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On the left and right joint spectra in Banach algebras

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Abstract. The main result of the paper says that the left joint spectrum of an arbitrary finite set of elements in a Banach algebra is always contained in the right joint spectrum of these elements if and only if the Banach algebra is commutative modulo the radical.

Throughout the paper Banach algebras are always assumed to be complex and unital. The unit element is denoted by 1 and we always write λ instead of $\lambda 1$ ($\lambda \in \mathbf{C}$). The symbol $\operatorname{rad} A$ stands for the (Jacobson) radical of a Banach algebra A, i.e., the intersection of the kernels of all irreducible representations of A or, what is the same, the intersection of all the maximal left ideals of A (see [1, p. 124]).

If $a_1, \ldots, a_n$ are elements of a Banach algebra A, then their *left joint spectrum*, denoted by $\sigma_l^A(a_1, \ldots, a_n)$ or simply by $\sigma_l(a_1, \ldots, a_n)$ if there is no confusion, is the subset of $\mathbf{C}^n$ defined as follows:

$$\sigma_l^A(a_1, \ldots, a_n) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n : \sum_{j=1}^{n} A(a_j - \lambda_j) \neq A\}.$$

The *right joint spectrum* $\sigma_r(a_1, \ldots, a_n)$ is similarly defined.

The *joint spectrum* or *Harte's spectrum* $\sigma(a_1, \ldots, a_n)$ is defined to be their union:

$$\sigma(a_1, \ldots, a_n) = \sigma_l(a_1, \ldots, a_n) \cup \sigma_r(a_1, \ldots, a_n).$$

Notice that for a single element $a \in A$ Harte's spectrum coincides with the usual spectrum of a.

The *left approximate point spectrum* of an n-tuple $(a_1, \ldots, a_n)$ of elements in a Banach algebra A, denoted by $\tau_l^A(a_1, \ldots, a_n)$ or simply by $\tau_l(a_1, \ldots, a_n)$, is defined to be the set

$$\tau_l^A(a_1, \ldots, a_n) = \{(\lambda_1, \ldots, \lambda_n) \in \mathbf{C}^n : \inf_{\|b\|=1} \sum_{j=1}^{n} \|(a_j - \lambda_j) b\| = 0\}.$$

The *right approximate point spectrum* $\tau_r(a_1, \ldots, a_n)$ can be defined in a similar manner. The union of the left and right approximate point spectra is denoted

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by $\tau(a_1, \ldots, a_n)$ and called the *joint approximate point spectrum*. (See [4] for the properties of all these joint spectra.)

A nonzero complex homomorphism of a Banach algebra is called a multiplicative functional in the sequel.

Let A be a Banach algebra. We begin with the following simple observation: If $A/\mathrm{rad}\,A$ is commutative, then $\sigma_l^A(a_1, \ldots, a_n) = \sigma_r^A(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$ and every $n = 1, 2, \ldots$, since they are the same as $\sigma^{A/\mathrm{rad}A}(a_1 + \mathrm{rad}\,A, \ldots, a_n + \mathrm{rad}\,A)$. The next theorem shows the converse is also true.

THEOREM. *Let A be a Banach algebra and let $\sigma_l^A(a_1, \ldots, a_n) \subset \sigma_r^A(a_1, \ldots, a_n)$ for all $a_1, \ldots, a_n \in A$ and all $n = 1, 2, \ldots$ Then $A/\mathrm{rad}\,A$ is commutative.*

Proof. Take a maximal left ideal M. Let MA be the right ideal generated by M, i.e., $MA = \{\sum_{j=1}^{k} m_j a_j\colon m_j \in M,\ a_j \in A,\ j = 1, \ldots, k;\ k = 1, 2, \ldots\}$. Then $M \subset MA$ since A is unital. We claim that $M = MA$. Suppose, on the contrary, $M \neq MA$. Then there exist $m_1, \ldots, m_n \in M$ and $a_1, \ldots, a_n \in A$ such that $a = m_1 a_1 + \ldots + m_n a_n \notin M$. Let M_1 be the left ideal generated by $M \cup \{a\}$. The maximality of M implies $M_1 = A$. So there exist $m \in M$ and $b \in A$ such that $1 = m + ba$. Hence $1 = m1 + \tilde{m}_1 a_1 + \ldots + \tilde{m}_n a_n$ where $\tilde{m}_j = bm_j \in M$, $j = 1, \ldots, n$. This means $(0, \ldots, 0) \notin \sigma_r(m, \tilde{m}_1, \ldots, \tilde{m}_n)$ and by the assumption there are $c_0, c_1, \ldots, c_n \in A$ such that $1 = c_0 m + c_1 \tilde{m}_1 + \ldots + c_n \tilde{m}_n$. Therefore, $1 \in M$, which is impossible since M is proper. Thus, $M = MA$, which means that it is a two-sided ideal. Moreover, M is closed since it is left maximal. Hence we may consider the quotient algebra A/M.

Let G be the set of all nonzero elements in A/M. The left maximality of M implies that every element in G has a left inverse and moreover, that G is a semigroup. Now we use the elementary fact that if G is a semigroup and every element in G is left invertible, then G is a group. Therefore A/M is a field. By Gelfand–Mazur's theorem A/M is isomorphic to $\mathbf{C}$. Thus

$$\mathrm{rad}\,A = \bigcap \{\text{the maximal left ideals of } A\} = \bigcap \{\text{the kernels of multiplicative functionals}\},$$

which implies that $A/\mathrm{rad}\,A$ is commutative. ∎

COROLLARY. *If $\sigma_l(a_1, \ldots, a_n) \subset \sigma_r(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in A^n$ and every $n = 1, 2, \ldots$, then $\sigma_l(a_1, \ldots, a_n) = \sigma_r(a_1, \ldots, a_n)$ for arbitrary $a_1, \ldots, a_n \in A$ and $n = 1, 2, \ldots$*

It would be interesting to know if the number of elements in the assumption of the above theorem can be bounded. Therefore, we pose the following:

PROBLEM 1. Does the Theorem remain true if $\sigma_l(a_1, \ldots, a_n) \subset \sigma_r(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in A^n$ with $1 \leqslant n \leqslant N$, $N > 1$ fixed?

We can give a positive answer to this problem only in a particular case:

PROPOSITION 1. *Let $\mathscr{A}$ be a von Neumann algebra. If $\sigma_l(a, b) \subset \sigma_r(a, b)$ for all $a, b \in \mathscr{A}$, then $\mathscr{A}$ is commutative.*

Proof. Suppose that $\mathscr{A}$ is noncommutative. Then $\mathscr{A}$ has a noncentral projection e. Let c_1 and c_2 be the central supports of e and $1-e$ respectively (i.e., c_1 is the smallest projection in the centre such that $e \leqslant c_1$, which means $ec_1 = c_1 e = e$). Then $c_1 c_2 \neq 0$, since otherwise $c_1 = e$ contradicting the fact that e is noncentral. By [2; Lemma 1, p. 217] there exist nonzero projections e_1, e_2 such that $e_1 \leqslant e$, $e_2 \leqslant 1-e$, and $e_1 \sim e_2$, i.e., $u^*u = e_1$ and $uu^* = e_2$ for some $u \in \mathscr{A}$. Replacing u by $v = e_2 u e_1$ if necessary, we may assume $e_2 u = ue_1 = u$.

Now $(1-e_1) + u^*u = 1$, which implies $\mathscr{A}(1-e_1) + \mathscr{A}u = \mathscr{A}$. But $(1-e_1)\mathscr{A} + u\mathscr{A} \neq \mathscr{A}$ since $u((1-e_1)\mathscr{A} + u\mathscr{A}) = (u - ue_1)\mathscr{A} + (ue_1 e_2 u)\mathscr{A} = \{0\}$. Thus $(0, 0) \notin \sigma_l(1-e_1, u)$ and $(0, 0) \in \sigma_r(1-e_1, u)$. Looking at $1-e_2$ and u we can see that $(0, 0) \notin \sigma_r(1-e_2, u)$ but $(0, 0) \in \sigma_l(1-e_2, u)$. ∎

In the remaining part of the paper we examine some connections between the approximate point spectra and the left and right spectra. We start with the following:

PROPOSITION 2. *Let A be a Banach algebra. If $\tau_l(a) \subset \tau_r(a)$ for every $a \in A$, then $\sigma_l(a) = \sigma_r(a)$ for all $a \in A$.*

Proof. The equality $\sigma_l(a) = \sigma_r(a)$ $(a \in A)$ is equivalent to the fact that whenever $ab = 1$, then $ba = 1$. So, assume $ab = 1$. Then $e = ba$ is an idempotent. We have $\|x\| = \|xab\| \leqslant \|xa\|\,\|b\|$, so $\|xa\| \geqslant \|b\|^{-1}\|x\|$ for all $x \in A$, which means $0 \notin \tau_r(a)$. By the assumption $0 \notin \tau_l(a)$ and so there exists $\delta > 0$ such that $\|ax\| \geqslant \delta\|x\|$ for all $x \in A$. But $ae = a(ba) = (ab)a = a$ and consequently $0 = \|a(1-e)\| \geqslant \delta\|1-e\|$, which implies $e = 1$. ∎

To show that the converse is not true we give

EXAMPLE 1 (cf. [5] or [6, Ex. 3.11, p. 33]). Let $\mathscr{S}$ be the free (noncommutative) semigroup with generators x, y, z, and with the empty word 1 as its unit element. Define a norm on $\mathscr{S}$ in the following way:

Put $\|1\|_{\mathscr{S}} = 1$. An arbitrary word $w \in \mathscr{S}$, $w \neq 1$, is a finite product of the "elementary" words x^n, y^m, z^p $(n, m, p \geqslant 1)$. The norm $\|w\|_{\mathscr{S}}$ is the product of factors corresponding to these elementary words according to the following rules:

– to x^n there corresponds the factor $1/n!$,
– to y^m one associates $1/(m+1)!$ unless the word w begins with y^m, in which case $1/m!$ corresponds to y^m,
– to z^p the factor $1/(p+1)!$ is attached unless w ends with z^p, in which case $1/p!$ corresponds to z^p.

Notice that $\|x^n\|_{\mathscr{S}} = 1/n!$, $\|y^m\|_{\mathscr{S}} = 1/m!$, $\|z^p\|_{\mathscr{S}} = 1/p!$, and, moreover, $\|w_1 w_2\|_{\mathscr{S}} \leqslant \|w_1\|_{\mathscr{S}}\|w_2\|_{\mathscr{S}}$ $(w_1, w_2 \in \mathscr{S})$. Let A be the l^1-algebra over $\mathscr{S}$, i.e.,

A consists of all sums $a = \sum_{w\in\mathscr{S}} \lambda_w w$ $(\lambda_w \in \mathbf{C})$ with $\|a\| = \sum_{w\in\mathscr{S}} |\lambda_w| \, \|w\|_{\mathscr{S}} < \infty$. Then A is a Banach algebra with generators x, y, z.

We claim that A has no idempotent except 0 and 1. Suppose $e \in A$ and $e^2 = e$. Let $e = \sum_{w\in\mathscr{S}} \lambda_w w$. Then

$$e^2 = \sum_{u,v\in\mathscr{S}} \lambda_u u \cdot \lambda_v v = \sum_{w\in\mathscr{S}} \Big(\sum_{uv=w} \lambda_u \lambda_v\Big) w$$

and so $\lambda_w = \sum_{uv=w} \lambda_u \lambda_v$ for all $w \in \mathscr{S}$.

If $w = 1$, then $\lambda_1 = \lambda_1^2$, which implies either $\lambda_1 = 1$ or $\lambda_1 = 0$. Further, if $w = x$, then $\lambda_x = \lambda_x \lambda_1 + \lambda_1 \lambda_x$. Hence, $\lambda_x(1-2\lambda_1) = 0$, which gives $\lambda_x = 0$. For the same reason $\lambda_y = 0$ and $\lambda_z = 0$. Now we proceed by induction with respect to the length of w.

Assume that $\lambda_u = 0$ for all words u with length less than n and $u \neq 1$. Let w be a word of length n. By the induction hypothesis,

$$\lambda_w = \sum_{uv=w} \lambda_u \lambda_v = \lambda_1 \lambda_w + \lambda_w \lambda_1.$$

This implies $\lambda_w(1-2\lambda_1) = 0$ and finally $\lambda_w = 0$. We conclude by induction that $\lambda_w = 0$ for all words $w \neq 1$. Hence either $e = 0$ or $e = 1$.

Now, suppose $ab = 1$. Then $e = ba$ is a nonzero idempotent, since $eb = b$. Therefore, $e = ba = 1$. So we have shown that $\sigma_l(a) = \sigma_r(a)$ for all $a \in A$.

Now look at $x+y$. We have $\|y^n\| = 1/n!$ and

$$\|(x+y)y^n\| \leqslant \|xy^n\| + \|y^{n+1}\| = \frac{2}{(n+1)!} = \frac{2}{n+1}\|y^n\|.$$

Thus, $0 \in \tau_l(x+y)$. On the other hand, if we write $a = \lambda + a_1 x + a_2 y + a_3 z$, then

$$\begin{aligned}\|a(x+y)\| &= \|\lambda x\| + \|\lambda y\| + \|a_1 x^2\| + \|a_1 xy\| + \|a_2 yx\| \\ &\quad + \|a_2 y^2\| + \|a_3 zx\| + \|a_3 zy\| \\ &\geqslant 2|\lambda| + \|a_1 xy\| + \|a_2 yx\| + \|a_3 zx\| \\ &= 2|\lambda| + \tfrac{1}{2}\|a_1 x\| + \|a_2 y\| + \|a_3 z\| \\ &\geqslant \tfrac{1}{2}\|\lambda + a_1 x + a_2 y + a_3 z\| = \tfrac{1}{2}\|a\|\end{aligned}$$

and so $0 \notin \tau_r(x+y)$. One can show in a similar way that $0 \notin \tau_l(x+z)$ but $0 \in \tau_r(x+z)$.

We do not know if $\tau_l(a_1, a_2) \subset \tau_r(a_1, a_2)$ for all $a_1, a_2 \in A$ implies that $\sigma_l(a_1, a_2) = \sigma_r(a_1, a_2)$. We do not even know whether the condition

$$(*) \qquad \tau_l(a_1, \ldots, a_n) \subset \tau_r(a_1, \ldots, a_n)$$

for all $(a_1, \ldots, a_n) \in A^n$ and every $n = 1, 2, \ldots$ implies that $\sigma_l(a_1, \ldots, a_n) = \sigma_r(a_1, \ldots, a_n)$ for all finite subsets $\{a_1, \ldots, a_n\}$ of A. Hence we pose

PROBLEM 2. Does condition $(*)$ imply that the algebra $A/\mathrm{rad}\,A$ is commutative?

The converse is not true. This can be seen by the following:

EXAMPLE 2. Take the following four matrices:

$$e = \begin{bmatrix} 0&1&0\\0&0&0\\0&0&0 \end{bmatrix}, \quad f = \begin{bmatrix} 1&0&0\\0&0&0\\1&0&0 \end{bmatrix}, \quad g = \begin{bmatrix} 0&1&0\\0&0&0\\0&1&0 \end{bmatrix}, \quad h = \begin{bmatrix} 0&0&0\\0&1&0\\0&0&0 \end{bmatrix}.$$

Then $e^2 = g^2 = ef = he = fh = hf = 0$, $f^2 = f$, $h^2 = h$, $eh = e$, $gh = g$, and $fe = g$.

Let A be the algebra generated by e, f, g, h, and the unit matrix 1. A typical element of A has the form

$$a = \lambda_0 + \lambda_1 e + \lambda_2 f + \lambda_3 g + \lambda_4 h = \begin{bmatrix} \lambda_0+\lambda_2 & \lambda_1+\lambda_3 & 0 \\ 0 & \lambda_0+\lambda_4 & 0 \\ \lambda_2 & \lambda_3 & \lambda_0 \end{bmatrix}.$$

It is easy to see that $A/\mathrm{rad}\,A$ is commutative. There are three multiplicative functionals on A:

$$\varphi(a) = \lambda_0, \qquad \psi(a) = \lambda_0 + \lambda_2, \qquad \chi(a) = \lambda_0 + \lambda_4.$$

Hence we have, for $a_1, \ldots, a_n \in A$,

$$\begin{aligned}\sigma_l(a_1, \ldots, a_n) &= \sigma_r(a_1, \ldots, a_n) = \sigma(a_1, \ldots, a_n) \\ &= \{(\varphi(a_1), \ldots, \varphi(a_n)), (\psi(a_1), \ldots, \psi(a_n)), (\chi(a_1), \ldots, \chi(a_n))\}.\end{aligned}$$

It is a matter of simple computations to show that $\tau_l(1+e, f) = \{(1, 0), (1, 1)\}$ while $\tau_r(1+e, f) = \{(1, 0)\}$ and moreover $\tau_l(1+e, h) = \{(1, 0)\}$ but $\tau_r(1+e, h) = \{(1, 1)\}$.

Remarks 1. It is easily seen (by looking at the elements $1+e, f$ and $1+e, h$) that for the algebra A of Example 2 the left and right joint approximate point spectra τ_l, τ_r do not have the projection property while their sum τ, being equal to the Harte spectrum σ, has this property, i.e., $P\tau(a_1, \ldots, a_{n+m}) = \tau(a_1, \ldots, a_n)$ where $P\colon \mathbf{C}^{n+m} \to \mathbf{C}^n$ is the canonical projection onto the first n coordinates (see [7] or [6, §3, pp. 34–40]).

2. If we take A to be the Banach algebra generated by e, f, g, and the unit, then still $A/\mathrm{rad}\,A$ is commutative and moreover $\tau_r(a_1, \ldots, a_n) \subset \tau_l(a_1, \ldots, a_n)$ for all $(a_1, \ldots, a_n) \in A^n$, $n = 1, 2, \ldots$ (see [7] or [6, Ex. 3.22, pp. 39–40]). But $\tau_r(1+e, f) = \{(1, 0)\} \neq \tau_l(1+e, f) = \{(1, 0), (1, 1)\}$. Thus the Corollary is not true if we replace the left and right joint spectra by the left and right approximate point spectra respectively.

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