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Spectral characterization of operators with precompact orbit

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Abstract. We characterize the Banach space operators A for which the orbit $\{A^m\}_{m\geq 1}$ has compact closure in the operator norm topology. It is shown that this occurs if and only if the spectrum of A consists of a closed set contained in the open unit disc and an at most finite number of eigenvalues of A on the unit circle which are poles of the resolvent $(\lambda I-A)^{-1}$ of order 1. The number of accumulation points of the orbit is finite if and only if all the eigenvalues on the unit circle are roots of unity.

1. Introduction. It is natural to describe the behaviour of the powers of a bounded linear operator A on a Banach space X in terms of its spectrum. In [2] Jamison proved that if A is power bounded and X is separable, then A can have an at most countable number of eigenvalues of modulus 1. The case when $\lim_{m\to\infty} A^m$ exists is well known and was fully characterized by Koliha in [3] and [4] and by Luecke in [6] (see also Li [5]). In these papers one can find many statements equivalent to the existence of the above limit, the most concise one saying that $\mathrm{Sp}A\setminus\{1\}\subset\{z\colon |z|\leqslant r<1\}$ and 1 is a pole of the resolvent $(\lambda I-A)^{-1}$ of order at most 1 ([3, Theorem 2]). Very interesting results about almost periodic operators were obtained by M. Yu. Lyubich and Yu. I. Lyubich. A part of their main structure theorem is quoted below as Proposition 1, and we shall use it in the sequel. However, no attempt has been made to describe the intermediate case when the powers of an operator form a precompact set with respect to the operator norm. In this paper we shall deal with this problem which seems to be very natural and connecting the previously obtained issues. We shall show that the compactness of $\overline{\{A^m\}}_{m\geq 1}$ can be characterized by spectral conditions similar to those obtained for the case when $\lim_{m\to\infty} A^m$ exists. For matrices the result was obtained by Mott and Schneider in [8].

2. Definitions and preliminaries. Let X denote a complex Banach space and $B(X)$ the Banach algebra of all bounded linear operators on X equipped with the operator norm. For T in $B(X)$ we shall write $N(T)$ for the null space $T^{-1}(0)$ and $R(T)$ for the range space TX. The smallest positive integer n such that $N(T^n)=N(T^{n+1})$ will be called the *ascent* of T and denoted by $\alpha(T)$. If no such

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integer exists we put $\alpha(T) = \infty$. Similarly, the smallest positive integer n such that $R(T^n) = R(T^{n+1})$ will be called the *descent* of T and denoted by $\delta(T)$. If there is no such integer we set $\delta(T) = \infty$. We shall write $\operatorname{Sp} T$ for the spectrum of T.

The following proposition is a part of [7, Theorem 3.1].

PROPOSITION 1. *Let $A \in B(X)$ be such that for every $x \in X$ the set $\overline{\{A^m x\}}_{m \geq 1}$ is compact. Then there is a direct sum decomposition $X = X_0 \oplus X_1$, where X_0 and X_1 are both closed subspaces, $X_0 = \{x \in X\colon \lim_{m\to\infty} A^m x = 0\}$ and X_1 is the topological direct sum of the eigenspaces of A corresponding to the eigenvalues of modulus 1.*

3. The result. The result of this paper is the following theorem.

THEOREM 2. *Let $A \in B(X)$. The set $\overline{\{A^m\}}_{m \geq 1}$ is compact in $B(X)$ if and only if there is an at most finite set of distinct complex numbers $\lambda_1, \ldots, \lambda_n$ with $|\lambda_1| = \ldots = |\lambda_n| = 1$ such that:*

(i) $\operatorname{Sp} A \setminus \{\lambda_1, \ldots, \lambda_n\} \subset \{z\colon |z| \leq r < 1\}$.

(ii) *$\lambda_1, \ldots, \lambda_n$ are poles of the resolvent $(\lambda I - A)^{-1}$ of order* 1.

Note that in this situation the points $\lambda_1, \ldots, \lambda_n$ are eigenvalues of A (see for example [1, Proposition 1.40]).

The essential step is contained in the following lemma which strengthens the result of Jamison [2] in our case.

LEMMA 3. *If the powers of an operator A form a precompact set in $B(X)$, then A has an at most finite number of eigenvalues on the unit circle.*

Proof. Let U be the set of these eigenvalues. Suppose, contrary to the claim, that U is infinite. Then there exists an accumulation point λ_0 of U. Fix $0 < \delta < \frac{1}{10}$. Put

$$m_0 = \min\{m \in \mathbf{N}\colon 0 \leq [\![m \arg \lambda_0]\!] < \delta\},$$

$$m_1 = \begin{cases} 1 & \text{if } [\![m_0 \arg \lambda_0]\!] = 0, \\ \min\{m \in \mathbf{N}\colon 2\pi \leq m [\![m_0 \arg \lambda_0]\!]\} & \text{otherwise,} \end{cases}$$

where $[\![a]\!] \in [0, 2\pi)$, $[\![a]\!] \equiv a \pmod{2\pi}$, and we understand that $\arg z \in [0, 2\pi)$.

Let $\{a_i\}_{i \geq 1}$ be the increasing sequence of the natural exponents a such that

$$|\lambda_0^a - \lambda_0| < \delta.$$

Note that if $|[\![a \arg \lambda_0]\!] - \arg \lambda_0| < \delta$ then $|\lambda_0^a - \lambda_0| < \delta$, so that obviously by the definitions of m_0 and m_1 we have

$$a_{i+1} - a_i \leq m_1 m_0 = L_0 \quad \text{for } i = 1, 2, \ldots \tag{1}$$

Without loss of generality we can assume that for every $\varepsilon > 0$ there is $\lambda \in U$ such that $|\arg \lambda - \arg \lambda_0| < \varepsilon$. This may not be true if $\lambda_0 = 1$ but we can get around this by setting $\arg z \in (-2\pi, 0]$ in that case.

Let then $\lambda_1 \in U \setminus \{\lambda_0\}$ be such that

$$(L_0 + 1)|\arg \lambda_1 - \arg \lambda_0| < \delta. \tag{2}$$

It follows from (1) and (2) that

$$0 < (a_{i+1} - a_i)|\arg \lambda_1 - \arg \lambda_0| < \delta \quad \text{for } i = 1, 2, \ldots \tag{3}$$

Because for every i the point $\lambda_0^{a_i}$ is in the δ-neighbourhood of λ_0 it is clear from (3) that every arc of length 5δ contains an infinite set of elements of the sequence $\{\lambda_1^{a_i}\}_{i \geq 1}$ (for every i we have $|\lambda_0^{a_{i+1}} - \lambda_0^{a_i}| < 2\delta$, therefore either $[\![(a_{i+1} - a_i) \arg \lambda_0]\!] < 4\delta$ or $[\![(a_{i+1} - a_i) \arg \lambda_0]\!] > 2\pi - 4\delta$). Since

$$\left[\frac{2\pi + \delta}{|\arg \lambda_1 - \arg \lambda_0|}\right] |\arg \lambda_1 - \arg \lambda_0| \geq 2\pi,$$

where $[a]$ denotes the integer part of a, it follows that if λ_1^k is in the 5δ-neighbourhood of $\pm\lambda_0$, then there must exist l,

$$k < l \leq k + \left[\frac{2\pi + \delta}{|\arg \lambda_1 - \arg \lambda_0|}\right], \tag{4}$$

such that $|\lambda_1^l \pm \lambda_0| < 5\delta$.

We now construct two subsequences $\{a_{-,i}\}_{i \geq 1}$, $\{a_{+,i}\}_{i \geq 1}$ of $\{a_i\}_{i \geq 1}$: $\{a_{-,i}\}_{i \geq 1}$ consists of the elements of $\{a_i\}_{i \geq 1}$ for which $|\lambda_1^{a_i} - \lambda_0| < 5\delta$ and $\{a_{+,i}\}_{i \geq 1}$ of those for which $|\lambda_1^{a_i} + \lambda_0| < 5\delta$. By (4) we have

$$a_{\mp,i+1} - a_{\mp,i} \leq \left[\frac{2\pi + \delta}{|\arg \lambda_1 - \arg \lambda_0|}\right] = L_1 \quad \text{for } i = 1, 2, \ldots$$

Assume that we have chosen the eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding pairwise disjoint sequences $\{a_{J,i}\}_{i \geq 1}$, $J = (j_1, \ldots, j_n)$, $j_k = +$ or $-$, such that for every $k = 1, \ldots, n$ we have for all $i \in \mathbf{N}$

$$|\lambda_k^{a_{J,i}} - \lambda_0| < 5\delta \quad \text{or} \quad |\lambda_k^{a_{J,i}} + \lambda_0| < 5\delta$$

according as $j_k = -$ or $+$, and

$$a_{J,i+1} - a_{J,i} \leq L_n. \tag{5}$$

We now proceed in the same manner as for λ_1. Choose $\lambda_{n+1} \in U \setminus \{\lambda_0, \ldots, \lambda_n\}$ such that

$$(L_n + 1)|\arg \lambda_{n+1} - \arg \lambda_0| < \delta. \tag{6}$$

Then (5) and (6) yield

$$(a_{J,i+1} - a_{J,i})|\arg \lambda_{n+1} - \arg \lambda_0| < \delta$$

for each J and $i \in \mathbf{N}$.

Hence (as for λ_1) for each J and for every arc of length 5δ there is an infinite set of elements of $\{a_{J,i}\}_{i\geqslant 1}$ contained in this arc. So for each fixed J we can choose two disjoint subsequences $\{a_{(J,-),i}\}_{i\geqslant 1}$, $\{a_{(J,+),i}\}_{i\geqslant 1}$ of $\{a_{J,i}\}_{i\geqslant 1}$ such that, setting $J' = (j_1, \ldots, j_n, j_{n+1}) = (J, -)$ or $(J, +)$, we have for all $i \in \mathbf{N}$

$$|\lambda_{n+1}^{a_{J',i}} - \lambda_0| < 5\delta \quad \text{or} \quad |\lambda_{n+1}^{a_{J',i}} + \lambda_0| < 5\delta$$

according as $j_{n+1} = -$ or $+$, and

$$a_{J',i+1} - a_{J',i} \leqslant \left[\frac{2\pi+\delta}{|\arg\lambda_{n+1} - \arg\lambda_0|}\right] = L_{n+1}.$$

This completes the induction step.

Having constructed these sequences we are able to finish our proof. For each $n \in \mathbf{N}$ we shall find n elements of the set $\{A^m\}_{m\geqslant 1}$ such that the distance between any two of them is greater than 1. Given $1 \leqslant k \leqslant n$ we choose any element b_k of $\{a_{J,i}\}_{i\geqslant 1}$ with $j_s = +$ for $s \neq k$, $j_k = -$, $s = 1, \ldots, n$, and an eigenvector x_k, with $\|x_k\| = 1$, corresponding to the eigenvalue λ_k.

Taking $A^{b_1}, \ldots, A^{b_n}$ we see that, if $1 \leqslant i \neq j \leqslant n$, then

$$\begin{aligned}\|A^{b_i} - A^{b_j}\| &\geqslant \|A^{b_i}x_i - A^{b_j}x_i\| \\ &\geqslant \|2\lambda_0 x_i\| - \|\lambda_i^{b_i}x_i - \lambda_0 x_i\| - \|\lambda_i^{b_j}x_i + \lambda_0 x_i\| \geqslant 2 - 5\delta - 5\delta > 1.\end{aligned}$$

This contradicts the compactness of $\overline{\{A^m\}}_{m\geqslant 1}$, hence the lemma is proved. ∎

We can now set about the proof of Theorem 2.

Proof of Theorem 2. Assume first that $\overline{\{A^m\}}_{m\geqslant 1}$ is compact. From Proposition 1 we get $X = X_0 \oplus X_1$, where X_0 and X_1 are as in the statement of Proposition 1. But by Lemma 3 we have an at most finite decomposition

$$X_1 = Y_1 \oplus \ldots \oplus Y_n,$$

where $Y_1, \ldots, Y_n$ are the eigenspaces of A corresponding to the eigenvalues $\lambda_1, \ldots, \lambda_n$ on the unit circle. Because $X_0, Y_1, \ldots, Y_n$ are invariant for A we have

$$\operatorname{Sp} A = \operatorname{Sp} A|_{X_0} \cup \{\lambda_1, \ldots, \lambda_n\}$$

([1, Proposition 1.37]), so in order to finish the proof of (i) it suffices to study $\operatorname{Sp} A|_{X_0}$. It is enough to show that $\|A^{m_0}|_{X_0}\| < 1$ for some m_0. From the compactness there is an operator $B \in B(X_0)$ such that a subsequence of $\{A^m|_{X_0}\}_{m\geqslant 1}$ converges to B in $B(X_0)$. By the form of X_0 we conclude that $B = 0$. Hence there must be powers $A^m|_{X_0}$ having arbitrarily small norm.

It remains to show (ii). By [1, Theorem 1.54] it suffices to prove that $\alpha(\lambda_i I - A) = \delta(\lambda_i I - A) = 1$ and $R(\lambda_i I - A)$ is closed for each $i = 1, \ldots, n$.

Let $Z_i = X_0 \oplus Y_1 \oplus \ldots \oplus Y_{i-1} \oplus Y_{i+1} \oplus \ldots \oplus Y_n$ for $i = 1, \ldots, n$. Since $\operatorname{Sp} A|_{Z_i}$ does not contain λ_i ([1, Proposition 1.37]), the operator $(\lambda_i I - A)|_{Z_i}$ is

invertible, and since $(\lambda_i I - A) Y_i = \{0\}$, we have

$$R(\lambda_i I - A)^2 = R(\lambda_i I - A) = Z_i \quad \text{for } i = 1, \ldots, n.$$

Thus $R(\lambda_i I - A)$ is closed and $\delta(\lambda_i I - A) = 1$.

If the ascent of $\lambda_i I - A$ is not equal to 1, then there exists $x \in N(\lambda_i I - A)^2 \setminus N(\lambda_i I - A)$, i.e. $\lambda_i x - Ax = y \neq 0$ and $Ay = \lambda_i y$. From the first equality we have $\lambda_i Ax - A^2 x = Ay$ and by an easy computation $\lambda_i^2 x - A^2 x = 2\lambda_i y$. Repeating this procedure we obtain for each $m \in \mathbf{N}$

$$\lambda_i^m x - A^m x = m\lambda_i^{m-1} y,$$

which contradicts the boundedness of $\{A^m x\}_{m\geqslant 1}$. Hence $\alpha(\lambda_i I - A) = 1$ for every $i = 1, \ldots, n$, and the proof of (ii) is complete.

The converse implication in Theorem 2 is a simple consequence of the Riesz decomposition $X = X_0 \oplus Y_1 \oplus \ldots \oplus Y_n$ corresponding to the given structure of $\operatorname{Sp} A$. Condition (ii) and [1, Proposition 1.40] imply that $(\lambda_i I - A) Y_i = \{0\}$ for $i = 1, \ldots, n$. We remark that, since the spectral radius of $A|_{X_0}$ is less than 1, $\|A^m|_{X_0}\|$ converges to zero. In view of this special structure of the operator A it is easy to construct a finite ε-net for $\overline{\{A^m\}}_{m\geqslant 1}$ for each $\varepsilon > 0$. ∎

Remark. It is worth noticing that the set of all accumulation points of $\{A^m\}_{m\geqslant 1}$ is finite if and only if all the λ_i in our theorem are roots of unity. This is a simple consequence of [3, Theorem 2]. Indeed, if all λ_i are roots of 1, then A^M, where M is the least common multiple of the degrees of λ_i, fulfills the assumptions of Koliha's theorem, therefore $\{A^l \lim_{m\to\infty} A^{mM}\}_{1\leqslant l\leqslant M}$ is the set of all accumulation points. Obviously, if some λ_i is not a root of 1, then the number of accumulation points cannot be finite.

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