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# GEOMETRY AND COMBINATORICS RELATED TO VECTOR PARTITION FUNCTIONS

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The aim of this talk is to give a review of some recent results of geometric and combinatorial nature arising from the study of partition functions associated to a finite family of vectors in a real finite-dimensional vector space (cf. [1–5]). These results are closely related to the general theory of hypergeometric functions developed by I. M. Gelfand and his coworkers but in order to make the exposition more elementary we shall not speak of this theory here.

## 1. Definition of vector partition function

Consider $\mathbf{R}^n$ and the "positive octant" $\mathbf{R}^n_+$. Suppose we are given a $k$-dimensional vector subspace $V_0 \subset \mathbf{R}^n$ such that $V_0 \cap \mathbf{R}^n_+ = 0$. Then for any $x \in \mathbf{R}^n$ the linear variety $V_0 + x$ intersects $\mathbf{R}^n_+$ in the compact (perhaps empty) convex polyhedron $\Delta_x = (V_0 + x) \cap \mathbf{R}^n_+$. The (vector) *partition function* is defined to be the "volume" of $\Delta_x$; clearly it depends only on the image of $x$ in $V = \mathbf{R}^n/V_0$.

The "volume" may be understood in different ways which leads to various versions of the partition function. In the "discrete" version the "volume" is the number of integral points in $\Delta_x$. In other words let $e_1, \ldots, e_n$ be the standard basis in $\mathbf{R}^n$, $p$: $\mathbf{R}^n \to V$ the natural projection and $v_i = p(e_i) \in V$. Then the discrete version of partition function is

$$P(v) = |\{(m_1, \ldots, m_n) \in \mathbf{Z}^n_+ : \sum_{1 \leqslant i \leqslant n} m_i v_i = v\}|, \qquad v \in V.$$

For example if $\{v_1, \ldots, v_n\}$ is the set of positive roots of a root system in $V$ then $P(v)$ is the Kostant partition function [6] playing the fundamental role in representation theory.

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This paper is in final form and no version of it will be submitted for publication elsewhere.

We shall deal here only with the "continuous" version of the (vector) partition function defined as follows. Choose a differential $k$-form $\omega$ on $\mathbf{R}^n$ with polynomial coefficients, and compatible orientations of all the varieties $V_0+x$; the corresponding partition function is defined by

$$P_\omega(v) = \int_{\Delta_x} \omega, \qquad v = p(x) \in V.$$

In fact the continuous version (in a different form) appeared even earlier than the discrete one in the work by Berezin and Gelfand [7] (for positive root systems).

We shall show that $P_\omega(v)$ is a piecewise polynomial function on $V$. Its behaviour is controlled by two kinds of geometric structures in a certain sense dual to each other. The first is connected with the system of simplicial cones generated by subsets of the set of vectors $\{v_1, \ldots, v_n\}$ in $V$, while the second one with arrangement of the hyperplanes $\{x_i = 0\}$ on the affine $k$-space $V_0+x$ (where $x_1, \ldots, x_n$ are coordinates in $\mathbf{R}^n$).

## 2. Parametric presentation of $P_\omega(v)$ and the space $\mathscr{H}$

Let $l = n-k$ and consider the $l$-dimensional vector space $V = \mathbf{R}^n/V_0$ and the system of vectors $v_1, \ldots, v_n \in V$ introduced above. Clearly $\operatorname{rk}\{v_1, \ldots, v_n\} = l$, and the property $V_0 \cap \mathbf{R}^n_+ = 0$ is equivalent to the fact that all $v_1, \ldots, v_n$ lie on one side of some hyperplane in $V$ passing through the origin (i.e. that $\langle p, v_i\rangle > 0$ for $1 \leqslant i \leqslant n$ and some $p \in V^*$).

Let $V' \subset V$ be the open set of vectors $v$ which are in general position with respect to $\{v_1, \ldots, v_n\}$, i.e. $V'$ is the complement in $V$ of the union of all proper subspaces spanned by some of the vectors $v_1, \ldots, v_n$. For any subset $I \subset [1, n]$ let $C_I$ denote the closed convex cone generated by $\{v_i : i \in I\}$. We are especially interested in *independent* subsets $I$, i.e. such that the corresponding vectors $\{v_i : i \in I\}$ are linearly independent. Clearly, for an independent $I$ the cone $C_I$ is simplicial, and $\dim C_I = |I|$. A maximal independent subset will be called a *base*. Clearly, each base $I$ is of cardinality $l$, and we denote by $\chi_I$ the characteristic function of the open set $(C_I \cap V') \subset V$.

THEOREM 1 ([1], [2]). *For $v \in V'$ the function $P_\omega(v)$ can be decomposed as*

$$P_\omega(v) = \sum_I P^I_\omega(v)\chi_I(v), \tag{1}$$

*where the sum is over all bases $I \subset [1, n]$, and $P^I_\omega$ are some polynomials on $V$.*

By Theorem 1, $P_\omega$ is piecewise polynomial. More precisely, let $v, v' \in V'$ be equivalent if $\chi_I(v) = \chi_I(v')$ for any base $I$ (in other words, either $v, v' \in C_I$ or $v, v' \notin C_I$). Obviously, the equivalence classes are open in $V'$, and $P_\omega$ is polynomial on each of these open subsets.

The decomposition (1) is not unique. Consider the simplest example: $n = 3$, $l = 2$, $V = \mathbf{R}^2$ with standard basis $\varepsilon_1, \varepsilon_2$ and coordinates $x_1, x_2$, $v_1 = \varepsilon_1$, $v_2 = \varepsilon_2$, $v_3 = \varepsilon_1 + \varepsilon_2$, and $\omega$ a suitably normalized 1-form on $\mathbf{R}^3$ with constant coefficients. Then it is easy to see that $P_\omega(x_1, x_2) = \min(x_1, x_2)$. It can be written in form (1) in several different ways, e.g.

$$P_\omega(x_1, x_2) = x_2 \chi_{13} + x_1 \chi_{23} = x_1 \chi_{12} + (x_2 - x_1)\chi_{13} = x_2 \chi_{12} + (x_1 - x_2)\chi_{23}.$$

This non-uniqueness is of course due to the fact that the functions $\chi_I$ are linearly dependent, viz., we have the obvious relation $\chi_{12} = \chi_{13} + \chi_{23}$.

Returning to the general case, let $\mathscr{H}$ denote the vector space of piecewise constant functions on $V'$ spanned by all $\chi_I$ corresponding to various bases $I \subset [1, n]$. By (1), $P_\omega$ can be thought of as an element of $\mathscr{H} \otimes_{\boldsymbol{C}} \mathrm{Pol}(V)$, where $\mathrm{Pol}(V)$ is the ring of polynomial functions on $V$. It is clear now that the following two problems are essential for description of $P_\omega$:

(A) Find a complete system of linear relations between the functions $\chi_I$.

(B) Construct some bases of $\mathscr{H}$ (in order to make the decomposition (1) unique).

Both problems will be solved explicitly but before doing this we describe several other realizations of $\mathscr{H}$.

## 3. Various realizations of $\mathscr{H}$

The original definition of $\mathscr{H}$ will be referred to as *geometric*. Now we give several other equivalent definitions.

ALGEBRAIC DEFINITION. Let us think of $v_i$ as linear forms on the dual space $V^*$. For any base $I \subset [1, n]$ define the rational function $f_I$ on $V^*$ by $f_I = (\prod_{i \in I} v_i)^{-1}$. Let $\mathscr{H}^{\mathrm{alg}}$ be the vector space of rational functions on $V^*$ spanned by all $f_I$. Choose a non-zero $l$-form $\omega_0 \in \Lambda^l(V^*)$.

PROPOSITION 1. [1]. *There is an isomorphism $\mathscr{H} \to \mathscr{H}^{\mathrm{alg}}$ sending each $\chi_I$ to $|\omega_0(\bigwedge_{i \in I} v_i)| \cdot f_I$.*

In fact this isomorphism is given by the Laplace transform:

$$\chi \mapsto f(p) = \int_V \chi(v) e^{-\langle p, v \rangle} \cdot \omega_0(v).$$

TOPOLOGICAL DEFINITION. Put $\mathscr{H}^{\mathrm{top}} = H^l(V_{\boldsymbol{C}}^* \setminus \bigcup_{1 \leqslant i \leqslant n} v_i^\perp, \boldsymbol{C})$.

This cohomology group was determined by Arnold [8] and Brieskorn [9]. Their results can be stated as follows:

PROPOSITION 2. *There is an isomorphism $\mathscr{H}^{\mathrm{alg}} \otimes \Lambda^l(V) \to \mathscr{H}^{\mathrm{top}}$ sending each element $f \otimes \omega$ to the cohomology class of the closed differential $l$-form $f\omega$ on $V_{\boldsymbol{C}}^* \setminus \bigcup_i v_i^\perp$.*

Combining Propositions 1 and 2 we get an explicit isomorphism $\mathscr{H} \simeq \mathscr{H}^{\mathrm{top}}$:

PROPOSITION 3. *There is an isomorphism $\mathscr{H} \to \mathscr{H}^{\mathrm{top}}$ sending each $\chi_I$ to the cohomology class of*

$$\varepsilon(v_{i_1}, \ldots, v_{i_l}) \cdot \frac{dv_{i_1}}{v_{i_1}} \wedge \ldots \wedge \frac{dv_{i_l}}{v_{i_l}}$$

*where $I = \{i_1, \ldots, i_l\}$ is a base, and $\varepsilon$ is a fixed orientation of $V$.*

COMBINATORIAL DEFINITION. It makes sense in the more general context of arbitrary matroids. Recall that a *matroid* structure on a finite set (say, $[1, n]$) is given by specifying a collection $\mathscr{I}$ of subsets of $[1, n]$ called *independent subsets*, which must satisfy the following axioms:

(I1) $\varnothing \in \mathscr{I}$; (I2) If $I' \subset I$ and $I \in \mathscr{I}$ then $I' \in \mathscr{I}$;

(I3) (*Exchange axiom*). If $I_1, I_2 \in \mathscr{I}$ and $|I_1| < |I_2|$ then $(I_1 \cup \{i\}) \in \mathscr{I}$, for some $i \in I_2 \backslash I_1$.

Clearly the matroid axioms are satisfied for the system of independent subsets associated as above to a family of vectors $v_1, \ldots, v_n$ (in a vector space over any field), i.e. in the case when $\mathscr{I}$ consists of all subsets $I$ such that $\{v_i \colon i \in I\}$ are linearly independent. This example suggests using for arbitrary matroids the terminology of linear algebra such as *dependent subsets*, *bases* (maximal independent subsets), *rank function* defined on all subsets of $[1, n]$ (for $J \subset [1, n]$ $r(J)$ is the cardinality of (any) maximal independent subset $I \subset J$) etc. The remarkable feature of the matroid theory is that any of these (and many other) notions satisfies natural properties which can be used for equivalent axiomatic definitions of a matroid. Clearly, in every matroid all bases are of the same cardinality called *rank* of the matroid. Basic facts about matroids can be found in [10].

Now suppose we have a matroid on $[1, n]$ of rank $l$. Let $E_m = \Lambda^m(C^n)$, and $E = \bigoplus_m E_m$ be the Grassmann algebra of $C^n$. For each $I \subset [1, n]$ denote by $E_I$ the one-dimensional space $C \cdot (\bigwedge_{i \in I} e_i)$ (where $e_1, \ldots, e_n$ is a standard basis in $C^n$), so $E_m = \bigoplus_{|I| = m} E_I$. Let $d \colon E \to E$ be the (super) derivation sending each $e_i$ to 1 (so $d(E_m) \subset E_{m-1}$ for all $m$). Put $B_m = \bigoplus E_I$, the sum over all $I \subset [1, n]$ such that $|I| = m$, $r(I) < l$; let $C_m = E_m / B_m$. Evidently, $C_m = 0$ for $m < l$. It is also clear that $d(B_m) \subset B_{m-1}$ so $d$ induces the mapping $d \colon C_m \to C_{m-1}$. We associate with our matroid the vector space $\mathscr{H}^{\mathrm{comb}} = C_l / dC_{l+1} = E_l / (B_l + dE_{l+1})$.

Suppose now that our matroid is associated as above with the family of vectors $v_1, \ldots, v_n$ in a $l$-dimensional vector space $V$. The following result is essentially proven in [11].

PROPOSITION 4. *The homomorphism* $E_l \to \mathscr{H}^{\mathrm{top}} = H^l(V_{\mathbf{C}}^* \setminus \bigcup v_i^\perp, \mathbf{C})$ *sending each* $e_{i_1} \wedge \ldots \wedge e_{i_l}$ *to the cohomology class of the form* $\frac{dv_{i_1}}{v_{i_1}} \wedge \ldots \wedge \frac{dv_{i_l}}{v_{i_l}}$ *is epimorphic with kernel* $(B_l + dE_{l+1})$ *so it induces an isomorphism* $\mathscr{H}^{\mathrm{comb}} \simeq \mathscr{H}^{\mathrm{top}}$.

Combining Propositions 3 and 4 we obtain an isomorphism $\mathscr{H} \simeq \mathscr{H}^{\mathrm{comb}} = C_l/dC_{l+1}$. This immediately gives us a complete system of linear relations between the functions $\chi_I$, which is indexed by subsets $J \subset [1, n]$ with $|J| = l+1$ and $r(J) = l$. This system will be given explicitly in the next section. In fact, the whole chain of syzygies is also available:

PROPOSITION 5. *For every matroid the sequence*

$$0 \to C_n \xrightarrow{d} C_{n-1} \xrightarrow{d} \ldots \xrightarrow{d} C_l \to \mathscr{H}^{\mathrm{comb}} \to 0$$

*is exact.*

This follows from another interpretation of $\mathscr{H}^{\mathrm{comb}}$ as the unique non-zero homology group of the geometric lattice associated to a matroid (see [12]).

## 4. Linear relations in $\mathscr{H}$ and Björner's theorem

Let $J \subset [1, n]$ be a subset such that $|J| = l+1$, $r(J) = l$. Then there is only one (up to a factor) linear relation $\sum_{j \in J} a_j v_j = 0$. Let

$$J_+ = \{j \in J\colon a_j > 0\}, \quad J_- = \{j \in J\colon a_j < 0\}, \quad J_0 = \{j \in J\colon a_j = 0\}.$$

Clearly, these subsets are determined uniquely up to interchanging $J_+$ and $J_-$. Note also that $J_+ \cup J_- = \{j \in J\colon J \setminus \{j\}$ is independent$\}$, and that our condition that $v_1, \ldots, v_n$ lie in a halfspace implies that $J_+, J_- \neq \emptyset$.

THEOREM 2. *For any subset* $J \subset [1, n]$ *with* $|J| = l+1$, $r(J) = l$, *the following relation holds:*

$$\sum_{j \in J_+} \chi_{J \setminus j} = \sum_{j \in J_-} \chi_{J \setminus j}. \tag{2}$$

*Moreover, the relations* (2) *form a complete system of linear relations between the elements* $\chi_I$ *in* $\mathscr{H}$.

The relation (2) has simple geometric meaning: both sides are equal to the characteristic function of $(C_J \cap V')$.

We turn now to Björner's theorem which gives $\dim \mathscr{H}$ and a class of special bases of $\mathscr{H}$. Minimal dependent subsets of $[1, n]$ are called *circuits* (the terminology comes from matroids associated to graphs). A subset $J' = \{j_1, \ldots, j_m\}$ is called a *broken circuit* if $J' \cup \{j\}$ is a circuit for some $j > j_1, \ldots, j_m$.

THEOREM 3. *$\dim \mathscr{H}$ is equal to the number of bases $I$ containing no broken circuits. Moreover, the $\chi_I$ corresponding to such bases form a basis in $\mathscr{H}$.*

This theorem is essentially proven in [13] for arbitrary matroids. In fact Björner constructs a more general class of bases in $\mathscr{H}^{\mathrm{comb}}$ which will be described later. A direct proof of Theorem 3 using only Proposition 5 is given in the Appendix to [1].

We remark that the definition of a broken circuit depends on an arbitrary choice of linear ordering of $[1, n]$. So Theorem 3 can give several various combinatorial interpretations of $\dim \mathscr{H}$.

EXAMPLES 1. Let $v_1, \ldots, v_n$ be in general position in $V$. Then broken circuits are just $l$-element subsets $I \subset [1, n]$ not containing $n$. So $\dim \mathscr{H} = \binom{n-1}{l-1}$, and a basis in $\mathscr{H}$ is formed by the $\chi_I$ for bases $I \ni n$.

2. If $\{v_1, \ldots, v_n\}$ is the set of positive roots of some root system $R$ in $V$ then $\dim \mathscr{H}$ is known to be $m_1 \cdot \ldots \cdot m_l$, the $m_i$ being the *exponents* of $R$ (see [14]). For example, when $R$ is of type $A_l$, $\dim \mathscr{H} = l!$. Theorem 3 gives the following "constructive" explanation of this fact.

In this case our vectors are indexed by pairs $(i, j)$ with $1 \leqslant i < j \leqslant l+1$; we have $v_{ij} = \varepsilon_i - \varepsilon_j$, where $\varepsilon_1, \ldots, \varepsilon_{l+1}$ is the standard basis in $\mathbf{R}^{l+1}$. Order these pairs $(i, j)$ lexicographically. Then it is easy to see that an $l$-subset $J$ of $\{(i, j)\}$ contains no broken circuits if and only if for any $i = 1, \ldots, l$, $J$ contains exactly one vector of the form $v_{ij}$. Clearly, the number of such subsets is $l!$, as required.

Returning to the vector partition function we see that the decomposition (1) becomes unique if we require $P_\omega^I = 0$ for all bases $I$ containing a broken circuit.

## 5. Dual description of $P_\omega$: chambers and flags

Now we present another approach to the description of $P_\omega$ developed in [4]. By (1), $P_\omega$ is uniquely determined by the family of polynomials $P_\omega^I$, where $I \subset [1, n]$ runs over all bases. We have seen that the family $(P_\omega^I)$ is not unique. But there is a canonical way to describe $P_\omega$ in terms of other polynomials. Namely, consider connected components of $V'$, which will be called *chambers*. Evidently, $P_\omega$ is equal to some polynomial $P_\omega^\Gamma$ on each chamber $\Gamma$ and is uniquely determined by the family of polynomials $P_\omega^\Gamma$.

It is a relatively easy task to compute each separate $P_\omega^\Gamma$. But the number of chambers is usually very large compared to $\dim \mathscr{H}$, and so there is a lot of "universal" (i.e. independent of $\omega$) linear relations between the corresponding polynomials $P_\omega^\Gamma$. We shall find explicitly a complete system of these linear relations and construct some families consisting of $\dim \mathscr{H}$ "universal" linear combinations of the $P_\omega^\Gamma$'s which enable us to evaluate all $P_\omega^\Gamma$.

To be more precise, each chamber $\Gamma$ determines a linear form $\psi_\Gamma \in \mathscr{H}^*$ defined by $\langle \psi_\Gamma, \chi_I \rangle = 1$ if $\Gamma \subset C_I$, and 0 otherwise. Extend $\psi_\Gamma$ to a $\mathrm{Pol}(V)$-linear form $\mathscr{H} \otimes \mathrm{Pol}(V) \to \mathrm{Pol}(V)$ (see Section 2). If $P_\omega$ is treated as an element of $\mathscr{H} \otimes \mathrm{Pol}(V)$ then by definitions we have $P_\omega^\Gamma = \langle \psi_\Gamma, P_\omega \rangle$. It is clear now that "universal" linear relations between the $P_\omega^\Gamma$'s are just linear relations between the $\psi_\Gamma$'s in $\mathscr{H}^*$. So we have two problems entirely analogous to problems (A) and (B) in Section 2:

(A*) Find a complete system of linear relations between the elements $\psi_\Gamma$.
(B*) Construct some bases of $\mathscr{H}^*$.

To state the results we need some definitions. Let $W_1, \ldots, W_m$ be all codimension 1 subspaces in $V$ spanned by some subset of $\{v_1, \ldots, v_n\}$ (so that $V' = V \setminus \bigcup_{1 \leq j \leq m} W_j$). The intersections $W_{j_1} \cap \ldots \cap W_{j_r}$ will be called *flats*. The flats spanned by some subsets of $\{v_1, \ldots, v_n\}$ will be called *essential*, and other flats *inessential*. For example, 1-dimensional essential flats are just $\mathbf{R}v_1, \ldots, \mathbf{R}v_n$. It was already said that the connected components of $V \setminus \bigcup_j W_j$ are called *chambers*. A chamber $\Gamma$ is *adjacent* to a flat $U$ if $\bar{\Gamma} \cap U$ is open in $U$, where $\bar{\Gamma}$ is the closure of $\Gamma$. A *flag* $F$ is a chain $(0 = U_0 \subset U_1 \subset \ldots \subset U_l = V)$, where each $U_s$ is an $s$-dimensional flat. A flag $F$ is said to be *oriented* if for any $s = 1, \ldots, l$ there is chosen one of two components $U_s^+$ of $U_s \setminus U_{s-1}$ (an oriented flag will be denoted by $\vec{F}$). We say that a chamber $\Gamma$ is adjacent to a flag $F$ if $\Gamma$ is adjacent to each of its flats $U_s$.

Clearly, there are exactly $2^l$ chambers adjacent to each flag $F$. For any of them let

$$\varepsilon(\Gamma, \vec{F}) = (-1)^{|\{s \in [1,l] : \Gamma \cap U_s^+ = \varnothing\}|}$$

and define the element $\psi_{\vec{F}} \in \mathscr{H}^*$ by

$$\psi_{\vec{F}} = \sum_\Gamma \varepsilon(\Gamma, \vec{F}) \psi_\Gamma,$$

the sum over all chambers $\Gamma$ adjacent of $\vec{F}$. Evidently, if $\vec{F}'$ is another orientation of the same flag then $\psi_{\vec{F}'} = \pm \psi_{\vec{F}}$.

THEOREM 4 ([4]). *For any oriented flag $\vec{F} = (0 \subset U_1 \subset \ldots \subset U_l = V)$ such that $U_1$ is inessential we have $\psi_{\vec{F}} = 0$. Moreover, these relations together with the obvious relations $\psi_\Gamma = 0$ for $\Gamma \notin \bigcup_I C_I$ form a complete system of linear relations between the elements $\psi_\Gamma$ in $\mathscr{H}^*$.*

Our next result is the analogue of Theorem 3. We associate to each base $I = \{i_1 < \ldots < i_l\} \subset [1, n]$ the oriented flag $\vec{F}(I) = (0 \subset U_1 \subset \ldots \subset U_l = V)$ as follows: $U_s = \langle v_{i_1}, \ldots, v_{i_s} \rangle$ is the linear span of vectors $v_{i_1}, v_{i_2}, \ldots, v_{i_s}$ oriented so that $U_s^+ \ni v_{i_s}$.

THEOREM 5 ([4]). *The elements $\psi_{\vec{F}(I)}$ corresponding to bases $I$ containing no broken circuits form a basis in $\mathscr{H}^*$.*

Theorems 3 and 5 admit the following generalization. Choose a mapping $\tau$: {Nonzero essential flats} $\to [1, n]$ so that if $\tau(U) = i$ then $v_i \in U$. To any such $\tau$ we associate the class $\mathscr{I}_\tau$ of independent subsets of $[1, n]$ uniquely determined by the following rules:

(a) $\emptyset \in \mathscr{I}_\tau$.

(b) Let $\emptyset \neq I \in \mathscr{I}$, and $\tau(\langle v_i\colon i \in I\rangle) = i_0$. Then $I \in \mathscr{I}_\tau$ if and only if $i_0 \in I$ and $(I \setminus i_0) \in \mathscr{I}_\tau$.

For example, it is easy to see that for $\tau$ defined by $\tau(U) = \max\{i\colon v_i \in U\}$ the class $\mathscr{I}_\tau$ consists of all subsets $I \subset [1, n]$ containing no broken circuits.

THEOREM 6. *Elements $\chi_I$ for $I \in \mathscr{I}_\tau$ form a basis in $\mathscr{H}$.*

This result generalizes Theorem 3; it is also essentially due to Björner [13].

Now let $\mathscr{F}_\tau$ be the class of all oriented flags $\vec{F} = (0 \subset U_1 \subset \ldots \subset U_l = V)$ satisfying two properties: (a) Any $U_s$ is essential; (b) If $\tau(U_s) = i_s$ then $v_{i_s} \in U_s^+$.

THEOREM 7 ([4]). *Elements $\psi_{\vec{F}}$ for $\vec{F} \in \mathscr{F}_\tau$ form a basis in $\mathscr{H}^*$.*

It is easy to see that Theorem 5 is a special case of Theorem 7 corresponding to $\tau(U) = \max\{i\colon v_i \in U\}$.

The basis given by Theorem 7 consists not of elements $\psi_\Gamma$ but of their certain linear combinations, so it seems to be more complicated than the basis from Theorem 6. But it has the following nice property.

THEOREM 8 ([4]). *For any chamber $\Gamma$ and any mapping $\tau$ as above all coefficients in the expansion of $\psi_\Gamma$ in the basis $\{\psi_{\vec{F}}\colon \vec{F} \in \mathscr{F}_\tau\}$ are 0 or 1.*

## 6. Dual geometric description of $\mathscr{H}$

In this section we consider another geometric approach to the space $\mathscr{H}$ developed in [2] (see also [3]). Recall that we have the natural projection $p\colon \mathbf{R}^n \to V$ such that $p(e_i) = v_i$, where $e_1, \ldots, e_n$ is the standard basis in $\mathbf{R}^n$. For any $v \in V$ let $K_v$ denote the affine $k$-space $p^{-1}(v) \subset \mathbf{R}^n$. The coordinates $x_1, \ldots, x_n$ in $\mathbf{R}^n$ define (affine) linear functions on each $K_v$. For $i \in [1, n]$ put $S_i(v) = K_v \cap \{x_i = 0\}$, and for any $J \subset [1, n]$ put $S_J(v) = \bigcap_{i \in J} S_i(v)$. Each $S_J(v)$ is a (perhaps, empty) affine subspace of $K_v$. The next proposition shows that for $v \in V'$ the rank of any of these subspaces is determined by the matroid structure on $[1, n]$ introduced above, and so does not depend on $v$.

PROPOSITION 6. *Let $v \in V'$. Then $S_J(v) = \emptyset$ if and only if $r([1, n] \setminus J) < l$, otherwise* $\operatorname{codim}_{K_v} S_J(v) = |J|$.

In particular, we see that (nonempty) affine hyperplanes $S_i(v)$ have normal crossings in $K_v$ for $v \in V'$. The points of the form $S_J(v)$ will be called *vertices*. We see that $S_J(v)$ is a vertex if and only if $[1, n] \setminus J$ is a base.

From now on we assume that $v \in V'$. Let $S_v = \bigcup_i S_i(v)$ and consider the relative homology $H_k(K_v, S_v; C)$. This can be realized as the space of formal linear combinations of bounded connected components of the space $K_v \backslash S_v$. Geometric structure of these components depends heavily on $v$. Nevertheless, A. N. Varchenko proved the following important

THEOREM 9 ([2]). *All spaces $H_k(K_v, S_v; C)$ for $v \in V'$ are naturally isomorphic to each other.*

The proof in [2] uses the Gauss–Manin connection on the homological bundle with fibers $H_k(K_v, S_v; C)$ over the complexification of $V'$, and shows that it has trivial monodromy.

Now consider the cohomology $H^k(K_v, S_v; C)$ which can be realized as the dual of $H_k(K_v, S_v; C)$. Following [2] we associate to each vertex $S_J(v)$ the element $\delta_{J,v} \in H^k(K_v, S_v; C)$ constructed as follows. For any vector $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n)$ with $\varepsilon_i = \pm 1$ put $\Delta_{\varepsilon,v} = K_v \cap \{\varepsilon_i x_i > 0 \text{ for } 1 \leqslant i \leqslant n\}$; clearly, the nonempty $\Delta_{\varepsilon,v}$ are just connected components of $K_v \backslash S_v$. For any bounded component $\Delta_{\varepsilon,v}$ we define $\langle \delta_{J,v}, \Delta_{\varepsilon,v} \rangle$ to be $\prod_{j \in J} \varepsilon_j$ if $S_J(v)$ belongs to the closure $\bar{\Delta}_{\varepsilon,v}$, otherwise $\langle \delta_{J,v}, \Delta_{\varepsilon,v} \rangle = 0$.

THEOREM 10 ([2], [3]). *For any $v \in V'$ there is the natural isomorphism $H^k(K_v, S_v; C) \to \mathscr{H}$ sending each $\delta_{J,v}$ to $\chi_{[1,n] \backslash J}$.*

In fact A. N. Varchenko showed in [2] that each element $\delta_{J,v}$ is covariantly constant with respect to the Gauss–Manin connection. This is a crucial step in his proof of Theorem 1 (see Section 2 above).

Theorems 3 and 10 immediately imply

PROPOSITION 7. *For any $v \in V'$ the number of bounded connected components of $K_v \backslash S_v$ is equal to the number of bases $I \subset [1, n]$ containing no broken circuits.*

A straightforward geometric proof of this fact is given in [2] (see also [3], Prop. 1.8.2).

Proposition 7 can be reformulated in terms of the family $v_1, \dots, v_n$. For this we give a characterization of bounded connected components of $K_v \backslash S_v$.

PROPOSITION 8. *The set $\Delta_{\varepsilon,v}$ is nonempty if and only if $v$ belongs to the open convex cone generated by $\varepsilon_1 v_1, \dots, \varepsilon_n v_n$; it is bounded if and only if this cone is proper, i.e. not equal to $V$.*

Propositions 7 and 8 give us

PROPOSITION 9. *For any $v \in V'$ the number of vectors $\varepsilon = (\varepsilon_1, \dots, \varepsilon_n) \in \{+1, -1\}^n$ such that the open convex cone generated by $\varepsilon_1 v_1, \dots, \varepsilon_n v_n$ is proper and contains $v$, is equal to the number of bases $I \subset [1, n]$ containing no broken circuits.*

## Concluding remarks

**1.** The natural combinatorial language for the problems discussed above is that of oriented matroids [15]. In fact most of the notions and results can be generalized to the case of arbitrary oriented matroids.

**2.** The ring of locally constant functions on the complement of a finite number of hyperplanes in a real finite dimensional vector space is systematically studied in [5]. Most of the results of Section 5 can be proved using the technique developed in [5]. In fact most of the notions and results of [5] also can be developed in the general setting of oriented matroids.

**3.** To study the "discrete" version of vector partition function one needs some refinement of the above geometric and combinatorial analysis. Namely, one must study the convex cones $\dot{C}_I$ and faces $\Gamma$ of arbitrary dimension $r = 0, 1, \ldots, l$ (not only $l$-dimensional ones as above). Some results in this direction are obtained in [2], [4], [5].

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(*) In Russian alphabetical order.