the topology of X. In this section we prove the following.

2.1. THEOREM. *Let X be a finite-dimensional complete C-manifold and K a compact subset of a metric vector space E. Let f: $K \to X$ be a continuous map such that $\sigma \circ f$ can be extended to a holomorphic function on a neighbourhood of K for all $\sigma \in H^\infty(X)$. Then f can be extended to a holomorphic map on a neighbourhood of K.*

Proof. (i) Consider the holomorphic map $\tilde{\delta} := \delta \circ e$, where δ: $X \to H^\infty(X)^*$ and e: $H^\infty(X)^* \to \tilde{B} := l^\infty(S(H^\infty(X)^*))$ are the canonical maps. Since X is complete, $\tilde{\delta}$: $X \to \tilde{\delta}(X)$ is a holomorphic homeomorphism and X is a Stein manifold [7].

Consider the exact sequence of holomorphic Banach bundles:

$$0 \to TX \to T\tilde{B} \xrightarrow{R} N \to 0$$

over X.

Since X is Stein, there exists a morphism S: $N \to \tilde{\delta}^* T\tilde{B}$ of holomorphic Banach bundles such that $R \circ S = \mathrm{id}_N$ [1]. Let Z denote the zero section of N and τ: $N \to X$ the canonical projection. Then τ: $Z \to X$ is a biholomorphism, and hence φ: $Z \to X$ is also a biholomorphism, where φ: $N \to \tilde{B}$ is given by

$$\varphi(n) = \tau(n) + \pi_2 S(n)$$

and π_2: $\tilde{\delta}^* T\tilde{B} \cong X \times \tilde{B} \to \tilde{B}$ is the canonical projection. Since for every $n \in Z$ we have

$$d\varphi(TN_n) = d\tau(TX_{\tau n}) + d\tau(TN_{\tau n}) = d\pi_2 \circ dS(TX_{\tau n}) + d\pi_2 \circ dS(TN_{\tau n})$$
$$= d\tau(TN_{\tau n}) + d\pi_2 \circ dS(TN_{\tau n}) = TX_{\tau n} \oplus N_{\tau n} \cong \tilde{B},$$

it follows that φ is a local biholomorphism at n. Thus φ can be extended to a biholomorphism $\hat{\varphi}$ from a neighbourhood W of Z in N onto a neighbourhood U of $\tilde{\delta}(X)$ in $\tilde{B}$. Define a holomorphic map ψ: $U \to X$ by

$$\psi(z) = \tau \circ \hat{\varphi}^{-1}(z) \quad \text{for } z \in U.$$

Obviously $\psi(\tilde{\delta} z) = z$ for every $z \in X$.

(ii) Assume now that f is as in the theorem. By Theorem 1.6, f can be extended to a holomorphic map $\tilde{f}$ of a neighbourhood G of K into $\tilde{B}$. By compactness of K we may assume that $\tilde{f}(G) \subseteq U$. Then $\hat{f} = \psi \circ \tilde{f}$: $G \to X$ is a holomorphic extension of f.

The theorem is proved.

§ 3. Extending separately holomorphic functions with values in Banach–Lie groups. In this section we consider the theorem of Siciak–Zakharyuta for continuous separately holomorphic functions with values in Banach–Lie groups. First we recall that a map f: $X \times F \cup E \times Y \to Z$, where X, Y, Z are complex spaces, $E \subseteq X$ and $F \subseteq Y$ are compact sets, is called *separately holomorphic* if

(i) $\forall w \in F$ the function $z \mapsto f(z, w)$ is holomorphic in X.

(ii) $\forall z \in F$ the function $w \mapsto f(z, w)$ is holomorphic in Y.

We shall say that a compact set K in a Stein manifold W is *P-regular* if

$$\omega^*(z, K \cap \tilde{B}(z, r), \tilde{B}(z, r)) = -1 \quad \forall r > 0 \text{ and } \forall z \in K,$$

where $z \mapsto \omega^*(z, A, D)$ is defined in [8], [11] for every subset A of a domain D in W.

3.1. THEOREM. *Let X and Y be Stein manifolds and let $E \subseteq X$, $F \subseteq Y$ be compact sets such that $\hat{E}$, $\hat{F}$, the holomorphically convex hulls of E, F respectively, are regular in X, Y respectively. Let Γ be a Banach–Lie group and f: $Z := X \times F \cup E \times Y \to \Gamma$ a continuous and separately holomorphic function. Then f can be uniquely extended to a holomorphic function $\hat{f}$ in the set*

$$\tilde{Z} := \{(z, w) \in X \times Y:\ \omega^*(z, E, X) + \omega^*(w, F, Y) < 0\}$$

containing Z.

Proof. (a) First we prove that f can be extended to a holomorphic map on a neighbourhood of Z. Let $\{X_n\}$ and $\{Y_n\}$ be two sequences of relatively compact Stein subsets of X and Y respectively such that

$$E \subset X_n \subset X_{n+1}, \quad F \subset Y_n \subset Y_{n+1} \quad \forall n \in N,$$
$$X = \bigcup_{n=1}^{\infty} X_n, \quad Y = \bigcup_{n=1}^{\infty} Y_n.$$

For each n, put

$$Z_n = X_n \times F \cup E \times Y_n, \quad f_n = f | Z_n.$$

Let $\exp$: $T\Gamma_e \to \Gamma$ denote the exponential map from the tangent space of Γ at the unit element e into Γ. Note that exp is biholomorphic at zero in $T\Gamma_e$. Let $H(Z, \Gamma)$ denote the topological group consisting of continuous and separately holomorphic maps from Z into Γ equipped with the compact-open topology and W a neighbourhood of zero in $T\Gamma_e$ such that $\exp$: $W \cong \exp W$. For each n, put

$$V_n = \{g \in H(Z, \Gamma):\ g(Z_n) \subseteq \exp W\}.$$

Then V_n is a neighbourhood of the unit element in $H(Z, \Gamma)$, hence V_n generates $H(Z, \Gamma)$. This yields $g_1, \ldots, g_p \in V_n$ such that $f_n = g_1 \cdots g_p$. By the theorem of Siciak–Zakharyuta [8], [11] the maps $g_1, \ldots, g_p$ can be extended to holomorphic maps on a neighbourhood G_n of Z_n in $\tilde{Z}$. Thus f_n can be extended to a holomorphic map $\tilde{f}_n$: $G_n \to \Gamma$. By uniqueness we have

$$\tilde{f}_n | G_n \cap G_{n+1} = \tilde{f}_{n+1} | G_n \cap G_{n+1} \quad \forall n \geq 1.$$

Hence the formula $\tilde{f} | G_n = \tilde{f}_n$ defines a holomorphic extension of f to $G = \bigcup_{n=1}^{\infty} G_n$, a neighbourhood of Z in $\tilde{Z}$.

(b) As in (a) we can prove that every holomorphic map f: $H_k(r) \to \Gamma$, where $H_k(r)$ denotes the k-dimensional Hartogs domain given by

$$H_k(r) = \{z \in D^k:\ |z_1|, \ldots, |z_{k-1}| < r\} \cup \{z \in D^k:\ |z_k| > 1 - r\} \quad (0 < r < 1)$$

can be extended to a holomorphic map $\hat{f}$: $(H_k(r))\hat{} \cong D^k \to \Gamma$, where $(H_k(r))\hat{}$ is the envelope of holomorphy of $H_k(r)$.

(c) From (b) it follows that every holomorphic map from a Riemann domain Ω over a Stein manifold into Γ can be extended to a holomorphic map from $\hat{\Omega}$, the envelope of holomorphy of Ω, into Γ.

(d) Since $\tilde{Z}$ is a domain of holomorphy and $\hat{G} \cong \tilde{Z}$, it follows that f can be extended to a holomorphic map $\hat{f}$: $\tilde{Z} \to \Gamma$.

The theorem is proved.

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Applications du théorème de factorisation pour des fonctions à valeurs opérateurs

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Abstract. In this paper we prove a factorization theorem for analytic functions with values in noncommutative L_p-spaces associated with a semifinite von Neumann algebra. Then we give two applications. The first is to show the Haagerup noncommutative L_p-spaces $(0 < p < 1)$ associated with an arbitrary von Neumann algebra are uniformly Hardy convex. In particular, they have the analytic Radon–Nikodym property. The second is to compute the interpolation spaces between the Hardy spaces of functions with values in noncommutative L_p-spaces associated with a semifinite von Neumann algebra.

§ 1. Introduction. On étudie dans cet article la convexité uniforme complexe des espaces non-commutatifs $L_p(M, \tau)$ $(0 < p < 1)$ associés à une algèbre de von Neumann M, munie d'une trace normale semi-finie et fidèle τ; on calcule aussi les espaces d'interpolation complexes entre les espaces de Hardy de fonctions à valeurs dans $L_p(M, \tau)$. L'outil que l'on utilise est des théorèmes de factorisation pour des fonctions à valeurs opérateurs. Le problème de factoriser une fonction en certaines composantes canoniques est bien classique. On présente, pour notre but, une forme de ces théorèmes. Notre théorème de factorisation concerne la factorisation pour des fonctions dans les espaces de Hardy à valeurs dans $L_p(M, \tau)$. Il se comporte presque comme le théorème classique de factorisation pour des fonctions scalaires dans les espaces de Hardy usuels. On l'applique ensuite pour démontrer que $L_p(M, \tau)$ $(0 < p < 1)$ est uniformément convexe complexe en un certain sens. On en déduit, en particulier, que $L_p(M, \tau)$ $(0 < p < 1)$ a la propriété de Radon–Nikodym analytique (en bref, ARNP). La deuxième application de ce théorème de factorisation est le calcul des espaces d'interpolation complexes entre les espaces de Hardy de fonctions à valeurs dans $L_p(M, \tau)$ $(0 < p < \infty)$. On démontre que pour $0 < p_0,\ p_1,\ q_0,\ q_1 < \infty$, $0 < \theta < 1$,

$$(H_{p_0}(L_{q_0}(M, \tau)),\ H_{p_1}(L_{q_1}(M, \tau)))_\theta = H_p(L_q(M, \tau)), \tag{1.1}$$

où $1/p = (1-\theta)/p_0 + \theta/p_1$, $1/q = (1-\theta)/q_0 + \theta/q_1$.

On présente le théorème de factorisation dans le deuxième paragraphe. Nous basons notre démonstration sur un théorème classique de factorisation pour des fonctions à valeurs opérateurs, dû à Devinatz [D].