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Harmonic, Gibbs and Hausdorff measures on repellers for holomorphic maps, II

by

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Abstract. We prove the hard part of the Refined Volume Lemma, postponed from Part I, leading to the following dichotomy:

For a simply connected domain $\Omega \subset \hat{\mathbb{C}}$ with the boundary $\partial\Omega$ preserved by a holomorphic map defined on its neighbourhood, repelling on the side of Ω, either $\partial\Omega$ is a real-analytic circle or interval or else a harmonic measure ω on $\partial\Omega$ viewed from Ω is singular with respect to the Hausdorff measure Λ_{Φ_c} with Makarov's function $\Phi_c(t) = t\exp\left(c\sqrt{\log(1/t)\log\log\log(1/t)}\right)$ for $c > c(\omega) = \sqrt{2\sigma^2/\chi} \neq 0$ ($\sigma^2 = \sigma^2(\omega)$ a certain asymptotic variance and χ a Lyapunov characteristic exponent) and ω is absolutely continuous for $c < c(\omega)$.

We also prove the above for $\partial\Omega$ a mixing piecewise repeller including the case of the limit set for a quasi-Fuchsian group, the boundary of the "snowflake" and more generally Carleson's fractal Jordan curves.

Finally, we study complex 1-parameter families of mixing repellers. In particular, if $\partial\Omega$ is the boundary of the basin of attraction to ∞ for the iteration of $z \mapsto z^2 + a$ we prove that $\sigma^2(\omega)$ is a subharmonic and real-analytic function of a, compute its quadratic part at $a = 0$ and estimate all other coefficients of the power series expansion with respect to a.

Contents

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6. Harmonic measure on fractal Jordan curves. Mixing piecewise repellers.
7. The dependence on a parameter.
8. (Appendix). Estimates of the asymptotic variance in the case of the boundary of the basin of attraction to ∞ for the iteration of $z \mapsto z^2+a$.

This paper (except the Appendix) is a revised version of a part of the Warwick preprint, July 1986 (referred to as [PUZ-preprint]). Part I appeared in Annals of Mathematics. To make this paper readable independently of Part I we recall in the Introduction the main theorems and necessary preliminaries.

Introduction. We study the relations of a harmonic measure ω on the boundary $\partial\Omega$ of a simply connected domain Ω in the Riemann sphere $\hat{\mathbf{C}}$ (Card $\hat{\mathbf{C}}\setminus\Omega > 1$), viewed from Ω, to various Hausdorff measures on $\partial\Omega$.

We recall the main definitions: Let $\Phi\colon \mathbf{R}_+ \to \mathbf{R}_+$ be an increasing continuous function, $\Phi(0) = 0$. We define an (outer) measure Λ_Φ for $A \subset \hat{\mathbf{C}}$ by

$$\Lambda_\Phi(A) = \lim_{\delta\to 0}\big(\inf\{\sum_j \Phi(\operatorname{diam} B_j)\}\big),$$

the infimum being taken over all countable coverings of A by discs B_j of diameter less than δ, the diameter taken with respect to the standard metric on $\hat{\mathbf{C}}$.

For every Borel set E its *Hausdorff dimension* is

$$\mathrm{HD}(E) = \inf\{\varkappa\colon \Lambda_{\Phi^{(\varkappa)}}(E) = 0\}, \quad \text{where} \quad \Phi^{(\varkappa)}(t) = t^\varkappa.$$

For every probability measure μ, $\mathrm{HD}(\mu) = \inf\{\mathrm{HD}(E)\colon \mu(E) = 1\}$.

If μ is invariant and ergodic, for a holomorphic mapping F from a neighbourhood in $\hat{\mathbf{C}}$ of the support of μ to $\hat{\mathbf{C}}$, we define the *characteristic Lyapunov exponent* by $\chi_\mu(F) = \int \log|F'|\,d\mu$. Also, $h_\mu(F)$ denotes measure-theoretic entropy. If $h_\mu(F) > 0$ then the crucial fact is: $\mathrm{HD}(\mu) = h_\mu(F)/\chi_\mu(F)$.

If μ is an invariant ergodic probability measure for a map F on a metric space X and ψ is a square integrable function on X with $\int\psi\,d\mu = 0$ and $\sum_{n=1}^\infty n\int\psi(\psi\circ F^n)\,d\mu < \infty$, then we define the asymptotic variance

$$\sigma_\mu^2(\psi, F) = \sigma_\mu^2(\psi) = \lim_{n\to\infty} n^{-1}\int\Big(\sum_{j=0}^{n-1}\psi\circ F^j\Big)^2 d\mu = \int\psi^2\,d\mu + 2\sum_{n=1}^{\infty}\int\psi(\psi\circ F^n)\,d\mu.$$

A domain Ω mentioned above is called an *RB-domain* (repelling boundary) if there exists a holomorphic map f defined on a neighbourhood U of $\partial\Omega$ such that $f(U\cap\Omega) \subset U\cap\Omega$, $f(\partial\Omega) = \partial\Omega$ and $\bigcap_{n=1}^\infty f^{-n}(U\cap\operatorname{cl}\Omega) = \partial\Omega$ (i.e. $\partial\Omega$ is repelling on the side of Ω). Then f is called an *RB-map*. If $R\colon \mathbf{D}\to\Omega$ is a Riemann map from the unit disc then $g = R^{-1}\circ f\circ R$ always extends holomorphically beyond $\partial\mathbf{D}$ and the last equality is equivalent to the expanding property for g (i.e. $|(g^n)'| > 1$, for n large enough). So g preserves a unique ergodic probability measure l' on $\partial\mathbf{D}$ equivalent to the normalized length measure l. The radial limit of R exists almost everywhere so we may consider the harmonic measure $\omega = R_*(l)$. Then $\omega' = R_*(l')$ is f-invariant and equivalent to ω.

The following is the main theorem of our theory:

THEOREM A. *Let Ω be an RB-domain in $\hat{\mathbf{C}}$. Then $\mathrm{HD}(\omega) = 1$ and for $c(\omega) = \sqrt{2\sigma_{\omega'}^2(\psi, g)/\chi}$, where $\psi = \log|f'\circ R| - \log|g'|$, $\chi = \chi_{l'}(g) = \chi_{\omega'}(f)$, we have the following possibilities:*

If $c(\omega) = 0$, then $\partial\Omega$ is an analytically embedded interval or a Jordan curve.

If $c(\omega) > 0$, then $\mathrm{HD}(\partial\Omega) > 1$ and

$$\text{(i)} \qquad \omega \perp \Lambda_{\Phi_c}, \quad \textit{for every } c > c(\omega),$$

$$\text{(ii)} \qquad \omega \ll \Lambda_{\Phi_c}, \quad \textit{for every } c < c(\omega) \quad (c > 0),$$

where $\Phi_c(t) = t\exp\big(c\sqrt{\log(1/t)\log\log\log(1/t)}\big)$ ($\perp$ means singular, $\ll$ means absolutely continuous).

Moreover, for l-a.e. $\zeta\in\partial\mathbf{D}$

$$\text{(iii)} \qquad G_R^\pm \equiv \limsup_{r\to 1} \pm\log|R'(r\zeta)| \Big/ \sqrt{\log\frac{1}{1-r}\log\log\log\frac{1}{1-r}} = c(\omega).$$

This theorem is a dynamics counterpart of general Makarov's theorem (see [Mk1, 2, 4, 5]). A part of Theorem A was proved in Part I. A final step of $c(\omega) = 0 \Rightarrow \partial\Omega$ is analytic was proved in [Z2] and $c(\omega) > 0 \Rightarrow \mathrm{HD}(\partial\Omega) > 1$ follows from [Z1]. More precisely, the case of $f|_{\partial\Omega}$ expanding was fully coped with in Part I. In the general case (i) was also proved in Part I (§4) but to have (ii) it remained to prove the upper part (ii) of the Refined Volume Lemma (we shall recall the statement below). This will be done in §5.

Consider $\Delta = (\delta_{ij})_{i,j=1}^J$, a 0, 1-matrix, which is transitive and aperiodic (all the entries of some power Δ^n are positive). Then

$$\Sigma(\Delta) = \{\alpha = (\alpha_n) \in \{1, \ldots, J\}^{\mathbf{Z}^+}\colon \delta_{\alpha_n\alpha_{n+1}} = 1 \text{ for every } n = 0, 1, \ldots\}$$

is called a *one-sided shift space*. By s we denote the shift on $\Sigma(\Delta)$ to the left, $(s(\alpha))_n = \alpha_{n+1}$. We consider the metric $\operatorname{dist}(\alpha, \hat{\alpha}) = \exp(-\inf\{n\colon \alpha_n \neq \hat{\alpha}_n\})$. The full shift space (when every $\delta_{ij} = 1$) is denoted by Σ^J. For the two-sided shift space (where $\mathbf{Z}^+$ is replaced by $\mathbf{Z}$) we use the notation $\tilde{\Sigma}(\Delta)$, $\tilde{\Sigma}^J$, $\tilde{s}$. Next, $\mathscr{P}\colon \tilde{\Sigma}(\Delta) \to \Sigma(\Delta)$ denotes the projection $\mathscr{P}((\alpha_n)_{-\infty}^\infty) = (\alpha_n)_0^\infty$. Finally, $\mathscr{A}$ is the partition of $\Sigma(\Delta)$ into cylinders $A_j = \{\alpha\colon \alpha_0 = j\}$ and

$$\mathscr{A}_k^n = \bigvee_{i=k}^n s^{-i}(\mathscr{A}) = \{\bigcap_{i=k}^n s^{-i}(A_{j_i})\colon A_{j_i}\in\mathscr{A}\}.$$

We add a tilde in the case of $\tilde{\Sigma}(\Delta)$, and write $\mathscr{A}^n$ for $\mathscr{A}_0^n$ and $\tilde{\mathscr{A}}^n$ for $\tilde{\mathscr{A}}_{-n}^n$.

Let φ be a Hölder continuous function on $\Sigma(\Delta)$. Then there exists a unique Gibbs measure (equilibrium state) μ_φ on $\Sigma(\Delta)$. It can be defined as a measure maximizing the functional $h_\mu(s) + \int\varphi\,d\mu$. The maximal value can be considered as the definition of the pressure $P(s, \varphi)$. Also, $\tilde{\mu}_\varphi$ denotes an $\tilde{s}$-invariant measure on $\tilde{\Sigma}(\Delta)$, the so-called natural extension of μ_φ.

There exists a "Jacobian" function $\mathscr{J}$ such that for every Borel set E on which s is injective, $\mu_\varphi(s(E)) = \int J\, d\mu_\varphi$. We write

$$\mathscr{J}(x) = \left(\frac{d(s_*(\mu_\varphi))}{d\mu_\varphi}(s(x))\right)^{-1}.$$

The function $-\log\mathscr{J}$ turns out to be Hölder continuous and homologous to $\varphi - P(s,\varphi)$ (there exists a Hölder continuous function h such that $-\log\mathscr{J} - \varphi + P(s,\varphi) = h\circ s - h$).

This implies for every $x\in A\in\mathscr{A}^n$

$$K^{-1} < \mu_\varphi(A)/\exp\Big(\sum_{j=0}^{n-1} G(s^j(x))\Big) < K \tag{1}$$

($K>0$ depending only on φ), for $G = -\log\mathscr{J}$ or $\varphi - P(s,\varphi)$.

For n large enough and every x we have

$$\prod_{j=0}^{n-1}\mathscr{J}(s^j(x)) > 1. \tag{2}$$

Now recall our coding which allows us to transport μ_φ from Σ^J to $\hat{\mathbf{C}}$. Let $f\colon U\to\hat{\mathbf{C}}$ be a holomorphic map defined on an open set $U\subset\hat{\mathbf{C}}$. Choose $z\in f(U)$ and curves $\gamma^1,\ldots,\gamma^d$ embedded into $f(U)$, joining z to some $d\leqslant 2$ preimages of z, provided they exist. For every $\alpha = (\alpha_n)_{n=0}^{\infty}\in\{1,\ldots,d\}^{\mathbf{Z}^+} = \Sigma^d$ we denote γ^{α_0} by $\gamma_0(\alpha)$ and the end point of $\gamma_0(\alpha)$ different from z by $z_0(\alpha)$. Assume that $\gamma_n(\alpha)$ and $z_n(\alpha)$ are already defined. Then define $\gamma_{n+1}(\alpha) = f_{\nu(\alpha)}^{-(n+1)}(\gamma^{\alpha_{n+1}})$, where $f_{\nu(\alpha)}^{-(n+1)}$ is the branch mapping z to $z_n(\alpha)$. We define $z_{n+1}(\alpha)$ as the end point of $\gamma_{n+1}(\alpha)$ different from $z_n(\alpha)$. (We assume that the above-mentioned branches exist.)

We call the graph with the vertices z and $z_n(\alpha)$ and edges $\gamma_n(\alpha)$ for all n and $\alpha\in\Sigma^d$ a *geometric coding tree* (g.c.t.) and denote it by $\mathscr{T}(z;\gamma^1,\ldots,\gamma^d)$. Every $\alpha\in\Sigma^d$ yields a subgraph $b(\alpha)$ called a *geometric branch*, composed of $z = z_{-1}(\alpha)$, $z_n(\alpha)$ and $\gamma_n(\alpha)$, $n\geqslant 0$. The subgraph composed of $z_{j-1}(\alpha)$, $\gamma_j(\alpha)$, $j\geqslant n$, is denoted by $b_n(\alpha)$. The branch $b(\alpha)$ is called *convergent* if $z_n(\alpha)$ is convergent in $\operatorname{cl} U$. We define the *coding map* $z_\infty\colon\mathscr{D}(z_\infty)\to\operatorname{cl} U$ as $z_\infty(\alpha) = \lim_{n\to\infty} z_n(\alpha)$ on the domain $\mathscr{D} = \mathscr{D}(z_\infty)$ of all α's for which $b(\alpha)$ is convergent.

It is sufficient for our aims to assume that the γ^j are smooth and

$$B\Big(\bigcup_{j=1}^{d}\gamma^j, Kn^{-2}\Big)\cap\big(f^n(\mathrm{Crit})\cup f^{n-1}(\mathrm{Asympt})\big) = \emptyset$$

for every $n\geqslant 0$ and a constant $K>0$, where $\mathrm{Crit} = \{x\in U\colon f'(x) = 0\}$,

$\mathrm{Asympt} = \{x\in U\colon \exists$ a continuous curve $\gamma\colon\mathbf{R}\to U$ such that $\gamma(t)\to\partial U$ and $f(\gamma(t))\to x$ as $t\to\infty\}$.

Such trees exist in the situations under consideration in Part I, namely for f an RB-map or a rational map on $\hat{\mathbf{C}}$.

It is crucial that the sets $\Sigma^d\setminus\mathscr{D}$ and $z_\infty^{-1}(x)$ for every x are "thin" (generalizing Beurling's classical theorem about radial limits of univalent functions). In particular, for every μ_φ considered above $\mu_\varphi(\Sigma^d\setminus\mathscr{D}) = 0$, which allows us to define the measure $(z_\infty)_*(\mu_\varphi)$ (denoted in the sequel by $h_{\mu_\varphi *}$) and $h_{\mu_\varphi}(s) = h_{\mu_\varphi *}(f)$ (provided f extends continuously to $z_\infty(\mathscr{D})$).

If f extends holomorphically to a neighbourhood of $z_\infty(\mathscr{D})$ we call $X = z_\infty(\mathscr{D})$ a *quasi-repeller*.

Now recall the above-mentioned

Refined Volume Lemma, RVL (Lemma 9, §4, Part I). *Let X be a quasi-repeller for f, let φ be a Hölder continuous function on Σ^d and $\psi = \varphi + \varkappa\log|f'\circ z_\infty| - P(s,\varphi)$, where $\varkappa = \mathrm{HD}(\mu_{\varphi *})$. Then for $c(\mu_{\varphi *}) = \sqrt{2\sigma^2_{\mu_\varphi}(\psi)/\chi_{\mu_{\varphi *}}(f)}$ we have for every c with $0 < c < c(\mu_{\varphi *})$ and $\mu_{\varphi *}$-a.e. $x\in X$*

$$\text{(i)}\qquad \limsup_{r\to\infty}\mu_{\varphi *}(B(x,r))/\big(r^\varkappa\exp\big(c\sqrt{\log(1/r)\log\log\log(1/r)}\big)\big) = \infty.$$

On the other hand, there exists a decomposition $\mu_{\varphi *} = \mu_1 + \mu_2$ into two measures, with $\mu_2(X) > 0$ arbitrarily small, such that if $c > c(\mu_{\varphi *})$ then for μ_1-a.e. $x\in X$

$$\text{(ii)}\qquad \limsup_{r\to\infty}\mu_1(B(x,r))/\big(r^\varkappa\exp\big(c\sqrt{\log(1/r)\log\log\log(1/r)}\big)\big) = 0.$$

Recall that ω' on $\partial\Omega$ in Theorem A is an example of $\mu_{\varphi *}$. Indeed, we can build a tree $\mathscr{T} = \mathscr{T}(z;\gamma^1,\ldots,\gamma^\alpha)$ in Ω close to $\partial\Omega$. Then $R^{-1}(\mathscr{T}) = \mathscr{T}(R^{-1}(z);R^{-1}(\gamma^1),\ldots,R^{-1}(\gamma^\alpha))$ is a tree in $\mathbf{D}$. We have $\omega' = \mu_{\varphi *}$ for $\varphi = -\log|g'|\circ(R^{-1}(z)_\infty)$.

Remark that $\mu_{\varphi *}$ is ergodic, hence for every $\varepsilon>0$, $x\in X$ and n large enough

$$\exp\big(n(\chi_{\mu_{\varphi *}}(f)-\varepsilon)\big)\leqslant|(f^n)'(x)|\leqslant\exp\big(n(\chi_{\mu_{\varphi *}}(f)+\varepsilon)\big). \tag{3}$$

An intermediate step between RVL and Theorem A is Theorem 6, §4, Part I. Here we shall refer only (in §§6–8) to its mixing repeller version, Theorem B. Recall that a compact set $X\subset\hat{\mathbf{C}}$ is a *mixing repeller* for f in Ruelle's sense if there exists a neighbourhood U of X such that $\bigcap_{n\geqslant 0} f^{-n}(U) = X$, for every open (in X) set $V\subset X$ there exists n for which $f^n(V) = X$ and $f|_X$ is expanding.

Theorem B. *Let $X\subset\hat{\mathbf{C}}$ be a mixing repeller for a holomorphic map f and let φ be a Hölder continuous function on X. Let μ_φ denote the Gibbs measure. Set $c(\mu_\varphi) = \sqrt{2\sigma^2_{\mu_\varphi}(\psi)/\chi_{\mu_\varphi}(f)}$, where $\psi = \varphi + \varkappa\log|f'| - P(f,\varphi)$, $\varkappa = \mathrm{HD}(\mu_\varphi)$. Then the following dichotomy takes place:*

If $c(\mu_\varphi) = 0$ then μ_φ is equivalent to $\Lambda_\varkappa$ and $\varkappa = \mathrm{HD}(X)$.

If $c(\mu_\varphi) > 0$ then for $\Phi_c^{(\varkappa)}(t) = t^\varkappa\exp\big(c\sqrt{\log(1/t)\log\log\log(1/t)}\big)$

$$\mu_\varphi \perp \Lambda_{\Phi_c^{(\varkappa)}} \quad \textit{for } c \leqslant c(\mu_\varphi),$$

$$\mu_\varphi \ll \Lambda_{\Phi_c^{(\varkappa)}} \quad \textit{for } c > c(\mu_\varphi),$$

In §6 we prove the above for mixing piecewise repellers. As an application we prove the following:

THEOREM C. *Let Ω be a simply connected domain in* $\mathbf{C}$ *with $\partial\Omega$ a Jordan curve. Let ∂_j, $j = 1, \ldots, J$, be a finite family of compact arcs in $\partial\Omega$ with pairwise disjoint interiors $\mathring{\partial}_j$. Denote $\bigcup \partial_j$ by ∂. (We do not assume that ∂ is connected.) Assume that there exists a family of conformal maps f_j (which may reverse the orientation on* $\mathbf{C}$) *on neighbourhoods U_j of ∂_j. For every j assume that $f_j(\Omega \cap U_j) \subset \Omega$, $|f_j'| > 1$ on U_j and*

$$(\#) \qquad f_j(\partial\Omega \cap U_j) \subset \partial\Omega.$$

Assume also the Markov partition property: the existence of a transitive and aperiodic matrix $\Delta = (\delta_{ij})_{i,j=1}^J$ with $\delta_{ij} = 0$ or 1 such that $f_i(\partial_i) = \bigcup_{j\in I_i} \partial_j$, where $I_i = \{j\colon \delta_{ij} = 1\}$.

Then there exists a transition parameter $c(\omega, \partial)$ such that for the harmonic measure ω restricted to ∂ the assertions of Theorem A: (i), (ii) (*for every $0 < c \leqslant c(\omega, \partial)$*) *and* (iii) *are satisfied. If $c(\omega, \partial) = 0$, then ∂ is a real-analytic curve.*

Of course dealing with a mixing (piecewise) repeller X one is not forced to use trees, one can define a Gibbs measure with the use of a coding $\pi\colon \Sigma(\Delta) \to X$ given by a Markov partition (cf. §1, Part I). One considers $\pi_*(\mu_\varphi)$ or $\pi_*(\mu_{\varphi\circ\pi})$ depending on whether φ is defined on $\Sigma(\Delta)$ or on X, and writes simply μ_φ. (Both ways of coding turn out to be useful, often they give the same.)

One example where the assumptions of Theorem C are satisfied with $\partial = \partial\Omega$ is of course any RB-domain, the Jordan expanding case. Another example is where $\partial\Omega$ is a quasi-circle invariant for the action on $\hat{\mathbf{C}}$ of a quasi-Fuchsian group (for a pair of isomorphic, compact surface, Fuchsian groups), as considered by R. Bowen [B2]. Theorem C is in fact motivated by this example. Also the proof is based on Bowen's ideas.

Other examples for which Theorem C yields $c(\omega) > 0$ are the celebrated von Koch "snowflake" and its self-similar generalizations described by L. Carleson [Ca2] or F. M. Dekking [De] (see [Fal] for more references). For these examples $\partial\Omega$ or almost all of $\partial\Omega$ can be covered by curves ∂ satisfying the assumptions of Theorem C.

Finally, in §6 we note that Sullivan's theorem [Su2], stating that for any two nonlinear mixing repellers for holomorphic maps in $\mathbf{C}$, a measurable conjugacy between Gibbs measures of maximal Hausdorff dimensions extends (up to a change on a set of measure 0) to a conformal one, holds in the case of mixing piecewise repellers.

In §7 we prove that for a complex 1-parameter family of mixing repellers X_λ with Gibbs measures μ_λ being images of the same μ_φ on $\Sigma(\Delta)$, provided $\mathrm{HD}(\mu_\lambda)$ is constant, $(c(\mu_\lambda))^2$ is a subharmonic and real-analytic function of λ. We estimate the coefficients of the power series expansion at any λ^0. We apply this to the harmonic measures ω_λ on $X_\lambda = \partial\Omega_\lambda$ for Ω_λ expanding RB-domains. We also consider the question of the existence of a holomorphic choice of Riemann maps $R_\lambda\colon \mathbf{D} \to \Omega_\lambda$ (cf. [PoRo]).

In §8 (Appendix) we make concrete estimates of the power series expansion coefficients and compute the quadratic part at $\lambda = 0$ for $(c(\omega_\lambda))^2$ for ω_λ viewed from the basin of attraction to ∞ for iteration of $z \mapsto z^2 + \lambda$. (We write there $z^2 + a$ leaving λ reserved to $z^2 + \lambda z$ according to Mandelbrot's notation. In fact we deal with σ_a^2 but $(c(\omega_a))^2 = 2\sigma_a^2/\chi$, $\chi = \log 2$.)

Remark on the notation. The letter K will be used to denote various positive constants, which may differ from one formula to the other.

Note. Sections 5, 6, 7 are independent of each other and may be read in any order.

5. Gibbs measures on quasi-repellers, continued

Remark 1. We shall rely in this section on the Law of Iterated Logarithm (LIL)

$$\limsup_{n\to\infty} S_n/\sqrt{n\log\log n} = \sqrt{2\sigma^2} \qquad \text{a.e.}$$

in the following situations:

1. For $S_n = \sum_{j=0}^{n-1} \psi\circ s^j$, on (Σ^d, μ_φ) where μ_φ, ψ and σ^2 are those from RVL (see Introduction).

2. For $S_n = \sum_{j=0}^{n-1} F_A \circ \tilde{s}^j$, on $(\tilde{\Sigma}^d, \tilde{\mu}_\varphi)$ with $\sigma^2 = \sigma_A^2$ where $F_A \equiv \chi_A - \tilde{\mu}_\varphi(A)$, $\chi_A \equiv 1$ on A, 0 outside, $\sigma_A^2 = \lim_{n\to\infty} n^{-1} \int \left|\sum_{j=0}^{n-1} F_A \circ \tilde{s}^j\right|^2 d\tilde{\mu}_\varphi$, and A is an arbitrary subset of $\tilde{\Sigma}^d$ with the following property: There exists a sequence of sets $A_n \in \sigma(\tilde{\mathscr{A}}^n_{-n})$ such that

$$(1) \qquad \tilde{\mu}_\varphi(A \div A_n) < \exp(n\delta_A) \quad \text{for a constant } \delta_A < 0.$$

Here $\sigma(\tilde{\mathscr{A}}^n_{-n})$ denotes the σ-algebra generated by $\tilde{\mathscr{A}}^n_{-n}$.

LIL in case 1 was proved in §4, Part I, Lemma 7. LIL in case 2 follows immediately from the LIL Theorem stated in §1, Part I.

Most of this section will be devoted to Lemma 1 which is a strong version of Lemma 8, §4, Part I. Recall that $\mathscr{P}$ is the standard projection from $\tilde{\Sigma}^d$ to Σ^d. For every $\alpha \in \tilde{\Sigma}^d$ such that $\mathscr{P}(\alpha) \in \mathscr{D}(z_\infty)$ set

$$x(\alpha) = z_\infty(\mathscr{P}(\alpha)), \qquad x_m(\alpha) = z_\infty(\mathscr{P}\tilde{s}^{-m}(\alpha)) \quad \text{for every } m \in \mathbf{Z}.$$

LEMMA 1. *In the situation of the Refined Volume Lemma* (see Introduction) *there exists a set $\Xi \subset \tilde{\Sigma}^d$ of $\tilde{\mu}_\varphi$ measure arbitrarily close to 1 and satisfying* (1), *with $\sigma_\Xi^2 > 0$ arbitrarily small, such that for a constant $R > 0$ for every $\alpha \in \Xi$, $m \leqslant 0$,*

(2) *there exists a branch $f_{\nu(\alpha)}^{-m}$ of f^{-m} on $B(x(\alpha), R)$ such that $f_{\nu(\alpha)}^{-m}(x(\alpha)) = x_{-m}(\alpha)$*

(*caution: this notation of branches is consistent with that of the Introduction or* § 3, *Part I, only up to $\mathscr{P}\mathscr{S}^{-m}$*);

$$|(f_{\nu(\alpha)}^{-m})'| \leqslant K \exp(-m\delta) \quad \text{on } B(x(\alpha), R) \tag{3}$$

for some constants K, δ independent of α;

$$\tfrac{1}{2} < |(f_{\nu(\alpha)}^{-m})'(x)/(f_{\nu(\alpha)}^{-m})'(y)| \leqslant 2 \quad \text{for every } x, y \in B(x(\alpha), R); \tag{4}$$

there exists a positive integer N (independent of $\alpha \in \Xi$) such that

$$b_N(\mathscr{P}(\alpha)) \subset B(x(\alpha), R/2) \tag{5}$$

(*the subgraphs b_n were defined in the Introduction and in* § 3, *Part I*).

Remark 2. Recall after § 4, Part I, that this lemma follows from Pesin's theory [Pe] except for the approximation condition (1) and the possibility to have σ_Ξ^2 arbitrarily small. We cannot get on without these properties of Ξ in the proof of RVL, the upper part (ii).

Proof of Lemma 1. We shall construct Ξ by consecutive removals of "bad" cylinders from $\tilde{\Sigma}^d$. As in (18), § 4, Part I, consider the set

$$\Xi_1 = \Xi_1(N) = \mathscr{P}^{-1}\big(\Sigma^d \setminus \bigcup_{n \geqslant N} \bigcup \mathscr{A}'_n\big) \tag{6}$$

where $\mathscr{A}'_n = \{A \in \mathscr{A}^n\colon \operatorname{diam} \gamma_n(\alpha) \geqslant \exp(-n\delta),\ \alpha \in A\}$ for a fixed $\delta > 0$.

Recall (§ 3, Part I) that $\operatorname{Card} \mathscr{A}'_n \leqslant K \exp(3n\delta)$.

Consider R satisfying the equality

$$R/2 = \exp(-N\delta)/(1 - \exp(-\delta)) \tag{7}$$

for some N to be specified later on.

By (1), (2), Introduction (or § 1, Part I) there exists $\beta < 1$ such that for every $A \in \mathscr{A}^n$, $\mu_\varphi(A) < K\beta^n$. Take δ such that

$$\beta \exp(6\delta) < 1. \tag{8}$$

Consider $\mathscr{A}'_n$ above with this δ. Then (7) immediately implies (5) for every $\alpha \in \Xi_1$. Also (1) is satisfied because

$$\mu_\varphi\big(\bigcup \{A \in \mathscr{A}^n\colon A \cap \mathscr{P}(\Xi_1) \neq \emptyset\} \setminus \mathscr{P}(\Xi_1)\big) \leqslant \sum_{j > n} \mu_\varphi\big(\bigcup \mathscr{A}'_j\big)$$
$$\leqslant \sum_{j > n} K\beta^j \exp(3j\delta) \to 0$$

exponentially fast as $n \to \infty$.

Now the idea will be to remove from every cylinder $A \in \tilde{\mathscr{A}}_0^N$ cylinders from $\tilde{\mathscr{A}}_{-n}^{-1}$ whose number grows exponentially very slowly, with the rate independent of the choice of A, as $n \to \infty$. This will ensure (1) and we shall remove just awkward cylinders to ensure (2)–(4) for the α's not removed. We shall use an idea from [FLM]. Some technical difficulties to overcome are caused by the fact that f is defined only on a neighbourhood of X rather than on all of the Riemann sphere (cf. Remark 3).

First restrict f to a smaller neighbourhood of X than the original domain so that we can assume that f is defined on a neighbourhood U of X and extends holomorphically to a neighbourhood of $\operatorname{cl} U$. So for every $r > 0$ there exists $L_r > 0$ such that for every $x \in X$, $r' \geqslant r$ and every connected component V of the set $f^{-1}(B(x, r))$ we have

$$\operatorname{diam} V \leqslant L_r r'. \tag{9}$$

Moreover,

$$L_r r \to 0 \quad \text{as } r \to 0. \tag{10}$$

Fix ε_0, ε_1, ε_2 such that

$$B(X, \varepsilon_0) \subset U, \quad \varepsilon_1 L_{\varepsilon_1} < \varepsilon_0, \quad \varepsilon_2 < (\varepsilon_1 K(\delta)^3/1000)^2 \tag{11}$$

(the constant $K(\delta)$ will be defined by (20) later on).

Fix some positive integer M, to be specified later on, depending only on structural constants related with f and φ, on the chosen δ and on an arbitrarily chosen constant with which we want to bound $1 - \mu_\varphi(\Xi)$. Set

$$\zeta_M = \bigcup_{n=1}^{M+1} f^n(\operatorname{Crit}(f)). \tag{12}$$

Since f extends holomorphically beyond $\operatorname{cl} U$, the set $\operatorname{Crit}(f)$ is finite. We have

$$\operatorname{Card} \zeta_M \leqslant (M+1) \operatorname{Card}(\operatorname{Crit}(f)) < \infty.$$

From now on we shall work for some time with

$$\hat{R} = 100R. \tag{13}$$

(The coefficient 100 is chosen here to yield the distortion in (4) bounded by 2. Demanded an arbitrary bound > 1 in (4), one must modify this coefficient accordingly.) In view of $\mu_{\varphi *}(B(x, r)) \leqslant Kr^\tau$ for a constant $\tau > 0$ and every $x \in X$, $r > 0$ (see Lemma 4, § 4, Part I) we get

$$\mu_{\varphi *}(B(\zeta_M, \tfrac{5}{2}\hat{R})) \leqslant (\operatorname{Card} \zeta_M) K \hat{R}^\tau \leqslant \hat{R}^{\tau/2} \tag{14}$$

if, say,

$$\hat{R} \leqslant \exp(-M\delta) \tag{15}$$

and M is sufficiently large. (Just to know that $\mu_{\varphi *}(B(\zeta_M, \frac{5}{2}\hat{R}))$ can be arbitrarily small if $\hat{R}$ is small enough one could again refer to "thinness" of the z_∞-preimages of points (see Introduction or (6), § 3, Part I). The exact estimates (14), (15) will, however, be needed to estimate σ_Ξ^2).

For every $\alpha\in\Xi_1$ if $z_\infty(\mathscr{P}\alpha)\notin B(\zeta_M,\frac{5}{2}\hat{R})$, then by (5), $z_N(\mathscr{P}\alpha)\notin B(\zeta_M,2\hat{R})$. So

$$\Xi_2=\{\alpha\in\Xi_1:\ \operatorname{dist}(z_N(\alpha),\zeta_M)\geqslant 2\hat{R}\}\tag{16}$$

has μ_φ measure close to 1, since Ξ_1 has had. (We have removed from Ξ_1 some cylinders from $\tilde{\mathscr{A}}_0^N$, intersecting it.)

Let us now specify $\hat{R}$ (and R), as promised after (7), as satisfying (15) and additionally small enough that for every $\alpha\in\Xi_2$, $k=1,\ldots,M$,

$$f^{-k}_{\nu(\alpha)}\big(B(z_N(\mathscr{P}\alpha),2\hat{R})\big)\subset B(X,\varepsilon_1).\tag{17}$$

This is possible due to (10). (Later on to estimate σ_Ξ^2 we shall need an estimate for $\hat{R}$ from below as well; now we do not bother about it.)

Fix for some time a cylinder $A\in\tilde{\mathscr{A}}_0^N$ containing $\alpha=\alpha(A)\in\Xi_2$. We shall define by induction on $n\geqslant M$ a sequence $\mathscr{A}(A,n)\subset\tilde{\mathscr{A}}_{-n}^{-1}$. (These will be families of cylinders not to be removed, in other words, "good" backward branches related to A.) All the time when A is fixed we shall denote $\mathscr{A}(A,n)$ by $\mathscr{A}(n)$.

Define $\mathscr{A}(M)=\tilde{\mathscr{A}}_{-M}^{-1}$. Suppose that for some $k\geqslant M$, $\mathscr{A}(k)$ is already defined and for every $\hat{A}\in\mathscr{A}(k)$ and every $\hat{\alpha}\in\hat{A}\cap A$ with $\mathscr{P}(\hat{\alpha})=\mathscr{P}(\alpha)$, the following properties (18) and (21) are satisfied:

(18) $f^{-k}_{\nu(\hat{\alpha})}$ is well-defined on the ball $B(z_N(\mathscr{P}\alpha),R_k)$, where

$$R_k=2\hat{R}\prod_{j=0}^{k-M}(1-K(\delta)\exp(-j\delta)),\tag{19}$$

$$K(\delta)=\tfrac{1}{4}(1-\exp(-\delta));\tag{20}$$

and

(21) $f^{-k}_{\nu(\hat{\alpha})}\big(B(z_N(\mathscr{P}\alpha),R_k)\big)\subset B(X,\varepsilon_1)$ and contains no critical value of f.

Observe that by (11), f_ν^{-1} is well-defined on $f^{-k}_{\nu(\hat{\alpha})}\big(B(z_N(\mathscr{P}\alpha),R_k)\big)$ and maps it into $B(X,\varepsilon_0)$ for every branch f_ν^{-1} involved in the definition of $\mathscr{T}$.

Define now $\mathscr{A}(k+1)$. This will be the set of all cylinders $\tilde{A}\in\tilde{\mathscr{A}}_{-(k+1)}^{-1}$ such that $\hat{A}\subset\bigcup\mathscr{A}(k)$ and that for every $\hat{\alpha}\in\hat{A}$ with $\mathscr{P}(\hat{\alpha})=\mathscr{P}(\alpha)$ the following conditions (22) and (23) are satisfied:

$$l_2\Big(f^{-(k+1)}_{\nu(\hat{\alpha})}\big(B(z_N(\mathscr{P}\alpha),\hat{R})\big)\Big)\leqslant\varepsilon_2\exp(-6\delta(k+1-M))\tag{22}$$

(this makes sense because $\hat{R}<R_n$ for every n; l_2 is the 2-dimensional Riemann measure on $\hat{\mathbf{C}}$);

(23) $f^{-(k+1)}_{\nu(\hat{\alpha})}\big(B(z_N(\mathscr{P}\alpha),R_k)\big)$ contains no critical value of f.

We need to prove (21) for $k+1$. By (22) for every $\hat{\alpha}\in\hat{A}\in\mathscr{A}(k+1)$ there exists $x\in B(z_N(\mathscr{P}\alpha),\hat{R})$ such that $|(f^{-(k+1)}_{\nu(\hat{\alpha})})'(x)|^2\leqslant\varepsilon_2\hat{R}^{-2}\exp(-6\delta(k+1-M))$ hence

$$|(f^{-(k+1)}_{\nu(\hat{\alpha})})'(x)|\leqslant\varepsilon_1K(\delta)^3(1000\hat{R})^{-1}\exp(-3\delta(k+1-M)).\tag{24}$$

By (19) and (20), $R_k\geqslant\frac{3}{2}\hat{R}$. The Distortion Theorem of Koebe (see [Hi], Theorem 17.4.6) applied to univalent functions on the disc $B(z_N(\mathscr{P}\alpha),R_k)$ gives, with the use of (19), the following estimate for every $\hat{\alpha}\in\hat{A}\in\mathscr{A}(k+1)$, $y\in B(z_N(\mathscr{P}\alpha),R_{k+1})$:

$$\begin{aligned}|(f^{-(k+1)}_{\nu(\hat{\alpha})})'(y)/(f^{-(k+1)}_{\nu(\hat{\alpha})})'(x)|&\leqslant\frac{2}{(1-R_{k+1}/R_k)^3}\cdot\frac{8}{1-\hat{R}/R_k}\\&\leqslant 48K(\delta)^{-3}\exp(3(k+1-M)\delta).\end{aligned}\tag{25}$$

This and (24) yield for $y\in B(z_N(\mathscr{P}\alpha),R_{k+1})$

$$|(f^{-(k+1)}_{\nu(\hat{\alpha})})'(y)|\leqslant\varepsilon_1/(3\hat{R}),\tag{26}$$

hence (21) for $k+1$. (We needed more in the denominator of the right-hand side of (26) than merely $2\hat{R}$, or more exactly R_{k+1}, since there was no reason for $f^{-(k+1)}_{\nu(\hat{\alpha})}(z_N(\mathscr{P}\alpha))$ to be in X. We only knew that $z_\infty(\mathscr{P}\alpha)\in X$; hence $f^{-(k+1)}(z_\infty(\mathscr{P}\alpha))\in X$ and we have used $z_\infty(\mathscr{P}\alpha)\in B(z_N(\mathscr{P}\alpha),R/2)$.)

The above distortion consideration is formally correct if U, the domain of f, is in $\mathbf{C}$ and the derivative is in the sense of the Euclidean metric on $\mathbf{C}$. However (in case we need to consider the whole Riemann sphere), the sets we consider have small diameters so the factor by which (25) changes under changes of metrics is close to 1.

Defining $\mathscr{A}(A,k+1)=\mathscr{A}(k+1)$ for every $k>M$ we have removed from $\tilde{\mathscr{A}}_{-(k+1)}^{-1}$ a finite number of cylinders not satisfying (23). This number is bounded independently of a cylinder A and number k by the number of critical values of f, since any two sets like in (23), for any two different cylinders $\hat{A}$, $\hat{A}'$ with $\hat{\alpha}\in\hat{A}$, $\mathscr{P}(\hat{\alpha})=\mathscr{P}(\alpha')=\mathscr{P}(\alpha)$ and $\hat{\alpha}'\in\hat{A}'$, are disjoint. Also by disjointness we remove at most

$$Q_k=(l_2(U)/\varepsilon_2)\exp(6\delta(k+1-M))\tag{27}$$

cylinders not satisfying (22).

By (1), (2), Introduction, for every $A'\in\tilde{\mathscr{A}}_{-k}^{-1}$, $A''\in\tilde{\mathscr{A}}_0^l$ and every $k,l\geqslant 1$ we have

$$\tilde{\mu}_\varphi(A'\cap A'')\leqslant K\beta^k\tilde{\mu}_\varphi(A'')$$

(β the same as in the line preceding (8)). Thus, for every $A\in\tilde{\mathscr{A}}_0^N$ intersecting Ξ_2 we have

$$\tilde{\mu}_\varphi\big(A\cap(\bigcup\mathscr{A}(k)\setminus\bigcup\mathscr{A}(k+1))\big)\leqslant 2K\beta^{k+1}Q_{k+1}\tilde{\mu}_\varphi(A).\tag{28}$$

(The factor 2 stands here because of the removal of cylinders containing critical values.)

Let us define

$$\Xi=\Xi_2\cap\bigcup_A\bigcap_{k>M}\big((\bigcup\mathscr{A}(A,k))\cap A\big).\tag{29}$$

We obtain

$$(30)\qquad \tilde{\mu}_\varphi(\Xi) \geqslant \tilde{\mu}(\Xi_2) - \sum_{n>M} 2K\beta^n Q_n.$$

This is close to 1 because if M is large, then $\sum_{n>M}\beta^n Q_n$ is small (see (8)).

Finally, for every $n_1 \geqslant M$, $n_2 \geqslant N$

$$(31)\quad \tilde{\mu}_\varphi\big(\bigcup\{\tilde{A}\in\mathscr{A}^{n_2}_{-n_1}\colon\ \tilde{A}\cap\Xi\neq\emptyset\}\setminus\Xi\big) \leqslant \sum_{k>n_2} K\beta^k\exp(3k\delta) + \sum_{k>n_1} 2K\beta^k Q_k,$$

so we obtain also the property (1) of fast approximation by cylinders for Ξ.

For fixed $\alpha\in\Xi$, from (5) we obtain $B(x(\alpha),R)) \subset B\big(z_N(\mathscr{P}\alpha),\tfrac{3}{2}R\big)$. As $\tfrac{3}{2}R < \hat{R} < \tfrac{3}{2}\hat{R} < R_k$ for every $k \geqslant M$, by the definition of Ξ we obtain (2). By Koebe's Distortion Theorem for every $x, y\in B\big(z_N(\mathscr{P}\alpha),\tfrac{3}{2}R\big)$ we have the estimate

$$|(f^{-m}_{\nu(\alpha)})'(x)/(f^{-m}_{\nu(\alpha)})'(y)| \leqslant \big(1+\tfrac{3}{2}R/(\tfrac{3}{2}\hat{R})\big)^4/\big(1-\tfrac{3}{2}R/(\tfrac{3}{2}\hat{R})\big)^4,$$

which together with (13) gives (4).

Meanwhile for $x, y\in B\big(z_N(\mathscr{P}\alpha),\hat{R}\big)$ we have the estimate by $(1+\tfrac{2}{3})^4/(1-\tfrac{2}{3})^4 < 1000$. This and (24) give (3). The proof of Lemma 1 except for the estimate of σ^2_Ξ is finished. (For a suspicious reader we summarize the order of choices: the constants ε_0, ε_1, ε_2, β, τ and δ depend only of f and φ, then an arbitrary number with which we want to bound $1-\tilde{\mu}_\varphi(\Xi)$ is chosen, then we choose consecutively the numbers M, R and N.)

SUBLEMMA 1. *There exists Ξ as constructed above with $\tilde{\mu}_\varphi(X\setminus\Xi)$ and σ^2_Ξ arbitrarily small* (>0).

Proof. *Step* 1. We shall prove that for a constant T arbitrarily large, for M large enough (depending on T) there exist N, R and $\hat{R}$ related with each other by (7), (13) satisfying (15), (17) and on the other hand satisfying

$$(32)\qquad M \geqslant T\log N.$$

Denote by d_1 the number of critical points of f in $\mathrm{cl}X$, by d_2 the maximal multiplicity of critical points plus 2. No trajectory $(f^n(x))$ in X hits the set of critical points for more than d_1 times. Otherwise it would contain a periodic sink so (see (7), §3, Part I) it would not belong to $z_\infty(\mathscr{D})$, a contradiction. So given T, there exists $\varepsilon_3 > 0$ such that $\varepsilon_3 \leqslant \varepsilon_1$ (see (11)) and for every $x\in U$

(33) $f^j\big(B(x,\varepsilon_3)\big)\cap B\big(\mathrm{Crit}(f)\cap\mathrm{cl}X,\varepsilon_3\big)\neq\emptyset$ happens for at most d_1 integers j from $\{j\colon 1\leqslant j\leqslant T.\}$.

We can assume that there are no f-critical points in $B(X,2\varepsilon_3)\setminus\mathrm{cl}X$. Let

$$\lambda = \big(\inf\{|f'(x)|\colon\ x\in B(X,\varepsilon_3)\setminus B\big(\mathrm{Crit}(f)\cap\mathrm{cl}X,\varepsilon_3\big)\}\big)^{-1}.$$

Let $\xi > 1$ be defined by the equality

$$(34)\qquad T-2d_1 = \log d_2/\log\xi.$$

Choose a constant ε_4 with $0<\varepsilon_4\leqslant\varepsilon_3$ such that $\lambda r \leqslant r^{\xi-1}$ for every $r\leqslant\varepsilon_4$. Thus, to have (17) satisfied, it is enough to have for $\hat{R}$ the estimate

$$(35)\qquad (4\hat{R})^L \leqslant \varepsilon_4 \quad\text{where}\quad L = d_2^{-Md_1/T}\,\xi^{-M(1-d_1/T)}.$$

This is so because for $\alpha\in\Xi_2$, in (17)

$$(36)\qquad \operatorname{diam} f^{-(k+1)}_{\nu(\alpha)}\big(B\big(z_N(\mathscr{P}\alpha),2\hat{R}\big)\big) \leqslant \big(\operatorname{diam} f^{-k}_{\nu(\alpha)} B\big(z_N(\mathscr{P}\alpha),2\hat{R}\big)\big)^a$$

where $a = d_2^{-1}$ for at most $2d_1/T$ per cent of time (see (33)) and $a=\xi^{-1}$ for the rest of time and because the sets whose diameter we measure intersect X. (A precise estimate would give the right-hand side of (36) multiplied by a constant but the exponent would be $a=(d_2-1)^{-1}$ rather than d_2^{-1}. Replacing d_2-1 by d_2 allows us to omit this constant if ε_4 is small enough. We put the percentage of time to be $2d_1/T$ rather than d_1/T because with the latter (36) would be true only for M a multiple of T.)

The inequality (35) is equivalent (due to (34)) to

$$\big(-M(2d_1/T)-M(1-2d_1/T)(T-2d_1)^{-1}\big)\log d_2 \geqslant \log(\log\varepsilon_4/\log 4\hat{R}),$$

then to

$$(37)\qquad M \leqslant -\frac{T}{2d_1+1}\cdot\frac{\log\log\varepsilon_4^{-1}}{\log d_2} + \frac{T\log\log(4\hat{R})^{-1}}{(2d_1+1)\log d_2}.$$

Thus, to have (15), (17) and (32) satisfied, it is enough (in view of (7) and (13)) to take M equal to the right-hand side of (37). (Of course T in (32) is not the same as in (37) but it is also large.)

Step 2. By (31) we have already $\tilde{\mu}_\varphi(\Delta^n_\Xi) \leqslant \beta_1^n$ for $n\geqslant N$, where $\Delta^n_\Xi = \bigcup\{\hat{A}\in\mathscr{A}^n_{-n}\colon\ \hat{A}\cap\Xi\neq\emptyset\}\setminus\Xi$, $\beta_1 < 1$ is a constant, N is large enough.

We shall now estimate $\tilde{\mu}_\varphi(\Delta^n_\Xi)$ for $n<N$. Unfortunately we do not know about any part of $\tilde{\Sigma}^d\setminus\Xi$ to belong to $\mathscr{A}^{N+1}_{-(N+1)}$. (Recall that we removed the sets $\hat{A}\cap A\in\mathscr{A}^N_{-(j+1)}$ for $j\geqslant M$, for. $\hat{A}\in\mathscr{A}(A,j+1)$ depending on A.) So to cope with $n<N$ we cannot help but estimate the whole $\tilde{\mu}_\varphi(\tilde{\Sigma}^d\setminus\Xi)$. By the estimate of Card $\mathscr{A}'_n$, (14) and (30), we have

$$\tilde{\mu}_\varphi(\tilde{\Sigma}^d\setminus\Xi) \leqslant \sum_{j\geqslant N} K\beta^j\exp(3j\delta) + \hat{R}^{\tau/2} + \sum_{j>M} 2K\beta^j Q_j.$$

The first two summands decrease exponentially fast with the growth of N. To estimate the last one we use (32) and obtain

$$K\sum\beta^j Q_j \leqslant K\beta^M \leqslant K\beta^{T\log N} = KN^{T\log\beta} \leqslant N^{-8}$$

if $T > 8/\log\beta^{-1}$ and N is large enough. It is convenient to write down the estimates for $n\geqslant N$ and $n<N$ in a joint formula

$$(38)\qquad \tilde{\mu}_\varphi(\Delta^n_\Xi) \leqslant \big(\max(N,n)\big)^{-8}.$$

Step 3. Having the uniform (independent of Ξ) estimate (38) we can estimate σ_{Ξ}^2. Suppose that $\tilde{\mu}_\varphi(\tilde{\Sigma}\setminus\Xi) < \varepsilon$. Recall that

$$\sigma_{\Xi}^2 = \int (F_\Xi)^2\, d\tilde{\mu}_\varphi + 2\sum_{n=1}^{\infty} \eta_n$$

where $\eta_n = \int F_\Xi (F_\Xi \circ \tilde{s}^n)\, d\tilde{\mu}_\varphi$, F_Ξ defined in Remark 1.2. A straightforward computation (which we omit) and the use of (38) give

$$\int |F_\Xi|\, d\tilde{\mu}_\varphi \leqslant 2\varepsilon, \qquad \int (F_\Xi)^2\, d\tilde{\mu}_\varphi \leqslant 2\varepsilon, \tag{39}$$
$$\int \big(E(F_\Xi \,|\, \tilde{\mathscr{A}}^n_{-n}) - F_\Xi\big)^2\, d\tilde{\mu}_\varphi \leqslant 3(\max(N,n))^{-4}.$$

To estimate η_n decompose F_Ξ in a standard way (see [B1] for example):

$$F_\Xi = \big(F_\Xi - E(F_\Xi \,|\, \tilde{\mathscr{A}}^{n/3}_{-n/3})\big) + E(F_\Xi \,|\, \tilde{\mathscr{A}}^{n/3}_{-n/3}). \tag{40}$$

We need also the following (cf. [B1], 1.25). There exists $\lambda < 1$ such that for any functions F and G measurable with respect to $\tilde{\mathscr{A}}^r_p$, $\tilde{\mathscr{A}}^t_s$ respectively, $p \leqslant r < s \leqslant t$, we have

$$\Big|\int FG\, d\tilde{\mu}_\varphi - \int F\, d\tilde{\mu}_\varphi \int G\, d\tilde{\mu}_\varphi\Big| < K\lambda^{s-r} \int |F|\, d\tilde{\mu}_\varphi \int |G|\, d\tilde{\mu}_\varphi.$$

The inequalities (39), the inequality above and the Hölder inequality give

$$\eta_n \leqslant 1000\varepsilon^{1/2}(\max(N,n))^{-2} + K\lambda^{n/3}\varepsilon^2 + 1000(\max(N,n))^{-4}.$$

The conclusion is that $\sigma_{\Xi}^2 \leqslant K\varepsilon^{1/2} + KN^{-3}$. This proves the Sublemma (hence Lemma 1) because ε can be arbitrarily small and N arbitrarily large. ∎

Remark 3. The proof of Lemma 1 would be much simpler if we assumed that f extends to a holomorphic (hence rational) function on $\hat{\mathbb{C}}$. An arbitrary M satisfying (15), in particular $M = N$ would fit to the construction of Ξ. One still must remove "bad" branches f_ν^{-n} for each $A \in \mathscr{A}_0^N$, $n > M$, as in [FLM], but the construction with the use of the sequence R_k is redundant. In the Sublemma Step 1 is also redundant and the approximation (38) turns out to be exponential.

Proof of the Refined Volume Lemma, the upper part (ii). Let $\Xi \subsetneq \tilde{\Sigma}^d$, N, R, δ be sets and numbers provided by Lemma 1. Consider any $\alpha \in \tilde{\Sigma}^d$ such that the LIL for the sequence of random variables $F_\Xi \circ \tilde{s}^n$ at α holds (see Remark 1.1). So if $j = m$ and $j = m'$ are two consecutive times when $\tilde{s}^j(\alpha) \in \Xi$ and say $m < m'$ we have

$$\sum_{j=0}^{m'-1} F_\Xi(\tilde{s}^j(\alpha)) = \Big(\sum_{j=0}^{m} F_\Xi(\tilde{s}^j(\alpha))\Big) - (m'-m-1)\tilde{\mu}_\varphi(\Xi)$$

and, if m and m' are large enough,

$$\sum_{j=0}^{m} F_\Xi(\tilde{s}^j(\alpha)) \leqslant 2\sqrt{\sigma_\Xi^2 m \log\log m}, \qquad \sum_{j=0}^{m'-1} F_\Xi(\tilde{s}^j(\alpha)) \geqslant -2\sqrt{\sigma_\Xi^2 m \log\log m}.$$

The conclusion is that if say $\tilde{\mu}_\varphi(\Xi) > 4/5$ and m, m' are large enough, then

$$m' - m \leqslant 5\sqrt{\sigma_\Xi^2 m \log\log m}. \tag{41}$$

Assume that $\sigma^2 = \sigma^2_{\mu_\varphi}(\psi) > 0$. The case $\sigma^2 = 0$ can be dealt with similarly. Only σ^2 in the estimates should be replaced by $\eta > 0$ arbitrarily small. This concerns also the estimates above with σ_Ξ^2. Given an arbitrary $\varepsilon > 0$ there exists a set $Q \subset \Sigma^d$ ($Q \subset \mathscr{D}z_\infty$), of μ_φ measure arbitrarily close to 1, on which the upper part of the LIL in Remark 1.1 holds uniformly, namely there exists $N_1 > 0$ such that for every $\alpha \in Q$ and $n \geqslant N_1$

$$\Big|\sum_{j=0}^{n-1} \psi(s^j(\alpha))\Big| \Big/ \sqrt{n \log\log n} \leqslant (1+\varepsilon)\sqrt{2\sigma^2_{\mu_\varphi}(\psi)}. \tag{42}$$

Denote $\tilde{\mu}_\varphi$ restricted to $\mathscr{P}^{-1}(Q)$ by $\tilde{\mu}$, μ_φ restricted to Q by μ and $(z_\infty)_*(\mu) = (z_\infty\mathscr{P})_*(\tilde{\mu})$ by μ_1. This μ_1 will be the measure standing in the assertion of the Refined Volume Lemma.

Take an arbitrary $\alpha \in \mathscr{P}^{-1}(Q)$ for which the LIL for $F_\Xi \circ \tilde{s}^n$ and (3), Introduction (or (23), § 1, Part I) for $x = z_\infty\mathscr{P}(\alpha)$ hold. Take any small $r > 0$. Let $n(r) \geqslant 0$ be the largest integer such that, for every j with $0 \leqslant j \leqslant n(r)$,

$$|(f^j)'(x(\alpha))| \leqslant R/(2r). \tag{43}$$

Let $m(r)$ be the largest integer less than or equal to $n(r)$ such that $\tilde{s}^{m(r)}(\alpha) \in \Xi$. Set

$$B_{m(r)} = f^{-m(r)}_{\nu(\tilde{s}^{m(r)}(\alpha))} B(x_{m(r)}(\alpha), R).$$

We have by (43) and (4)

$$B_{m(r)} \supset B(x(\alpha), r). \tag{44}$$

The consideration which follows the inequality (2) in the proof of Lemma 4 in § 4, Part I, implies that $s^{m(r)}$ on $B'_m(r) = z_\infty^{-1}(B_{m(r)}) \cap Q$ is injective. This is so because $f^{m(r)}$ is injective on $B_{m(r)}$ and has no critical points there. Hence for $\mathscr{J}$ the Jacobian (see Introduction or (6), § 1, Part I) we have

$$\mu_\varphi(s^{m(r)}(E)) = \int_E \prod_{j=0}^{m(r)-1} \mathscr{J}(s^j(\hat{\alpha}))\, d\mu_\varphi(\hat{\alpha}).$$

So, as $-\log \mathscr{J}$ is homologous to $\varphi - P(s,\varphi)$ in bounded functions (cf. Introduction or Remark 5 in § 1, Part I), we obtain

$$\mu_1(B_{m(r)}) = \mu_\varphi(B'_{m(r)}) \leqslant K \sup_{\hat{\alpha} \in B_{m(r)}} \exp\Big(\sum_{k=0}^{m(r)-1} \varphi(s^k(\hat{\alpha})) - m(r)P(s,\varphi)\Big)$$

and by the definition of ψ, by (42) and by (4),

$$\log \mu_1(B_{m(r)}) \leqslant -\varkappa \log |(f^{m(r)})'(x(\alpha))| \tag{45}$$
$$+ (1+\varepsilon)\sqrt{2\sigma^2_{\mu_\varphi}(\psi)\, m(r) \log\log m(r)} + \log K.$$

With the use of (44), the inequality opposite to (43) for $j = n(r)+1$, replacing under the square root $m(r)$ by $n(r)$ in (45) and also applying (3), Introduction (or (23), § 1, Part I) we obtain (as in the proof of (28), § 1, Part I) for r small enough

$$\log\left(\mu_1\left(B(x(\alpha),r)\right)/r^{\varkappa}\right) \leqslant \varkappa \log\left|(f^{n(r)-m(r)})'(x_{m(r)}(\alpha))\right| \tag{46}$$
$$+(1+2\varepsilon)\sqrt{(2\sigma^2_{\mu_\varphi}(\psi)/(\chi-2\varepsilon))\log(1/r)\log_{(3)}(1/r)}$$
$$= \mathrm{I}+\mathrm{II}.$$

(The coefficient 2 before ε absorbs all constants.) With the use of (41) we obtain

$$\mathrm{I} \leqslant \varkappa\left(\log(\sup|f'|)\right)\left(n(r)-m(r)\right)$$
$$\leqslant \varkappa\left(\log(\sup|f'|)\right)\cdot 5\sqrt{(\sigma^2_{\Xi}/(\chi-2\varepsilon))\log(1/r)\log_{(3)}(1/r)}.$$

Since ε and σ^2_{Ξ} can be taken arbitrarily close to 0 (by Lemma 1) this proves RVL(ii). ∎

Remark 4. Let us come back to the inequality (i) in RVL and explain the trouble we cannot overcome with proving it for $c = c(\mu_{\varphi *})$. In §4, Part I we arrived, with the use of Kolmogorov's test as in § 1, (33)–(36), at the inequality

$$\log\left(\mu_{\varphi *}\left(B(x(\alpha),r)\right)/r^{\varkappa}\right) \geqslant \sqrt{2\sigma^2 m(\log\log m+\tfrac{3}{2}\log_{(3)} m)}+\sum_{j=m}^{n}\psi\left(f^j(x(\alpha))\right)$$
$$= \mathrm{I}+\mathrm{II}$$

for $\tilde{\mu}_\varphi$-a.e. α and a sequence of r's converging to 0. Here m depends on r and n is the first time after m when $\tilde{s}^n(\alpha)$ hits a "good" set Ξ. We would arrive at

$$\log\left(\mu_{\varphi *}\left(B(x(\alpha),r)\right)/r^{\varkappa}\right) \geqslant \sqrt{(2\sigma^2/\chi)\log(1/r)\log_{(3)}(1/r)}$$

if we knew that $m-n \leqslant n^\delta$, $\delta < 1$, which would allow the error terms in I to be absorbed by the term $\frac{3}{2}\log_{(3)} m$ as in (33)–(35), § 1, Part I, and if we knew that $\mathrm{II} \leqslant n^\delta$, $\delta \leqslant \frac{1}{2}$, in order for II to be also absorbed. To get $\mathrm{II} \leqslant n^{1/2}$ the approximation property (1) for Ξ and the uniform upper part of LIL for the sequence $\psi \circ \tilde{s}^{-j}$, $j \to \infty$, on Ξ would be sufficient. Unfortunately we cannot assure both the properties at the same time.

6. Harmonic measure on fractal Jordan curves. Mixing piecewise repellers. We shall here prove Theorem C and discuss the examples mentioned in the Introduction. We shall divide the proof into steps in the same way as in §§ 1 and 2, Part I. Along the proof we shall often draw attention to the importance of the assumption # ((16), §0, Part I). In examples, where we want to apply Theorem C, we need to construct sophisticated Markov partitions rather than the natural ones in order to have # satisfied (see the snowflake example). On the other hand, we obtain the conclusion about the relation between ω and Hausdorff measures easier than Carleson [Ca2] and Makarov [Mk3] (cf. [Mk4]).

Proof of Theorem C. *Step* 1. For every j there exists a conformal extension g_j of the map $R^{-1}\circ f_j \circ R$ beyond $\partial\mathbf{D}$, by the Symmetry Principle (analogously to g, see Introduction). Denote the domain of g_j by V_j. Due to # every V_j is a neighbourhood of $R^{-1}(\partial_j)$. By construction $g_j(\partial\mathbf{D}\cup V_j) \subset \partial\mathbf{D}$. By [P2], §7, or [B2], L. 3, the family g_j is expanding, namely there exist $n > 0$, $\lambda > 1$ such that

$$\prod_{k=0}^{n-1}\left|g'_{j_k}(g^k(x))\right| > \lambda > 1$$

for every $x \in \bigcup_j R^{-1}(\partial_j)$ and j_k such that $g^k(x) \in \delta_{j_k}$.

The partition $\{\partial'_j\} = \{R^{-1}(\partial_j)\}$, $j = 1, \ldots, J$, together with the maps g_j on V_j satisfies the similar Markov partition properties as $\{\partial_j\}$. R conjugates the two Markov partitions. So, as in § 1, there exist coding maps π_1 and π_2 from $\Sigma(\Delta)$ onto the sets $\partial' = \bigcup \partial'_j$ and $\partial = \bigcup \partial_j$ respectively, $\pi_2 = R\circ\pi_1$ and π_1, π_2 semiconjugate the shift s with $g = g_j$, $f = f_j$ on the cylinder $\{\alpha_0 = j\}$ for every j.

Step 2. Let μ_φ be the Gibbs measure on $\Sigma(\Delta)$ for the function $\varphi(\alpha) = -\log|(g'_{\alpha_0})\circ\pi_1(\alpha)|$. (The function φ is Hölder continuous because π_1 is, due to (1).) Set $(\pi_i)_*(\mu_\varphi) = \mu_{\varphi i}$, for $i = 1, 2$. Then

$$\mu_{\varphi 1} \text{ is equivalent to } l_1 \text{ on } \partial', \tag{1}$$

hence $\mu_{\varphi 2}$ is equivalent to ω on ∂ (with bounded densities).

(1) follows from (1), Introduction, up to one detail: (1), Introduction, implies that for every "cylinder" curve $\gamma = \pi_1(A)$ for $A \in \mathscr{A}^n$, $\alpha \in A$ and a.e. $\zeta \in \gamma$

$$K^{-1} \leqslant \mathscr{J}(\zeta, g^n)/|(g_{\alpha_n}\circ\ldots\circ g_{\alpha_0})'(\zeta)| \leqslant K \tag{2}$$

so by the bounded distortion property for iterations $K^{-1} \leqslant \mu_{\varphi 1}(\gamma)/l_1(\gamma) \leqslant K$. Here $\mathscr{J}(\zeta, g^n)$ denotes the Jacobian $d\left(((g_{\alpha_n}\circ\ldots\circ g_{\alpha_0})^{-1})_*(\mu_{\varphi 1})\right)/d\mu_{\varphi 1}(\zeta)$.

However, we need to know this for $\hat{\gamma}$ an arbitrary arc in ∂. The trouble is that even if $\hat{\gamma}$ is an extension of our "cylinder" γ on which $g_{\alpha_n}\circ\ldots\circ g_{\alpha_0}$ is still defined we cannot write (2) for $\zeta \in \hat{\gamma}\setminus\gamma$ because $\mathscr{J}(\zeta, g^n)$ involves a sequence $\hat{\alpha}_0, \ldots, \hat{\alpha}_n$ which may be different from $\alpha_0, \ldots, \alpha_n$.

We cope with this by observing that, by the bounded distortion property, for every arc $\eta \in \pi_1(\mathscr{A}^n)$ divided into arcs $\eta_1, \ldots, \eta_k \in \pi_1(\mathscr{A}^{n+1})$ we have $\inf_j \operatorname{diam}\eta_j > K^{-1}\operatorname{diam}\eta$ for $K > 0$ independent of n and η. So $\hat{\gamma}$ contains a "cylinder" η with $\operatorname{diam}\eta \geqslant K^{-1}\operatorname{diam}\hat{\gamma}$ and can be covered by a finite number of "cylinders" $\gamma_t \in \pi_1(\mathscr{A}^{n_t})$, $t = 1, \ldots, T$, $\operatorname{diam}\gamma_t \leqslant K\operatorname{diam}\hat{\gamma}$, with pairwise disjoint interiors. So we arrive at

$$K^{-1}\operatorname{diam}\gamma \leqslant \mu_{\varphi 1}(\hat{\gamma}) \leqslant \sum_t \mu_{\varphi 1}(\gamma_t) \leqslant \sum_t \operatorname{diam}\gamma_t \leqslant K\operatorname{diam}\gamma.$$

(The last inequality is true because we work inside the circle $\partial\mathbf{D}$ where the diameter of the union is roughly the sum of diameters. However, it turns out that this inequality is also true in a more general situation, because T is bounded by a constant independent of $\hat{\gamma}$, see Remark 5 after the proof of Theorem C.)

Step 3. The formula (1) from [P2] is valid, namely for $\mu_{\varphi 1}$-a.e. $\zeta \in \partial'$ the nontangential limit

$$\chi(R)(\zeta) = \lim_{x\to\zeta} \operatorname{nontan} \frac{\log|R'(x)|}{-\log|\zeta - x|} = 1 - \chi_{\mu_{\varphi 2}}(f)/\chi_{\mu_{\varphi 1}}(g) \tag{3}$$

exists, $f\colon \partial \to \partial$ is defined as f_j on every ∂_j, g is g_j on δ'_j. The choice at the end-points is of no importance. ((3) is true for the Gibbs measure μ_φ of any Hölder function φ on $\Sigma(\Delta)$ in fact.)

Indeed, the idea from [P2] works. Given a point x inside a Stolz cone at $\zeta = \pi_1(\alpha)$ for $\alpha = (\alpha_j) \in \Sigma(\Delta)$, map it forward by g_{α_0}, g_{α_1}, and so on until the image x' is well inside $\mathbf{D}$, then map $R(x')$ backward by $f_{\alpha_n}^{-1}, \ldots, f_{\alpha_0}^{-1}$.

This uses $g_j(V_j \cap \partial\mathbf{D}) \subset \partial\mathbf{D}$ and the bounded distortion property for the iterations. This allows us to replace $\log|R'(x)|$ by an expression containing $|\log(f^n)'(R(\zeta))|$ and $\log|(g^n)'(\zeta)|$ which can be replaced (roughly) by $n\chi_{\mu_{\varphi 2}}(f)$, $n\chi_{\mu_{\varphi 1}}(g)$ giving (3). The reader will find details in [P2].

Step 4. By Makarov's theory [Mk2]

$$\chi(R)(\zeta) = 0 \quad \text{for } l_1\text{-a.e., } \zeta \in \partial\mathbf{D}. \tag{4}$$

(See also Remark 6.) Hence

$$\chi = \chi_{\mu_{\varphi 1}}(g) = \chi_{\mu_{\varphi 2}}(f),$$

so for $\psi(\alpha) = -\log|g'_{\alpha_0} \circ \pi_1(\alpha)| + \log|f'_{\alpha_0} \circ \pi_2(\alpha)|$ we have $\int \psi\, d\mu_\varphi = 0$ and we consider the asymptotic variance $\sigma^2 = \sigma^2_{\mu_\varphi}(\psi)$ as in §2, Part I.

Now one proceeds as in §1, Part I. The proof that $c(\omega, \partial) = \sqrt{2\sigma^2/\chi}$ is the transition parameter for which (i)–(iii) of Th. A ((8)–(11), §0, Part I) hold reduces to the proof of the Refined Volume Lemma (RVL) (Lemma 2 in §1, Part I, or RVL in Introduction with μ_1 replaced by $\mu_{\varphi 2}$ and $c \geqslant c(\mu_{\varphi 2})$ in (ii)).

For $x \in \pi_2(\alpha)$, $\alpha = (\alpha_j) \in \Sigma(\Delta)$, one considers a ball $B(x, r)$ and its images, first under f_{α_0}, then under f_{α_1} etc., until $f_{\alpha_n} \circ \ldots \circ f_{\alpha_0}(B(x, r))$ becomes large (it is almost a ball by bounded distortion). One can relate n to r as in (24), §1, Part I. So one has

$$K^{-1} \leqslant r \Big/ \prod_{j=0}^{n} |(f_{\alpha_j}^{-1})'(f^j(x))| \leqslant K, \tag{5}$$

$$K^{-1} \leqslant l_1\big(R^{-1}(B(x,r))\big) \Big/ \prod_{j=0}^{n} \big|(g_{\alpha_j}^{-1})'\big(g^j(R^{-1}(x))\big)\big| \leqslant K, \tag{6}$$

by bounded distortion for the iterations $f_{\alpha_n} \circ \ldots \circ f_{\alpha_0}$ and $g_{\alpha_n} \circ \ldots \circ g_{\alpha_0}$. (By the continuity of R and R^{-1} on $\operatorname{cl}\mathbf{D}$ and $\operatorname{cl}\Omega$ the sets $f_{\alpha_j} \circ \ldots \circ f_{\alpha_0}(B(x,r))$ and $g_{\alpha_j} \circ \ldots \circ g_{\alpha_0}\big(R^{-1}(B(x,r))\big)$ come out with large diameters for the same $j = n$. This n is also the same if instead of the set $R^{-1}(B(x,r))$ we consider only its

component containing $\pi_1(\alpha)$.) Thus one obtains the RVL as in §1, Part I, by using the LIL (or Kolmogorov's test if $c = c(\omega)$) for the sequence $\psi \circ s^n$.

Step 5. Consider the case $\sigma^2 = 0$. Then by Lemma 1, §1, Part I, ψ is homologous to 0 in Hölder functions, in particular in bounded functions in this case. So R and R^{-1} on ∂' and ∂ respectively are Lipschitz continuous, due to estimates similar to (5) and (6). In particular, we conclude that ∂ is rectifiable.

Consider a Riemann mapping $\hat{R}$: $\hat{\mathbf{C}} \setminus \operatorname{cl}\mathbf{D} \to \hat{\mathbf{C}} \setminus \operatorname{cl}\Omega$ and $\hat{g}_j$ the conformal extension of $\hat{R}^{-1} \circ f_j \circ \hat{R}$. Let us hat also all other symbols in the construction analogous to those for R. We claim that

$$\hat{\sigma}^2 = \sigma^2_{\mu_{\hat{\varphi}}}(\hat{\psi}) = 0. \tag{7}$$

Otherwise, by the part of Theorem C already proven applied to the domain $\hat{\mathbf{C}} \setminus \operatorname{cl}\Omega$ we would have $\hat{\omega} \perp \Lambda_1$ on ∂. On the other hand ∂, being rectifiable, can be extended to a rectifiable Jordan curve $\partial^\vee \subset \operatorname{cl}\Omega$. Denote the harmonic measure on $\partial^\vee$ viewed from ∞ by $\omega^\vee$. Since the component of $\hat{\mathbf{C}} \setminus \partial^\vee$ containing ∞ contains $\hat{\mathbf{C}} \setminus \operatorname{cl}\Omega$ we have $\omega^\vee \gg \hat{\omega}$ on δ. By the F. and M. Riesz Theorem ([Go], Ch. 10, §1, Th. 2), $\omega^\vee$ is equivalent to Λ_1, so $\hat{\omega} \ll \Lambda_1$ on δ. We have come to a contradiction.

By (7), $\hat{R}$ and $\hat{R}^{-1}$ are Lipschitz continuous on $\hat{\partial}'$ and $\hat{\partial}$ respectively.

Now one proceeds as in [P3] (see also the references given in §2, Part I). $\hat{R}^{-1} \circ R$ is absolutely continuous and the measures $(\pi_1)_*(\mu_\varphi)$ and $(\hat{\pi}_1)_*(\mu_{\hat{\varphi}})$ are equivalent to l_1 and ergodic, so $(\hat{R}^{-1} \circ R)_*$ maps the former to the latter. These measures have real-analytic densities with respect to l_1 on the closed (!) arcs ∂'_1 and ∂'_j for every j, by the argument similar to Krzyżewski's [Krz]: Lift everything from a neighbourhood of $\partial\mathbf{D}$ to a neighbourhood of $\mathbf{R}$ in $\mathbf{C}$ and consider, on a neighbourhood of $\zeta = \pi_1(\alpha) \in \partial'_{\alpha_0}$, iterations of the Perron–Frobenius operator on the function say 1, namely the functions $P^n(1) = \bigcap_{\beta: S^n(\beta)=\alpha} \big((g_{\beta_{n-1}} \circ \ldots \circ g_{\beta_0})'\big)^{-1}$. (In case $g_{\beta_{n-1}} \circ \ldots \circ g_{\beta_0}$ reverses orientation, by $(g_{\beta_{n-1}} \circ \ldots \circ g_{\beta_0})'$ we mean the holomorphic function $-(g_{\beta_{n-1}} \circ \ldots \circ g_{\beta_0})'$.) The sequence $P^n(1)$ converges to a complex-analytic function, real and positive on $\mathbf{R}$, which is just the density.

The conclusion is that $r = \hat{R}^{-1} \circ R$ (and its inverse) are real-analytic on every ∂'_j ($\hat{\partial}'_j$ resp.). Hence, for arbitrary j, r extends to a complex-analytic, injective map on a neighbourhood of ∂'_j. If $\hat{R}$ on a neighbourhood of $\hat{\partial}'_j$ in $\hat{\mathbf{C}} \setminus \mathbf{D}$ is replaced by $\check{R} = \hat{R} \circ r$ on a neighbourhood of ∂'_j in $\hat{\mathbf{C}} \setminus \mathbf{D}$, then R and $\check{R}$ are equal to each other in a neighbourhood of ∂'_j in $\partial\mathbf{D}$. They glue together to a holomorphic map H, so a neighbourhood in $\partial\Omega$ of $\partial_j = H(\partial'_j)$ is an analytic curve. The conclusion is that ∂ (and even its neighbourhood in $\partial\Omega$) is analytic. ∎

Remark 5. One can generalize the notion of the mixing repeller for a holomorphic map of §1, Part I, and consider a *mixing piecewise repeller* in $\hat{\mathbf{C}}$. This is a compact set $X \subset \hat{\mathbf{C}}$ for which there exists a covering $\mathscr{A} = \{A_j\}_{j=1}^{J}$ by compact sets with pairwise disjoint interiors such that $\mathrm{cl}_X(\mathrm{int}_X A_j) = A_j$, together with conformal injective maps f_j, each defined on a neighbourhood U_j of A_j, such that $f_j(A_j)$ is a union of elements of $\mathscr{A}$, $|f_j'| > 1$ and

$$f_j(U_j \cap X) \subset X. \tag{8}$$

(The index X at cl and int denotes the closure and interior in X.)

We also assume that for the standard coding π_X: $\Sigma(\Delta) \to X$ by a one-sided shift space, the matrix Δ is transitive, aperiodic.

For a mixing piecewise repeller Theorem B stays true. To see that, it is enough to prove the Refined Volume Lemma with the assertion as in the version from the Introduction (including the case $c = c(\mu_{\varphi *})$, with φ a Hölder function on $\Sigma(\Delta)$ and $\mu_{\varphi *} = (\pi_X)_*(\mu_\varphi)$ and to prove $\mu_{\varphi *} \ll \Lambda_{\mathrm{HD}(X)}$ if the function $\psi(\alpha) = \varphi(\alpha) + \varkappa \log |f_{\alpha_0}'(\pi_X(\alpha))| - P(s, \varphi)$ is homologous to 0.

The only delicate place in the proof is an estimate corresponding to (27), §1, Part I. This relies on the following lemma told us by Caroline Series.

LEMMA 2. *Every ball $B(x, r)$ in X for $x \in X$, $r > 0$, can be covered by at most M cylinders $\hat{A}_t \in \mathscr{A}^{n_t}$ with $K^{-1} r < \operatorname{diam} \hat{A}_t < Kr$, for constants K, $M > 0$ independent of x and r.*

Proof. By the definition of the open set for every $A_j \in \mathscr{A}$ there exists a set V_j open in $\hat{\mathbb{C}}$ such that

$$\mathrm{int}_X A_j = V_j \cap X \tag{9}$$

and $\mathrm{cl}_{\hat{\mathbb{C}}} V_j \subset U_j$.

Fix $a_j \in \mathrm{int}_X A_j$ for every j. Set $D = \min_j \mathrm{dist}(a_j, \mathrm{Fr}_{\hat{\mathbb{C}}} V_j)$.

Consider an arbitrary ball $B(x, r)$ in X with $x \in X$, $r > 0$. For every α such that $\pi_X(\alpha) \in B(x, r)$ choose $n = n(\alpha)$ such that

$$K^{-1} \leqslant r \left| (f_{\alpha_n} \circ \ldots \circ f_{\alpha_0})' (\pi_X(\alpha)) \right| < K.$$

Then by the bounded distortion property

$$K^{-1} r < \operatorname{diam} (f_{\alpha_n} \circ \ldots \circ f_{\alpha_0})^{-1} (V_{\alpha_{n+1}}) < Kr. \tag{10}$$

One can choose a finite number of sets $\hat{A}_t$, $t = 1, \ldots, T$, of the form $F_t(A_{\alpha^t_{n+1}})$, where we have set $F_t = (f_{\alpha^t_n} \circ \ldots \circ f_{\alpha^t_0})^{-1}$ for $n = n(\alpha^t)$ with $\pi_X(\alpha^t) \in B(x, r)$ which cover $B(x, r)$ and whose interiors are pairwise disjoint in X. This is so because the only possible relations between the interiors of cylinders are inclusion or disjointness.

By the bounded distortion property and (10), for $\hat{a}_t = F_t(a_{\alpha^t_{n+1}}) \in \hat{A}_t$ and $\hat{V}_t = F_t(V_{\alpha^t_{n+1}})$ we have

$$\mathrm{dist}(\hat{a}_t, \mathrm{Fr}\, \hat{V}_t) > K^{-1} D \operatorname{diam} \hat{V}_t / \operatorname{diam} V_{\alpha^t_{n+1}} > K^{-1} r$$

(the latter term with another K).

By (8) and (9), $\hat{V}_t \cap X = \mathrm{int}_X \hat{A}_t$. So for every $t \neq t'$, $\mathrm{dist}(\hat{a}_t, \hat{a}_{t'}) > K^{-1} r$. So the balls $B(\hat{a}_t, \frac{1}{2} K^{-1} r)$ are pairwise disjoint and are contained in $B(x, r(K+1))$ (here the balls in $\hat{\mathbb{C}}$). So

$$T \leqslant \pi (K+1)^2 r^2 / (\tfrac{1}{4} \pi K^{-2} r^2) < 4 (K+1)^4. \ \blacksquare$$

This lemma was used in fact in [B2] in the proof that the measures $(\pi_X)_*(\mu_{-\mathrm{HD}(X) \log |f' \circ \pi_X|})$ and $\Lambda_{\mathrm{HD}(X)}$ are equivalent. We put the detailed proof here as the corresponding consideration in [B2] seemed to us obscure. Let us remark that in the case where X is a Jordan curve, in particular Bowen's quasi-circle, one can have $T = 2$.

The above considerations prove the following geometric theorem lying half way between the abstract Theorem B and the concrete Theorem C.

THEOREM 1. *Suppose that $(X, \mathscr{A}, f_j)$ and $(Y, \mathscr{B}, g_j)$ are two mixing piecewise repellers in $\hat{\mathbb{C}}$ conjugated by a homeomorphism h: $Y \to X$. Denote by π_Y: $\Sigma(\Delta) \to Y$, π_X: $\Sigma(\Delta) \to X$ the corresponding codings with the property $\mathrm{h} \circ \pi_Y = \pi_X$.*

Denote by μ the Gibbs measure for the function φ on $\Sigma(\Delta)$ defined by

$$\varphi(\alpha) = -\mathrm{HD}(Y) \cdot \log |g_{\alpha_0}' \circ \pi_Y(\alpha)|$$

for $\alpha = (\alpha_j) \in \Sigma(\Delta)$. Set $\mu_Y = (\pi_Y)_*(\mu)$, $\mu_X = (\pi_X)_*(\mu)$, $\varkappa = \mathrm{HD}(\mu_X)$ and

$$\psi(\alpha) = -\mathrm{HD}(Y) \cdot \log |g_{\alpha_0}' \circ \pi_Y(\alpha)| + \varkappa \log |f_{\alpha_0}' \circ \pi_X(\alpha)|.$$

Then μ_Y is equivalent to $\Lambda_{\mathrm{HD}(Y)}$ and the following dichotomy appears:

1. If $\sigma^2 = \sigma_\mu^2(\psi) > 0$ then $\mu_X \perp \Lambda_\varkappa$, more exactly there exists a nonzero transition parameter $c(\mu_X)$ for the family of functions $\Phi_c^{(\varkappa)}$ given by the formula $c(\mu_X) = \sqrt{2\sigma^2 / \chi_{\mu X}}$.

Geometrically, for $\Lambda_{\mathrm{HD}(Y)}$-a.e. $y_1 \in Y$ the set of limit values as $y_2 \to y_1$ ($y_2 \in Y$) for

$$\log \frac{\left(\mathrm{dist}(\mathrm{h}(y_1), \mathrm{h}(y_2))\right)^\varkappa}{\left(\mathrm{dist}(y_1, y_2)\right)^{\mathrm{HD}(Y)}} \Bigg/ \sqrt{\frac{2\sigma^2}{\chi_\mu(g)} \log \frac{1}{\mathrm{dist}(y_1, y_2)} \log_{(3)} \frac{1}{\mathrm{dist}(y_1, y_2)}}$$

is the interval $[-1, 1]$.

2. If $\sigma^2 = 0$ then $\varkappa = \mathrm{HD}(X)$ and the measure $\mathrm{h}_*(\mu_Y) = \mu_X = \mathrm{h}_*(\Lambda_{\mathrm{HD}(Y)})$ is equivalent to $\Lambda_{\mathrm{HD}(X)}$.

Geometrically

$$K^{-1} \leqslant \frac{\mathrm{dist}(\mathrm{h}(y_1), \mathrm{h}(y_2))^{\mathrm{HD}(X)}}{\left(\mathrm{dist}(y_1, y_2)\right)^{\mathrm{HD}(Y)}} \leqslant K$$

for a constant $K > 0$ and every y_1, $y_2 \in Y$.

Note. In [PUZ-preprint] we asked whether $\sigma^2 = 0$ implies h extends to a holomorphic map. Shortly afterward we got acquainted with an answer by

D. Sullivan [Su2] (for continuous f, g). Adapting Sullivan's theorem to our piecewise expanding case we obtain the following

THEOREM 2. *For $(X, \mathcal{A}, f_j)$ and $(Y, \mathcal{B}, g_j)$ as in Theorem 1 and a measurable one-to-one map* h *such that* $\mathrm{h}(\mathcal{B}) = \mathcal{A}$, $\mathrm{h}_*(\mu_Y) = \mu_X$ *and* h *conjugates g and f a.e., if $\sigma^2 = 0$ and if $(X, \mathcal{A}, f_j)$ or $(Y, \mathcal{B}, g_j)$ is not linear* (see Def. below) *then* $\mathrm{HD}(X) = \mathrm{HD}(Y)$ *and* $\mathrm{h}|_{B_j}$ *and* $\mathrm{h}^{-1}|_{A_j}$ *extend to conformal diffeomorphisms on neighbourhoods of B_j, A_j in $\hat{\mathbf{C}}$ for every $j = 1, \ldots, J$ (after a change on sets of measure 0 if necessary).*

DEFINITION. We call a mixing piecewise repeller $(X, \mathcal{A}, f_j)$ *linear* if there exist conformal charts $\varphi_{j,t}\colon U_{j,t} \to \mathbf{C}$ such that $\bigcup_t U_{j,t}$ is a neighbourhood of A_j for every $j = 1, \ldots, J$ and all the maps $\varphi_{j,t'} \circ \varphi_{j,t}$, $\varphi_{j',t'} \circ f_j \circ \varphi_{j,t}$ are affine.

In the situation of Theorem 2 it can be proved that if $(X, \mathcal{A}, f_j)$ is not linear then $(Y, \mathcal{B}, g_j)$ is not linear. If both the repellers are linear then the assertion of Theorem 2 is false (even if we assume h to be continuous; see [MkV]).

EXAMPLES. 1. *The snowflake.* To every side of an equilateral triangle, in the middle, we glue from outside an equilateral triangle three times as small. To every side of the resulting polygon we glue again an equilateral triangle three times as small and so on infinitely many times. The triangles do not overlap in this construction and the boundary of the resulting domain Ω is a Jordan curve. This Ω is called the *snowflake* (see Fig. 1). It was first described by Helge von Koch in 1904.

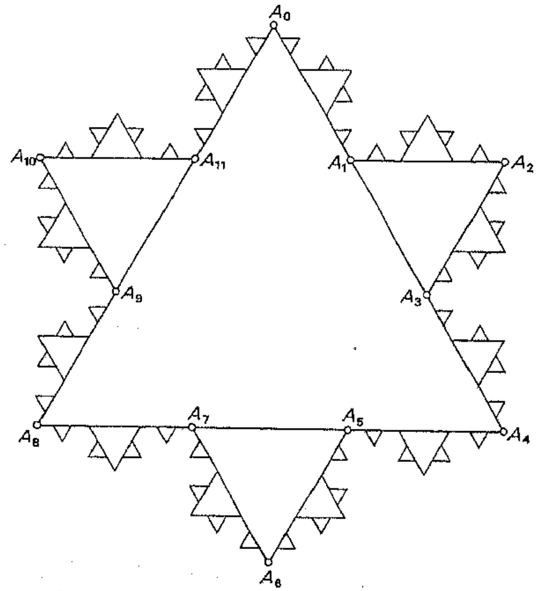

Fig. 1

Denote the curve in $\partial\Omega$ joining a point $x \in \partial\Omega$ to $y \in \partial\Omega$ in the clockwise direction just by xy. For every $\partial_i = A_i A_{i+1 \,(\mathrm{mod}\, 12)} \subset \partial\Omega$, $i = 0, \ldots, 11$, we consider its covering by the curves 12, 23, 34, 45, 56 in $\partial\Omega$ (see Fig. 2). This covering together with the affine maps

12, 34 $\to$ 16 (preserving orientation on $\partial\Omega$)
23 $\to$ 61 (reversing orientation)
56 $\to$ 36 (preserving orientation)
45 $\to$ 63 (reversing orientation)

gives a Markov partition of ∂_i satisfying the assumptions of Theorem C.

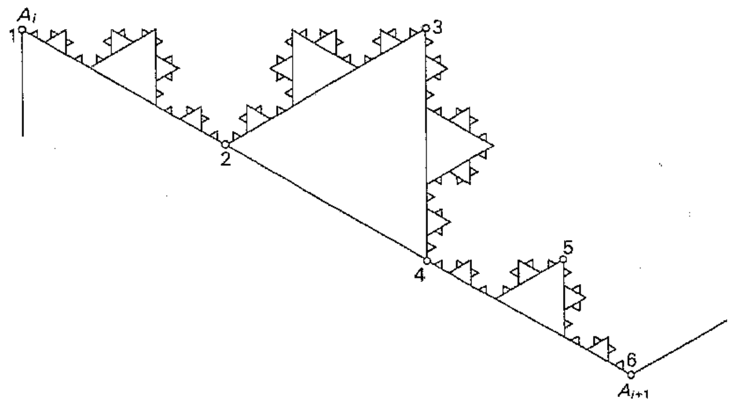

Fig. 2

Because $\partial\Omega$ (and every subcurve) is definitely not real-analytic ($\mathrm{HD}(\partial\Omega) = \log 4/\log 3$), the assertion of Theorem C is valid with $c(\omega, \partial_i) > 0$.

We may denote $c(\omega, \partial_i)$ by $c(\omega)$ because it is independent of ∂_i by symmetry.

PROBLEM. Compute $c(\omega)$.

Remark 6. In the proof of Theorem C applied to the snowflake (and also to Carleson's domains described below), one does not need to refer to [Mk2] to have (4). Indeed, $\chi(R)$ is l_1-a.e. constant on each $R^{-1}(A_i A_{i+1})$ by (3). Denote it by χ_i. We have (cf. [P2] and (3))

$$0 = \lim_{r\to 1} \int_{\partial \mathbf{D}} \log|R'(r\zeta)|/(\log(1-r))\, dl_1(\zeta) = \int_{\partial \mathbf{D}} \chi(R)(\zeta)\, dl_1(\zeta)$$
$$= \sum_{i=0}^{11} \chi_i \omega(A_i A_{i+1}),$$

with the first equality true because $\log|R'|$ is a harmonic function, the second one due to the Distortion Theorem of Koebe, which allows us to change the order of lim and $\int$.

Since the conformal maps: the rotation by the angle $\pi/3$ and the reflection transform Ω onto itself we deduce that χ_i and $\omega(A_i A_{i+1})$ are independent of i, hence $\chi(R) = 0$ a.e.

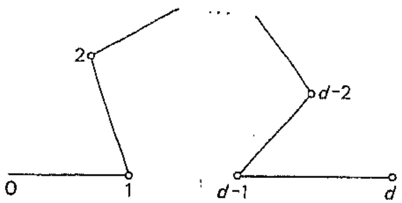

Fig. 3

2. *Carleson's domain.* We recall Carleson's construction from [Ca2]. We fix a broken line γ with the first and last segments lying in the same straight line in $\mathbf{R}^2$, with no other segments intersecting the segment $\overline{1, d-1}$ (see Fig. 3). Then we take a regular polygon Ω^1 with vertices $T_0, \ldots, T_n$ and glue to every side of it, from outside, the rescaled, not mirror reflected, curve γ so that the ends of the glued curve coincide with the ends of the side. The resulting curve bounds a second polygon Ω^2. Denote its vertices by $A_0, A_1, \ldots$ (Fig. 4). Then we glue again the rescaled γ to all the sides of Ω^2 and obtain a third order polygon Ω^3 with vertices $B_0, B_1, \ldots$ Then we build Ω^4 with vertices $C_0, C_1, \ldots, \Omega^5$ with

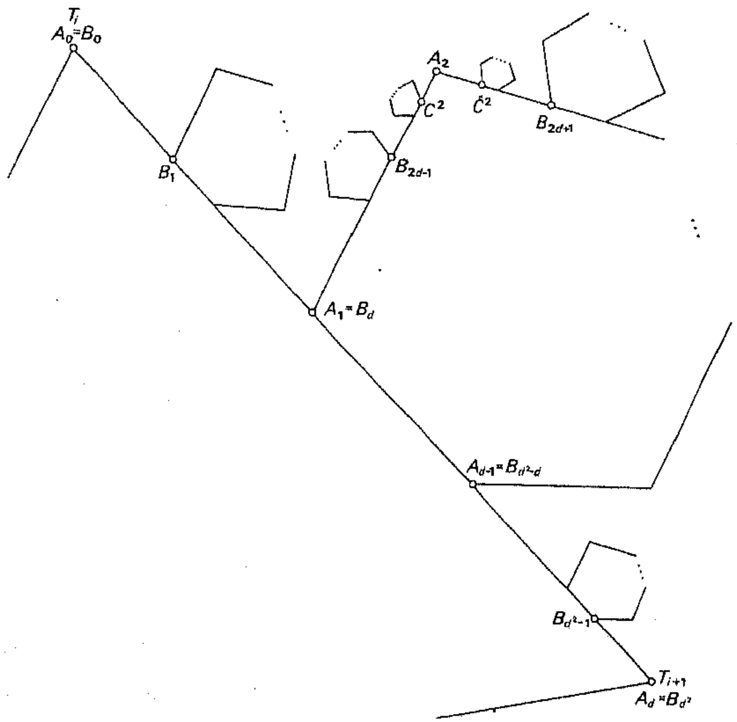

Fig. 4

$D_0, D_1, \ldots$ etc. Assume that there is no selfintersecting of the curves $\partial\Omega^n$ in this construction, moreover assume that in the limit we obtain a Jordan curve $\mathscr{L} = \mathscr{L}(\Omega^1, \gamma) = \partial\Omega$.

The natural Markov partition of each curve $T_i T_{i+1}$ in $\mathscr{L}$ into curves $A_j A_{j+1}$ with $f(A_j A_{j+1}) = T_i T_{i+1}$ considered by Carleson does not satisfy the property # of the Introduction ((16), § 0, Part I) so we cannot succeed with it.

We proceed as follows: Define

$$f(B_{d(j-1)+1} B_{dj-1}) = A_1 A_{d-1}$$

for every $j = 1, \ldots, d$.

Divide now every $B_{dj-1} A_j$ for $j = 1, \ldots, d$, and $A_j B_{dj+1}$ for $j = 0, \ldots, d-1$, into two curves with ends in the vertices of the polygon Ω^4: $C^j \in B_{dj-1} A_j$, $\hat{C}^j \in A_j B_{dj+1}$ respectively, the closest to $A_j (\neq A_j)$. Let, for $j = 1, \ldots, d-1$,

$$f(C^j A_j) = B_{dj-1} A_j, \quad f(B_{dj-1} C^j) = A_{d-1} B_{d^2-1},$$

$$f(A_j \hat{C}^j) = A_j B_{dj+1}, \quad f(\hat{C}^j B_{dj+1}) = B_1 A_1.$$

This gives a transitive aperiodic Markov partition of $B_1 B_{d^2-1}$.

We can consider, instead of the broken line γ in the construction of Ω, the line $\gamma^{(2)}$ consisting of d^2 segments which arises by glueing to every side of γ a rescaled γ. Consecutive glueing of the rescaled $\gamma^{(2)}$ to the polygon Ω^1 gives consecutively Ω^3, Ω^5 etc. The same construction as above gives a Markov partition of $D_1 D_{d^4-1}$ in $T_i T_{i+1}$.

By continuing this procedure we approximate $T_i T_{i+1}$ so from Theorem C and from symmetry we deduce that there exists a transition parameter $c(\omega) = 0$ (independent of $T_i T_{i+1}$) such that the properties (i)–(iii) of Th. A ((8)–(11), § 0, Part I) are satisfied.

Observe that Carleson's assumption that the broken line $1, \ldots, d-1$ does not intersect $\overline{1, d-1}$ (except for the ends) has not been needed in these considerations. Also the assumption that Λ^1 is a regular polygon can be omitted: One can prove that $c(\omega)$ does not depend on $T_i T_{i+1}$ by considering a transitive, aperiodic Markov partition which involves all the sides of Ω^1 simultaneously.

7. The dependence on a parameter. One can ask about how $c(\omega(a))$ depends on a for a family of domains $\Omega(a)$ (and respective harmonic measures $\omega(a)$) analytically depending on a complex parameter a. In particular, what can be proved if $\Omega(a)$ is the basin of attraction to ∞ for a polynomial $z^d + a$? How large $c(\omega(a))$ can be?

Let us start by considering an abstract situation as in Theorem B (see Introduction or § 1, Part I). Recall that if $f\colon U \to M$ is a C^1-map defined on a neighbourhood U of a mixing repeller X and $g\colon U \to M$ is sufficiently close to f in the C^1-topology then, in a small C^0-neighbourhood of the inclusion of X into U, there exists a unique injective map $\mathrm{h}_{f,g}\colon X \to U$ such that $g \circ \mathrm{h}_{f,g} = \mathrm{h}_{f,g} \circ f$. The maps $\mathrm{h}_{f,g}$ and $\mathrm{h}_{f,g}^{-1}$ are Hölder continuous. These facts are well known: see [Sh] (structural stability of expanding maps), [P6], p. 72, and [Su1] (telescopes). $\mathrm{h}_{f,g}(X)$ is of course a mixing repeller for g so Theorem B is applicable to it. Finally observe the one-to-one correspondence between the Gibbs measures of Hölder continuous functions on X and on $\mathrm{h}_{f,g}(X)$, namely $(\mathrm{h}_{f,g})_*(\mu_\varphi) = \mu_{\varphi\circ\mathrm{h}_{f,g}^{-1}}$.

PROPOSITION 1. *Let X be a mixing repeller for a holomorphic mapping $f\colon U \to \mathbf{C}$ and let φ be a Hölder continuous function on X (as in Theorem B). Consider a family of mappings $f_\lambda\colon U \to \mathbf{C}$ for a complex parameter λ in a neighbourhood V of $0 \in \mathbf{C}$, such that $f_0 = f$ and $f_\lambda(x)$ is a holomorphic function of λ, x. Assume that $\varkappa = \mathrm{HD}\big((\mathrm{h}_\lambda)_*(\mu_\varphi)\big)$ is constant (does not depend on λ); here h_λ denotes h_{f,f_λ}. Then $c_\lambda^2 = \big(c\big((\mathrm{h}_\lambda)_*(\mu_\varphi)\big)\big)^2$ is a subharmonic and real-analytic function of λ. The domain in $\mathbf{C}^2$ to which c_λ^2 extends complex-analytically contains for each $\lambda^0 = (\lambda_1^0, \lambda_2^0) \in V$ the set*

$$\{(\lambda_1,\lambda_2)\in\mathbf{C}^2\colon |\lambda_1-\lambda_1^0|+|\lambda_2-\lambda_2^0| < \operatorname{dist}(\lambda^0,\partial V)\}.$$

Proof. Remark first that $\mathrm{h}_\lambda(x)$ is continuous in λ, x and that for every $x\in X$, $\mathrm{h}_\lambda(x)$ is a holomorphic function of λ. The latter follows from $\acute{o}\mathrm{h}_\lambda = \lim_{n\to\infty} f_\lambda^{-n} f^n$ (for appropriate branches f_λ^{-n}), the limit of holomorphic functions of λ for every x. (One could also refer to Mañé–Sad–Sullivan's λ-lemma [MSS]. Indeed, $\mathrm{h}_\lambda(x)$ is holomorphic for every periodic source x by the Implicit Function Theorem and periodic sources are dense in mixing repellers.)

We recall that $c_\lambda^2 = 2\sigma_\lambda^2/\chi_\lambda$, where $\sigma_\lambda^2 \equiv \sigma^2_{(\mathrm{h}_\lambda)*(\mu_\varphi)}(\psi_\lambda) = \sigma^2_{\mu_\varphi}(\bar\psi_\lambda)$, where ψ_λ is a function on $\mathrm{h}_\lambda(X)$ defined by

$$\psi_\lambda = \varphi\circ\mathrm{h}_\lambda^{-1} + \varkappa\log|f_\lambda'| - P(f_\lambda|_{\mathrm{h}_\lambda(X)}, \varphi\circ\mathrm{h}_\lambda^{-1}),$$

$\bar\psi_\lambda = \psi_\lambda\circ\mathrm{h}_\lambda = \varphi + \varkappa\log|f_\lambda'\circ\mathrm{h}_\lambda| - P(f,\varphi)$ is the corresponding function on X and $\chi_\lambda = \chi_{(\mathrm{h}_\lambda)_*(\mu_\varphi)}(f_\lambda|_{\mathrm{h}_\lambda(X)})$. Observe, however, that χ_λ is constant, independent of λ, because $\chi_\lambda = h_\lambda/\varkappa$ and the entropy $h_\lambda = h_{(\mathrm{h}_\lambda)_*(\mu_\varphi)}(f_\lambda|_{\mathrm{h}_\lambda(X)})$ is constant, equal to $h_{\mu_\varphi}(f)$. Thus it is enough to study σ^2. Recall that

$$\sigma_\lambda^2 = \lim_{n\to\infty} n^{-1}\int (S_n\bar\psi_\lambda)^2\,d\mu_\varphi, \quad \text{where} \quad S_n\bar\psi_\lambda = \sum_{j=0}^{n-1} \bar\psi_\lambda\circ f^j.$$

The function $S_n\bar\psi_\lambda(x)$ is a harmonic function of λ for every $x\in X$, $n>0$, hence its square, as a composition with the convex function $t\mapsto t^2$, is subharmonic (see [HK], Th. 2.2). In consequence the average (integral) and the limit σ_λ^2 are subharmonic too.

Now let us pass to the analyticity question. We shall rely on the following

LEMMA 2 (see [H], Th. 2). *Let u be a harmonic function on the disc $\mathbf{D}_r = B(0,r) \subset \mathbf{R}^2 = \{(z_1,z_2)\in\mathbf{C}^2\colon z_1, z_2 \text{ are real}\}$, continuous on $\mathrm{cl}\,\mathbf{D}_r$. Then u extends to a complex-analytic function on a neighbourhood of $\mathbf{D}_r$ in $\mathbf{C}^2$ and*

$$u(z_1,z_2) = \sum_{i,j=0}^{\infty} b_{ij} z_1^i z_2^j \quad \textit{with} \quad |b_{ij}| \leqslant \sup|u|\cdot 2\binom{i+j}{i} r^{-(i+j)}. \tag{1}$$

In particular, the series is absolutely convergent in $|z_1|+|z_2|<r$.

Proof. Suppose that $r=1$. Write the Poisson formula $u(z) = (2\pi)^{-1}\int P(z,\zeta)u(\zeta)\,d|\zeta|$ with the kernel $P(z,\zeta) = \mathrm{Re}(\zeta+z)/(\zeta-z)$. We have

$$P(z,\zeta)-1 = 2\,\mathrm{Re}\sum_{n=1}^{\infty}(z/\zeta)^n = 2\,\mathrm{Re}\sum_{n=1}^{\infty}(1/\zeta^n)(\mathrm{Re}\,z + i\,\mathrm{Im}\,z)^n$$
$$= \sum_{n=1}^{\infty}\sum_{k=0}^{n} 2\binom{n}{k}\mathrm{Re}\frac{i^{n-k}}{\zeta^n}(\mathrm{Re}\,z)^k(\mathrm{Im}\,z)^{n-k}.$$

Integrating according to the Poisson formula gives (1). ■

Proof of Proposition 1 (continued). Take an arbitrary $\lambda^0\in V$. Let $r>0$ be such that $\mathrm{cl}\,B(\lambda^0,r)\subset V$. Set $A = \sup_{x\in X,\,\lambda\in B(\lambda^0,r)}|\bar\psi(\lambda,x)|$. (We write $\bar\psi(\lambda,x) = \bar\psi_\lambda(x)$.) Then by (1), $\bar\psi(\lambda^0+\lambda,x) = \sum b_{ij}(x,\lambda^0)(\mathrm{Re}\,\lambda)^i(\mathrm{Im}\,\lambda)^j$ with $|b_{ij}(x,\lambda^0)| \leqslant A2\binom{i+j}{i}r^{-(i+j)}$. Consequently for every $k\geqslant 0$ the function

$$\eta_k(\lambda) = \int\bar\psi_\lambda\cdot(\bar\psi_\lambda\circ f^k)\,d\mu_\varphi$$

is real-analytic with the coefficients of the power series expansion at λ^0 satisfying

$$|b_{ij}^{(k)}(\lambda^0)| \leqslant A^2L_{ij}r^{-(i+j)}, \quad \text{where} \quad L_{ij} = 4\sum_{t=0}^{i}\sum_{l=0}^{j}\binom{t+l}{t}\binom{(i+j)-(t+l)}{i-t}.$$

As this estimate is independent of k, all η_k are defined on a common neighbourhood of λ^0 in $\mathbf{C}^2$, $\{(\lambda_1,\lambda_2)\colon |\lambda_1-\lambda_1^0|+|\lambda_2-\lambda_2^0|<r\}$.

For $\lambda\in V$ (in $\mathbf{R}^2$) the series

$$\sigma_\lambda^2 = \eta_0(\lambda) + 2\sum_{k=1}^{\infty}\eta_k(\lambda) \tag{2}$$

is uniformly (exponentially fast) convergent because of the uniform Hölder continuity of $\bar\psi_\lambda$ on X. To know that σ_λ^2 is real-analytic one wants to know that the convergence of (2) holds also for every λ in a neighbourhood of V in $\mathbf{C}^2$. It is sufficient to know that $\bar\psi_\lambda$ is Hölder continuous on X.

We have

$$G_\lambda \equiv \bar\psi(\lambda^0+\lambda,x) - \bar\psi(\lambda^0+\lambda,y) < K|x-y|^s$$

for every $\lambda = (\lambda_1,\lambda_2)\in B(\lambda^0,r)\subset V$ and $x,y\in X$. With the help of Lemma 2 we deduce that

$$G(\lambda_1,\lambda_2) = \sum b_{ij}(\lambda^0,x,y)\lambda_1^i\lambda_2^j \quad \text{with} \quad |b_{ij}(\lambda^0,x,y)| \leqslant K|x-y|^s 2\binom{i+j}{i} r^{-(i+j)}.$$

So $G(\lambda_1,\lambda_2) \leqslant 2K(1-\hat r/r)^{-1}|x-y|^s$ if $|\lambda_1-\lambda_1^0|+|\lambda_2-\lambda_2^0| \leqslant \hat r < r$. ■

Remark 7. The real-analyticity part of this proposition immediately follows from [R2], we can even omit the assumption that $\varkappa$ is constant and consider φ depending on λ (i.e. instead of $(\mathrm{h}_\lambda)_*(\mu_\varphi)$ consider on $\mathrm{h}_\lambda(X)$ the Gibbs measures for $\varphi_\lambda|_{\mathrm{h}_\lambda(X)}$ with $\varphi_\lambda(z)$ a real-analytic function of (λ, z), $\lambda \in \mathbf{R}^n$). Indeed, by [R1], p. 99 (cf. Remark 3, Part I),

$$\sigma_\lambda^2 = \frac{d^2}{dt^2} P(f, \varphi_\lambda \circ \mathrm{h}_\lambda + t\bar{\psi}_\lambda)\Big|_{t=0}, \qquad \chi_\lambda = \frac{d}{dt} P(f, \varphi_\lambda \circ \mathrm{h}_\lambda + t \log|f_\lambda' \circ \mathrm{h}_\lambda|)\Big|_{t=0},$$

and by [R2], P above is a real-analytic function of λ, t.

A reason we prove Proposition 1 instead of referring to Ruelle's theory is that Ruelle used a machinery of ζ-functions while our case is rather trivial. By the way we get concrete estimates of the coefficients of the power series expansion. Our method will give concrete estimates of σ^2 in §8.

One can reformulate Proposition 1 in the language of a coding either through a Markov partition or a geometric coding tree (g.c.t.) (or a "unified" Markovian g.c.t., see Remark 9 in §3, Part I). Let us do this for the g.c.t. case:

PROPOSITION 2. *Let f_λ be a family of holomorphic maps on a neighbourhood U of a mixing repeller X as in Proposition* 1 (*and Theorem* 4, *Part* I). *Suppose that U is small enough that for every λ, $\mathrm{cl} f_\lambda^{-1}(U) \subset \mathrm{int}\, U$, $\bigcap_{n=0}^{\infty} f_\lambda^{-n}(U) = \mathrm{h}_\lambda(X)$ and U contains no critical value for f_λ. Let $\mathcal{T}_0 = \mathcal{T}(z; \gamma^1, \ldots, \gamma^d)$ be a g.c.t. for f_0 in U. Let $z_\lambda \in U$ be a holomorphic function of λ. Then there exists a family of g.c.t.'s $\mathcal{T}_\lambda = \mathcal{T}(z_\lambda; \gamma_\lambda^1, \ldots, \gamma_\lambda^d)$ corresponding to f_λ with the vertices $z_{\lambda,n}$ continuously depending on λ. Consider an arbitrary such family $\mathcal{T}_\lambda$. Then the functions $z_{\lambda,n}$ and the family of coding maps $z_{\lambda,\infty} = \lim_{n\to\infty} z_{\lambda,n}$ are holomorphic with respect to λ. For φ an arbitrary Hölder continuous function on the shift space Σ^d the assertion of Proposition* 1 *holds, with h_λ replaced by $z_{\lambda,\infty}$ and $\mathrm{h}_\lambda(X)$ replaced by $z_{\lambda,\infty}(\Sigma^d)$.*

Proof. The function $z_{\lambda,n}(\alpha)$, for every $\alpha \in \Sigma^d$, is holomorphic by the Implicit Function Theorem applied to $f_\lambda^n(x) = z_\lambda$. Each function $z_{\lambda,\infty}$ is holomorphic by the convergence of $z_{\lambda,n}$ to $z_{\lambda,\infty}$ holding by the expanding property. The rest of the proof is the same as for Proposition 1. ■

COROLLARY 1. *Let Ω be an RB-domain with closure in $\mathbf{C}$ (maybe not Jordan!) with $f|_{\partial\Omega}$ expanding. Let f_λ: $U \to \mathbf{C}$ be a family of holomorphic maps defined on a neighbourhood U of $\partial\Omega$, holomorphically depending on λ, with $f_0 = f$ for λ in a neighbourhood of $0 \in \mathbf{C}$ small enough that h_λ exist. Denote $\mathrm{h}_\lambda(\partial\Omega)$ by $\partial\Omega_\lambda$, the corresponding simply connected domain by Ω_λ and the harmonic measure on $\partial\Omega_\lambda$ by ω_λ. Assume finally that there exists a continuous choice of Riemann maps R_λ: $\mathbf{D} \to \Omega_\lambda$ such that the mappings*

(3) $\qquad g_\lambda = R_\lambda^{-1} \circ f_\lambda \circ R_\lambda$ *are independent of* λ.

Then $(c(\omega_\lambda))^2$ is a subharmonic, real-analytic function of λ (*with domain in* $\mathbf{C}^2$ *described in Proposition* 1).

Proof. Take z_λ holomorphically depending on λ, $z_\lambda \in \Omega_\lambda$ close to $\partial\Omega_\lambda$. (The assertion of the corollary is local so we can consider λ from a neighbourhood of 0 small enough that z_λ is constant, independent of λ.) Then there exists a family of trees $\mathcal{T}_\lambda = \mathcal{T}(z_\lambda; \gamma_\lambda^1, \ldots, \gamma_\lambda^d)$ satisfying the conditions of Proposition 2. Consider the preimage family $\mathcal{T}(\hat{z}_\lambda; \hat{\gamma}_\lambda^1, \ldots, \hat{\gamma}_\lambda^d)$ in $\mathbf{D}$ for $g = g_\lambda$, where $\hat{z}_\lambda = R_\lambda^{-1}(z_\lambda)$, $\hat{\gamma}_\lambda^j = R_\lambda^{-1}(\gamma_\lambda^j)$. By the continuity of $R_\lambda(z)$, the $\hat{z}_{\lambda,n}$ continuously depend on λ, so the coding maps $\hat{z}_{\lambda,\infty}$ are independent of λ (because the diameters of $\hat{z}_{\lambda,n}$ shrink to 0 as g^{-n} contract exponentially). Denote $\hat{z}_{\lambda,\infty}$ by $\hat{z}_\infty$. Take $\varphi = -\log|g'| \circ \hat{z}_\infty$. Observe now that $(\hat{z}_\infty)_*(\mu_\varphi) = l'$ (recall that l' is a g-invariant measure on $\partial\mathbf{D}$ equivalent to the length measure). We have $R_\lambda \circ \hat{z}_{\lambda,n} = z_{\lambda,n}$ so $R_\lambda \circ \hat{z}_\infty = z_{\lambda,\infty}$ so $(z_{\lambda,\infty})_*(\mu_\varphi) = (R_\lambda)_*(l')$ which is equivalent to the harmonic measure ω_λ. In particular, $\varkappa = \mathrm{HD}\big((z_{\lambda,\infty})_*(\mu_\varphi)\big) = 1$ (is independent of λ). Thus by Proposition 2, $\big(c\big((z_{\lambda,\infty})_*(\mu_\varphi)\big)\big)^2 = \big(c(\omega_\lambda)\big)^2$ is a subharmonic real-analytic function of λ.

Remark 8. If $\partial\Omega$ is a Jordan curve then Corollary 1 follows from Proposition 1. Namely from the fact that $R_\lambda R_0^{-1}$ conjugates f_0 with f_λ in Ω we can deduce the same on $\partial\Omega$, more precisely that $R_\lambda R_0^{-1} = \mathrm{h}_\lambda$. This can be done as in [DH], Exposé VIII, Proposition 3 (Stabilité). The above g.c.t. method corresponds to Douady–Hubbard's idea.

EXAMPLE. Consider a complex-analytic family of polynomials f_λ ($\lambda \in \mathbf{C}$) such that for every f_λ forward orbits of all critical points (except ∞) converge to periodic sinks different from ∞. It is known (P. Fatou, G. Julia) that in this case Ω_λ, the basin of attraction to ∞, is connected, simply connected and f_λ is expanding on $\partial\Omega_\lambda$. So by Corollary 1, $\big(c(\omega_\lambda)\big)^2$ on $\partial\Omega_\lambda$ is a subharmonic, real-analytic function of λ. One takes $R_\lambda(z) = \lim_{n\to\infty} f_\lambda^{-n}(z^{d^n})$ (an appropriate branch of f_λ^{-n}) and obtains $g(z) = z^d$ ($d = \deg f_\lambda$).

Let R_λ be a continuous family of Riemann maps R_λ: $\mathbf{D} \to \Omega_\lambda$, where the Ω_λ are RB-domains with closures in $\mathbf{C}$ for a holomorphic family of maps f_λ expanding on $\partial\Omega_\lambda$ and the conjugacies h_λ: $\partial\Omega_0 \to \partial\Omega_\lambda$ as in Corollary 1. Then the following fact helps to understand the meaning of the condition (3):

PROPOSITION 3. *The following conditions are equivalent*:

(α) *the maps $g_\lambda = R_\lambda^{-1} \circ f_\lambda \circ R_\lambda$ are independent of λ,*

(β) *h_λ is the boundary function of $R_\lambda \circ R_0^{-1}$,*

(γ) *R_γ holomorphically depends on λ.*

Before proving this recall that each g_λ extends holomorphically beyond $\partial\mathbf{D}$ and that we denote this extension by the same symbol g_λ (cf. Introduction).

Observe that the continuity of the family R_λ implies the continuity of g_λ. Indeed, $R_\lambda(z)$ continuous with respect to z, λ implies the continuity of $g_\lambda(z)$ with respect to λ for $z \in \mathbf{D}$, hence by the Symmetry Principle the continuity for $z \in \mathbf{C} \setminus \mathrm{cl}\,\mathbf{D}$, hence by the Cauchy Formula for $z \in \partial\mathbf{D}$.

Another observation is that

$$z_{\lambda,\infty} = \mathrm{h}_\lambda \circ z_{0,\infty} \tag{4}$$

for a choice of g.c.t.'s $\mathscr{T}_\lambda$ as in the proof of Corollary 1.

Indeed, $z_{\lambda,\infty}$ gives a factorization of (Σ^d, s) to $(\partial\Omega_\lambda, f_\lambda)$ so each trajectory $s^n(\alpha)$ is mapped under $z_{\lambda,\infty}$ to a trajectory $z_{\lambda,\infty}(s^n(\alpha))$ for f_λ, moving continuously with λ (because $z_{\lambda,\infty}(\beta)$ for $\beta\in\Sigma^d$ is a family of holomorphic functions of λ by the proof of Corollary 1, equicontinuous because uniformly bounded). The only choice is $\mathrm{h}_\lambda\big(z_{0,\infty}(s^n(\alpha))\big)$.

Remark that if we interpret $z_{\lambda,n}$ as a "dynamics Riemann map" then the analogues of (α)–(γ) are automatically true: g_λ translates to s on Σ^d so (α) translates to $z_{\lambda,n}\circ s = f_\lambda \circ z_{\lambda,n+1}$, ($\beta$) to (4), ($\gamma$) to the holomorphy of $z_{\lambda,n}$.

Proof of Proposition 3, $(\alpha)\Rightarrow(\gamma)$. Consider the trees $\mathscr{T}_\lambda$ in Ω_λ and $\mathscr{T}(\hat z_\lambda;\ \hat\gamma_\lambda^1, \ldots, \hat\gamma_\lambda^d)$ in $\mathbf{D}$ as in the proof of Corollary 1. Let $\zeta\in\partial\mathbf{D}$. We have $\zeta = \hat z_\infty(\alpha)$ for an $\alpha\in\Sigma^d$. Next

$$R_\lambda(\zeta) = z_{\lambda,\infty}(\alpha) = \mathrm{h}_\lambda\big(z_{0,\infty}(\alpha)\big) = \mathrm{h}_\lambda\big(R_0(\zeta)\big),$$

which is holomorphic with respect to λ. (The first and third equalities follow from definitions. The second one is (4).) Now with the use of Cauchy's Formula we prove the holomorphic dependence of $R_\lambda(z)$ on λ for $z\in\mathbf{D}$ (cf. [PoRo], Th. 5). (By the way we proved $(\alpha)\Rightarrow(\beta)$.)

$(\gamma)\Rightarrow(\beta)$. This follows from Mañé–Sad–Sullivan's λ-lemma [MSS] in view of which the holomorphic motions $R_\lambda R_0^{-1}$ and h_λ are compatible (cf. [PoRo], Th. 4).

$(\beta)\Rightarrow(\alpha)$. We have on $\partial\mathbf{D}$, $R_\lambda\circ g_\lambda = f_\lambda\circ R_\lambda$. Then $\mathrm{h}_\lambda^{-1}R_\lambda g_\lambda = \mathrm{h}_\lambda^{-1} f_\lambda R_\lambda = f_0 \mathrm{h}_\lambda^{-1} R_\lambda$, so $R_0 g_\lambda = f_0 R_0$. But for every $\zeta\in\partial\mathbf{D}$ the R_0-preimage of $f_0 R_0(\zeta)$ contains no arc (Fatou's theorem) so $g_\lambda(\zeta)$ is constant. Thus $g_\lambda = g_0$ on $\partial\mathbf{D}$ hence by the analyticity $g_\lambda = g_0$ in $\mathbf{D}$.

($(\gamma)\Rightarrow(\alpha)$ is immediately visible directly. Indeed, $g_\lambda(z)$ complex-analytic with respect to λ for every $z\notin\partial\mathbf{D}$ implies the same for $z\in\partial\mathbf{D}$. But for $z\in\partial\mathbf{D}$, $g_\lambda(z)\in\partial\mathbf{D}$, which is a nowhere dense set, hence $g_\lambda(z)$ is constant.) ∎

Remark 9. From the improved λ-lemma ([SuT], [BeR]) it follows that the only holomorphic motion $\partial\Omega_0\to\partial\Omega_\lambda$ is h_λ (see [PoRo], Th. 1). A question arises: are all holomorphic curves $\lambda\to z(\lambda)$ with $z(\lambda)\in\partial\Omega_\lambda$ of the form $\mathrm{h}_\lambda(z_0)$?

Remark 10. Of course the condition (3) in Corollary 1 (or (α)–(γ)) may not be satisfied. Take for example Ω_λ to be the basin of attraction to 0 for iteration of $z^2+\lambda z$. If the condition (3) is skipped in Corollary 1, we do not know whether the assertion $\big(c(\omega_\lambda)\big)^2$ is real-analytic is true in general. In particular, we do not know this for the above example.

8. (Appendix). Estimates of the asymptotic variance in the case of the boundary of the basin of attraction to ∞ for the iteration of $z\mapsto z^2+a$. The aim of this section is to study σ_a^2 for Ω_a the basin of attraction to ∞ for the polynomial $f_a(z) = z^2+a$, a inside the "cardioid" $\big(-(\lambda/2)^2+\lambda/2,\ |\lambda|=1\big)$ bounded region $\mathring{M}\subset\mathbf{C}$. Recall (§7) that

$$\sigma_a^2 = \lim_{n\to\infty}\frac{1}{n}\cdot\frac{1}{2\pi}\int_{\partial\mathbf{D}}\Big(\sum_{k=0}^{n}\tilde\psi_a(g^k(\zeta)\Big)^2\,d|\zeta| = \eta_0(a) + 2\sum_{k=1}^{\infty}\eta_k(a) \tag{1}$$

where $\eta_k(a) = (2\pi)^{-1}\int\tilde\psi_a(\zeta)\tilde\psi_a(g^k(\zeta))\,d|\zeta|$, $g(\zeta)=\zeta^2$, $\tilde\psi_a(\zeta) = \tilde\psi(a,\zeta) = \log|f_a'(\mathrm{h}_a(\zeta))| - \log 2 = \log|\mathrm{h}_a(\zeta)|$. Here h_a is the conjugacy between z^2 on $\partial\mathbf{D} = \partial\Omega_0$ and f_a on the quasi-circle $\partial\Omega_a$.

First let us compute the quadratic part of the power series expansion of σ_a^2 at $a=0$ (analogously to Ruelle's $\mathrm{HD}(\partial\Omega_a)$, [R2]). We shall do this for $f_a(z) = z^q + a$, $q\leqslant 2$, keeping the same notation as for z^2+a.

Proposition 4. *For $f_a(z) = z^q+a$, $\sigma_a^2 = |a|^2/2 + O(|a|^3)$.*

(Remark that this is independent of q, unlike Ruelle's $\mathrm{HD}(\partial\Omega_a) = |a|^2/(4\log q) + O(|a|^3)$.)

Proof. (It mostly repeats Ruelle's arguments but we give it complete for the convenience of the reader.) First compute $\zeta_a = \mathrm{h}_a(\zeta)$ for every $\zeta\in\partial\mathbf{D}$. We have $\zeta_a = \lim_{n\to\infty} f_a^{-n}(\zeta^{q^n})$ (for the appropriate branches of f_a^{-n}), so

$$\left.\frac{d\zeta_a}{da}\right|_{a=0} = \lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{d(f_0^{-k})}{dz}(\zeta^{q^k})\cdot\frac{df_0^{-1}}{dz}(\zeta^{q^{k+1}}-a)\cdot(-1)$$

$$= -\lim_{n\to\infty}\sum_{k=0}^{n-1}\frac{df_0^{-(k+1)}}{dz}(\zeta^{q^{k+1}}) = \sum_{k=0}^{\infty}\frac{1}{q^{k+1}}\zeta^{1-q^{k+1}},$$

$$\zeta_a = \zeta - \zeta\left(\sum_{n=1}^{\infty}\frac{1}{q^n}(\zeta^{-1})^{q^n}\right)a + O(|a|^2),$$

$$\tilde\psi_a(\zeta) = (q-1)\log\left(|\zeta|\cdot\left|1-\left(\sum_{n=1}^{\infty}\frac{1}{q^n}(\zeta^{-1})^{q^n}\right)a + O(|a|^2)\right|\right)$$

$$= (q-1)\operatorname{Re}\left(\sum_{n=1}^{\infty}\frac{1}{q^n}(\zeta^{-1})^{q^n}a\right) + O(|a|^2),$$

$$S_{a,N} = \sum_{k=0}^{N-1}\tilde\psi_a(\zeta^{q^k}) = (q-1)\sum_{n=1}^{\infty}t_n\operatorname{Re}\big(a(\zeta^{-1})^{q^n}\big) + O(|a|^2),$$

where

$$t_n = \begin{cases}\dfrac{(1/q)\big(1-(1/q)^n\big)}{1-1/q} & \text{for } n\leqslant N,\\[2ex] \dfrac{(1/q)^{n-N+1}\big(1-(1/q)^N\big)}{1-1/q} & \text{for } n>N.\end{cases}$$

To compute $\sigma_a^2 = \lim_{n\to\infty}(2\pi N)^{-1}\int|S_{a,N}|^2\,d|\zeta|$ observe that

$$(2\pi)^{-1}\int\operatorname{Re}\big(a(\zeta^{-1})^{q^n}\big)\operatorname{Re}\big(a(\zeta^{-1})^{q^m}\big)\,d|\zeta| = \begin{cases}0 & \text{if } n\neq m,\\ |a|^2/2 & \text{if } n=m.\end{cases}$$

Thus

$$\sigma_a^2 = \lim_{N\to\infty} (2\pi N)^{-1} \int \sum_{n=0}^{N-1} \big((q-1)t_n \operatorname{Re} a(\zeta^{-1})^{q^n}\big)^2 \, d|\zeta|$$
$$+ \lim_{N\to\infty} (2\pi N)^{-1} \int \sum_{n=N}^{\infty} \big((q-1)t_n \operatorname{Re} a(\zeta^{-1})^{q^n}\big)^2 \, d|\zeta|.$$

The second summand is obviously 0, the first one is equal to $|a|^2/2$. ∎

Now it would be desirable to compute all other coefficients of the power series expansion of σ_a^2 at $a=0$; we hope to do this in future (something is done in that direction for $\mathrm{HD}(\partial\Omega_a)$, see [WBKS]). Here we shall only estimate these coefficients from above using the methods from §7. This together with the knowledge about the quadratic part of σ_a^2 (Proposition 4) allows us to give some estimates of σ_a^2 from below.

We restrict the considerations to $\{|a| < 1/4\}$, the largest disc in $\mathring{M}$ with centre at 0. (Considering the family $z^2+\lambda z$, $|\lambda| < 1$, did not lead us to better estimates.)

One can check that if $A_r - \varepsilon \le |z| < A_r$ or $A'_r < |z| \le A'_r + \varepsilon$ for $A_r = (1+\sqrt{1-4r})/2$, $A'_r = (1+\sqrt{1+4r})/2$, $\varepsilon > 0$ small enough, then for $|a| = r$, $|f_a(z)| < |z|$ or $|f_a(z)| > |z|$ respectively. So

$$\partial\Omega_a \subset P_r = \{z\colon A_r \le |z| \le A'_r\}. \tag{2}$$

It is crucial that $r < 1/4$ implies $|f'_a| \ge 2A_r > 1$ on P_r.

Define

$$\bar{\Psi}(r) = \sup\{|\bar{\Psi}_a(\zeta)|\colon |a| \le r,\ \zeta \in \partial\mathbf{D}\},$$
$$\operatorname{var}\bar{\Psi}(r) = \sup\{|\bar{\Psi}_a(\zeta_1) - \bar{\Psi}_a(\zeta_2)|\colon |a| \le r,\ \zeta_1,\zeta_2 \in \partial\mathbf{D}\}.$$

Let $\mathscr{B}^n$ denote the partition of $\partial\mathbf{D}$ into arcs $\pi k/2^n \le \operatorname{Arg}\alpha \le \pi(k+1)/2^n$, $n = 0, 1, \ldots$ For an arbitrary $n \ge 0$ set

$$\Delta_n(r) = \sup\{|\bar{\Psi}_a(\zeta_1) - \bar{\Psi}_a(\zeta_2)\colon |a| \le r,\ \zeta_1,\zeta_2 \in B \in \mathscr{B}^n\}.$$

LEMMA 3. (i) $\bar{\Psi}(r) \le \log(1/A_r)$, $\operatorname{var}\bar{\Psi}(r) \le \log(A'_r/A_r)$.

(ii) $\Delta_n(r) \le \min\big(\log(A'_r/A_r),\ \log(1 + T_{r,n}(2A_r)^{-n}/A_r)\big)$, *where*

$$T_{r,n} = \min_{0\le k\le n} \sqrt{(\pi\cdot 2^{-k} + 2\alpha_r)^2 A_r'^2 + (A'_r - A_r)^2}\cdot(2A_r)^k, \qquad \alpha_r = \arcsin(r/A_r).$$

Proof. (i) follows immediately from (2) and from $\log A'_r < \log(1/A_r)$. To prove (ii) we shall estimate $\operatorname{dist}_{P_r}(\mathrm{h}_a(\zeta_1), \mathrm{h}_a(\zeta_2))$ for ζ_1, ζ_2 belonging to the same $B \in \mathscr{B}^k$. Here $|a| = r$ and $\operatorname{dist}_{P_r}(\ldots)$ denotes the length of the shortest curve joining $\mathrm{h}_a(\zeta_1)$ to $\mathrm{h}_a(\zeta_2)$ inside the annulus P_r, homotopic to $\gamma_1 * \gamma_2 * \gamma_3$ where γ_2 joins ζ_1 to ζ_2 in B, $\gamma_1(t) = \mathrm{h}_{ta}(\zeta_1)$, $\gamma_2(t) = \mathrm{h}_{(1-t)a}(\zeta_2)$.

Suppose that $\alpha < \pi/2$. For every $\zeta \in \partial\mathbf{D}$ set

$$\mathscr{K}(\zeta,\alpha) = \{z\colon A_r \le |z| \le A'_r,\ \operatorname{Arg}\zeta - \alpha \le \operatorname{Arg} z \le \operatorname{Arg}\zeta + \alpha\}.$$

Observe (Fig. 5) that

$$\inf\{|z_1 - z_2|\colon \operatorname{Arg} z_1 = \operatorname{Arg}\zeta^2 + 2\alpha,\ \operatorname{Arg} z_2 = \operatorname{Arg}\zeta^2 + \alpha,\ A_r \le |z_2| \le A'_r\} = A_r \sin\alpha.$$

The analogous observation holds for $\operatorname{Arg} z_1 = \operatorname{Arg}\zeta^2 - 2\alpha$, $\operatorname{Arg} z_2 = \operatorname{Arg}\zeta^2 - \alpha$.

So if α is such that $r < A_r \sin\alpha$ then $f_a(\mathscr{K}(\zeta,\alpha)) \supset \mathscr{K}(\zeta^2,\alpha)$ with $\partial\mathscr{K}(\zeta,\alpha)$ mapped by f_a outside $\mathscr{K}(\zeta^2,\alpha)$, going once around it. (The sequence $\mathscr{K}(\zeta^{2^n},\alpha)$ forms a "telescope" for f_a, see [Su1].) Hence $\mathrm{h}_a(\zeta) \in \mathscr{K}(\zeta,\alpha)$, for $\alpha > \alpha_r = \arcsin(r/A_r)$.

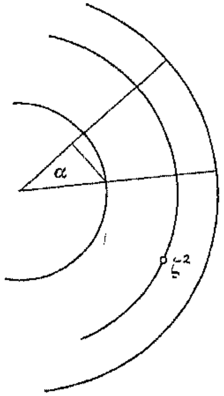

Fig. 5

So for every $\zeta_1, \zeta_2 \in B \in \mathscr{B}^k$

$$\operatorname{dist}_{P_r}(\mathrm{h}_a(\zeta_1), \mathrm{h}_a(\zeta_2)) \le \operatorname{diam}\mathscr{K}(B,\alpha_r), \tag{3}$$

where $\mathscr{K}(B,\alpha_r) = \{z\colon A_r \le |z| \le A'_r,\ \exp(i\operatorname{Arg} z) \in B\}$ and diam is measured in the euclidean metric restricted to $\mathscr{K}(B,\alpha_r)$.

We have

$$\operatorname{diam}\mathscr{K}(B,\alpha_r) \le \sqrt{(\pi 2^{-k} + 2\alpha_r)^2 A_r'^2 + (A'_r - A_r)^2}. \tag{4}$$

(This is not the best estimate but we want to have in Lemma 3 something concrete.)

Now as $|f'_a| \ge 2A_r$ on P_r, using (3) we have for every ζ_1, $\zeta_2 \in \mathscr{B}^n$ and $0 \le k \le n$

$$|\mathrm{h}_a(\zeta_1) - \mathrm{h}_a(\zeta_2)| \le (2A_r)^{k-n} \operatorname{dist}_{P_r}\big(\mathrm{h}_a(\zeta_1^{2^{n-k}}), \mathrm{h}_a(\zeta_2^{2^{n-k}})\big)$$
$$\le (2A_r)^{-n}\big(\operatorname{diam}\mathscr{K}(f_0^{n-k}(B),\alpha)\big)(2A_r)^k.$$

This in view of (4) gives (ii). ∎

Remark 11. We can give now a concrete estimate of σ_a^2 from above. Let $|a| = r < 1/4$. To use (1) observe that for $k \ge 1$

$$\eta_k(a) = (2\pi)^{-1}\int (\bar{\psi}_a - E(\bar{\psi}_a|\mathcal{B}^{k-1}))\cdot(\bar{\psi}_a\circ g^k)(\zeta)\,d|\zeta| \leqslant \tfrac{1}{2}\Delta_{k-1}(r)\bar{\psi}(r).$$

($E(\cdot|\cdot)$ is the conditional expectation value.)

So

$$\sigma_a^2 \leqslant \bar{\psi}(r)^2 + \bar{\psi}(r)\sum_{n=0}^{\infty}\Delta_n(r). \tag{5}$$

One computes for example:

$r =$ 0.2	0.205	0.21	0.215	0.22	0.225	0.23	0.235	0.24	0.245
$\sigma^2 \leqslant$ 1.504	1.701	1.924	2.196	2.527	2.955	3.524	4.35	5.73	8.62

Of course when $|a| \to 1/4$ the estimate tends to ∞, so it gets worthless because of the universal estimate $\sigma^2 \leqslant 18\log 2$ (which follows from $c(\omega) = \sqrt{2\sigma^2/\chi}$, $\chi = \log 2$ and $c(\omega) \leqslant 6C$ where C is the universal Makarov constant $C \leqslant 1$, [Po]).

Now the time is to use Lemma 2, §7. For every $\zeta \in \partial\mathbf{D}$, $a = (a_1, a_2) \in \mathbf{C}^2$, $|a_1|+|a_2| < 1/4$, $n \geqslant 0$ we obtain $\bar{\psi}_a(\zeta) = \sum_{i,j=0}^{\infty} b_{i,j}(\zeta)a_1^i a_2^j$, where

$$b_{00} = 0 \equiv M_{00}, \quad |b_{ij}| \leqslant \inf_{0<r\leqslant 1/4} \operatorname{var}\bar{\psi}(r)\cdot\binom{i+j}{i} r^{-(i+j)} \equiv M_{ij}. \tag{6}$$

(We got rid of the coefficient 2 due to the replacement of $u = \bar{\psi}_a(\zeta)$ by $u - \frac{1}{2}(\sup_{|a|\leqslant r} u + \inf_{|a|\leqslant r} u)$.) We have

$$\bar{\psi}_a(\zeta, n) \equiv \bar{\psi}_a(\zeta) - E(\bar{\psi}_a|\mathcal{B}^n)(\zeta) = \sum_{i,j=0}^{\infty} b_{ijn}(\zeta)a_1^i a_2^j,$$

where

$$b_{00n} = 0 \equiv M_{00}^{(n)}, \quad |b_{ijn}| \leqslant \inf_{0<r<1/4} \Delta_n(r)\binom{i+j}{i} r^{-(i+j)} \equiv M_{ij}^{(n)}. \tag{7}$$

Thus, writing $\bar{\psi}_a(\zeta)$ and $\bar{\psi}_a(\zeta, n)$ in the power series expansion forms (6), (7), multiplying, integrating along $\partial\mathbf{D}$ and summing according to the formula

$$\sigma_a^2 = (2\pi)^{-1}\int(\bar{\psi}_a(\zeta))^2\,d|\zeta| + 2\sum_{n=1}^{\infty}(2\pi)^{-1}\int\bar{\psi}_a(\zeta)(\bar{\psi}_a - E(\bar{\psi}_a|\mathcal{B}^{n-1}))\,d|\zeta|$$

we arrive at

PROPOSITION 5. *$\sigma_a^2 = \sum_{i,j=0}^{\infty} c_{ij}a_1^i a_2^j$, where*

$$|c_{ij}| \leqslant \sum_{r=0}^{i}\sum_{s=0}^{j} M_{rs}\Big(M_{i-r,j-s} + 2\sum_{n=0}^{\infty} M_{i-r,j-s}^{(n)}\Big) \equiv N_{ij} \tag{8}$$

and M_{rs}, $M_{rs}^{(n)}$ are given by (6), (7).

Remark 12. Observe that $N_{ij} = N_{ji}$ and that $c_{j0} = 0$ for every j odd. The first assertion follows from the definitions, the latter from the fact that $f_a(z) = \overline{f_{\bar{a}}(\bar{z})}$, hence $\sigma_a^2 = \sigma_{\bar{a}}^2$.

COROLLARY 2. *For every $a \in \mathring{M} \subset \mathbf{C} = \mathbf{R}^2$*

$$\sigma_a^2 \geqslant |a|^2/2 - \sum_{i=3}^{\infty} N_{i0}|a|^i. \tag{9}$$

(This is valuable only if $|a|$ is small enough, otherwise the right-hand side is negative or even divergent.)

Proof. For $a = (a_1, 0)$, (9) immediately follows from Propositions 4 and 5. To cope with an arbitrary a we should suitably rotate the coordinates on $\mathbf{R}^2$ before writing the power series expansions. ∎

COROLLARY 3. *For a imaginary* ($a_1 = 0$)

$$\sigma_a^2 \geqslant |a|^2/2 - \sum_{i=2}^{\infty} N_{2i,0}|a|^{2i}. \tag{10}$$

Proof. Use Propositions 4 and 5 and Remark 12. ∎

Remark 13. Leszek Zdunik computed (with the help of PC) that in Lemma 3 the minimum in $T_{r,n}$ is taken for $k = 1$ if $r \leqslant 0.13$, for $k = 2$ if $r \leqslant 0.255$, and for $k = 3$ if $r \leqslant 0.2425$. (Of course $k \to \infty$ if $r \to 0.25$.) The integer n for which the second term in min in the definition of $\Delta_n(r)$ gets less than the first one also goes to ∞ if $r \to 0.25$ and is for example equal to 6, 6, 7, 8, 9, 11 for $r = 0.2$, 0.215, 0.225, 0.235, 0.24, 0.245 respectively.

For each $M_{ij}^{(n)}$ and M_{ij} the infimum over r in (7) and (6) is taken for a different $r = r(n, i, j)$, $r(i, j)$ respectively (for example for $n = 0$, $i = 1, 2, \ldots$, we have $r = 0$, 0.2232, 0.2393, 0.2443, 0.2464, 0.2476, … The results for $n \leqslant 5$ are the same because of the domination of the first term in min in Lemma 3(ii)).

Finally, for $i = 3, 4, 5, \ldots$, $N_{i0} \leqslant 11.959\cdot 4^3$, $22.698\cdot 4^4$, $35.308\cdot 4^5$, $49.438\cdot 4^6$, $64.89\cdot 4^7$, $81.512\cdot 4^8, \ldots$ This gives in (10) the maximum at $|a| \approx 0.00655$. Then $\sigma_a^2 \geqslant 0.00001074$ (a poor result).

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