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Constructions of $B_h[g]$-sequences

by

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1. Introduction. A (finite) $B_h[g]$-*sequence* is a sequence $\bar{a} = (a_0, a_1, a_2, \ldots, a_J)$ of integers such that no integer has more than g representations as sums of h summands from $\bar{a}$. A survey of results on $B_h[1]$-sequences (up to 1966) is given in Chapter II of [5]. Recent papers on $B_h[g]$-sequences include [4].

Without loss of generality we may assume that

$$0 = a_0 \leqslant a_1 \leqslant a_2 \leqslant \ldots \leqslant a_J.$$

Assuming that a sum contains the summand a_j x_j times, the sum may be written as $\sum_{j=0}^{J} x_j a_j$ where $\sum_{j=0}^{J} x_j = h$. Since $a_0 = 0$, we have $\sum_{j=0}^{J} x_j a_j = \sum_{j=1}^{J} x_j a_j$. Further $\sum_{j=1}^{J} x_j = h - x_0 \leqslant h$. Hence we get the following precise definition of a $B(g,h,J)$-sequence ($B_h[g]$-sequence with $J+1$ elements): Let

$$C(h, J) = \{\bar{x} = (x_1, x_2, \ldots, x_J) |\ x_j \text{ non-negative integers and } \sum_{j=1}^{J} x_j \leqslant h\}.$$

Further, let

$$D_k = D_k(\bar{a}) = \{\bar{x} \in C(h,J) |\ \sum_{j=1}^{J} x_j a_j = k\}.$$

The sequence $\bar{a} = (a_0, a_1, a_2, \ldots, a_J)$ is a $B(g,h,J)$-*sequence* if $|D_k| \leqslant g$ for all integers $k \geqslant 0$.

Let

$$N(g,h,J) = \min\{a_J |\ (a_0, a_1, a_2, \ldots, a_J) \text{ is a } B(g,h,J)\text{-sequence}\}.$$

A $B(g,h,J)$-sequence $(a_0, a_1, a_2, \ldots, a_J)$ where $a_J = N(g,h,J)$ is called *optimal*.

The main emphasis in the published literature has been on the behaviour of $N(g,h,J)$ for fixed g and h and varying J, in particular the asymptotic behaviour when $J \to \infty$. In this paper we consider mainly the situation with varying h, in particular the asymptotic behaviour when $h \to \infty$.

It is easy to see that $(0, 1, h+1, (h+1)^2, \ldots, (h+1)^{J-1})$ is a $B(1,h,J)$-sequence. Hence

(1) $$N(g,h,J) \leqslant N(1,h,J) \leqslant (h+1)^{J-1}.$$

Let

(2) $$\underline{c}(g,J)=\liminf_{h\to\infty}\frac{N(g,h,J)}{h^{J-1}},\qquad \bar{c}(g,J)=\limsup_{h\to\infty}\frac{N(g,h,J)}{h^{J-1}}.$$

Clearly, (1) implies that $\bar{c}(g,J)\leqslant 1$.

2. Known bounds. Krückeberg [6] proved both lower and upper bounds on $N(1,h,J)$. His lower bound generalizes to the following.

THEOREM 1. *For all g, h, J we have*

$$N(g,h,J)\geqslant\frac{1}{h}\left\{\frac{1}{g}\binom{J+h}{h}-1\right\}.$$

In particular $\underline{c}(g,J)\geqslant 1/(gJ!)$.

Proof. Let $(a_0,a_1,\ldots,a_J)$ be a $B(g,h,J)$-sequence. If $(x_1,x_2,\ldots,x_J)\in C(h,J)$, then $0\leqslant\sum_{j=1}^{J}x_ja_j\leqslant ha_J$. Hence $|C(h,J)|\leqslant g(ha_J+1)$ and so

$$a_J\geqslant\frac{1}{h}\left(\frac{1}{g}|C(h,J)|-1\right)=\frac{1}{h}\left\{\frac{1}{g}\binom{J+h}{h}-1\right\}.$$

Chen [2] gave stronger bounds on $N(1,h,J)$:

$$N(1,h,J)\geqslant\frac{1}{h}\left\{\sum_{k=1}^{h/2}\sum_{m=1}^{k}\binom{J+1}{m}\binom{k-1}{m-1}\binom{J-m+k}{k}\right\}\qquad\text{if } h \text{ is even,}$$

$$N(1,h,J)\geqslant\frac{1}{h}\left\{J+\sum_{k=1}^{(h-1)/2}\sum_{m=1}^{k+1}\binom{J+1}{m}\binom{k}{m-1}\binom{J-m+k}{k}\right\}\qquad\text{if } h \text{ is odd.}$$

From this we can derive

(3) $$\underline{c}(1,J)\geqslant\frac{1}{J!2^J}\binom{2J}{J}.$$

The best known lower bounds on $N(g,2,J)$ are due to Chen and Kløve [3].

Recently, Hajela [4] proved a lower bound on $N(g,h,J)$ using trigonometrical polynomials. His formulation of the bound is such that it can not be immediately compared to the bound in Theorem 1. However, we will show that Hajela's bound is weaker than the bound in Theorem 1.

Let $F=J+1$ and $n=N(g,h,J)+1$. Then Hajela's result is

$$F\leqslant\frac{4g^{1/h}(h!)^{1/h}}{(3n)^{1/h}}\inf_{m\in N}\left(\frac{(2mn^2+1/m)^{1/(2h)}}{2-1/m}\right)^2$$

or equivalently

(4) $$F^h\leqslant\frac{2^{2h}gh!}{3n}\cdot\frac{2mn^2+1/m}{(2-1/m)^{2h}}=\frac{gh!}{3n}\cdot\frac{2mn^2+1/m}{(1-1/(2m))^{2h}}$$

for all integers $m\geqslant 1$.

From Theorem 1 we get

$$n\geqslant 1+\frac{1}{h}\left\{\frac{1}{g}\binom{J+h}{h}-1\right\}\geqslant 1+\frac{1}{h}\left\{\frac{1}{g}\cdot\frac{(J+1)^h}{h!}-1\right\}$$
$$=1-\frac{1}{h}+\frac{1}{hg}\cdot\frac{F^h}{h!}\geqslant\frac{1}{hg}\cdot\frac{F^h}{h!}$$

and so

(5) $$F^h\leqslant hgh!n.$$

We will show that (5) is stronger than (4). First, we note that for all integers $m\geqslant 1$ we have

$$\frac{1}{m}\left(1-\frac{1}{2m}\right)^{2h}\leqslant\frac{1}{h+1/2}\left(1-\frac{1}{2h+1}\right)^{2h}=\frac{1}{h}\left(1-\frac{1}{2h+1}\right)^{2h+1}<\frac{1}{he}$$

and so

$$h<\frac{m}{e(1-1/(2m))^{2h}}.$$

Hence, from (5) we get

$$F^h<gh!n\frac{m}{e(1-1/(2m))^{2h}}=\frac{gh!}{2en}\cdot\frac{2mn^2}{(1-1/(2m))^{2h}}<\frac{gh!}{2en}\cdot\frac{(2mn^2+1/m)}{(1-1/(2m))^{2h}}$$

for all $m\geqslant 1$, which gives an improvement over Hajela's bound (4) by a factor $2e/3\approx 1.8$.

Bose and Chowla [1] proved that if q is a prime power, then

$$N(1,h,q-1)\leqslant q^h-1\quad\text{ and }\quad N(1,h,q)\leqslant(q^{h+1}-1)/(q-1).$$

These upper bounds are quite weak when h is large compared to J. E.g. they give $N(1,h,2)\leqslant 2^{h+1}-1$, whereas (1) gives $N(1,h,2)\leqslant h+1$.

3. Some new constructions. Our first construction is a simple observation.

CONSTRUCTION 1. *If $(a_0=0,a_1,a_2,\ldots,a_J)$ is a $B(g,h,J)$-sequence, then so is*

$$(0,a_J-a_{J-1},a_J-a_{J-2},\ldots,a_J-a_1,a_J).$$

Proof. Define $\bar{x}^*$ by

$$x_j^*=x_{J-j}\quad\text{ for } 1\leqslant j\leqslant J-1,$$

$$x_J^*=h-\sum_{j=1}^{J}x_j.$$

We note that if $\bar{x}\in C(h,J)$, then $\bar{x}^*\in C(h,J)$. Further,

$$\sum_{j=1}^{J}x_j(a_J-a_{J-j})=k\quad\text{ if and only if }\quad\sum_{j=1}^{J}x_j^*a_j=ha_J-k.$$

Hence

$$|\{\bar{x}\in C(h,J)|\ \sum_{j=1}^{J} x_j(a_J-a_{J-j})=k\}| = |D_{ha_J-k}| \leqslant g \quad \text{for all } k.$$

CONSTRUCTION 2. *Let $h \geqslant 2$. Let $(0, a_1, \ldots, a_J)$ be a $B(g_1, h, J)$-sequence and $(0, b_1, \ldots, b_K)$ a $B(g_2, h, K)$-sequence with $b_1 > 1$. Let $A > (h-1)a_J$. Then*

$$(0, a_1, a_2, \ldots, a_J, Ab_1, Ab_2, \ldots, Ab_K)$$

is a $B(g_1 g_2, h, J+K)$-sequence.

Proof. The elements of $C(h, J+K)$ are of the form $(\bar{x}, \bar{y})$ where $\bar{x} = (x_1, x_2, \ldots, x_J)$ and $\bar{y} = (y_1, y_2, \ldots, y_K)$. In particular, $\bar{x}\in C(h,J)$ and $\bar{y}\in C(h,K)$. Let $k = rA+s$ where $0 \leqslant s < A$. If $(\bar{x},\bar{y})\in D_k$, then

$$\sum_{j=1}^{J} x_j a_j \equiv s \pmod{A}.$$

Since $0 \leqslant \sum_{j=1}^{J} x_j a_j \leqslant \sum_{j=1}^{J} x_j a_J \leqslant h a_J \leqslant 2(h-1)a_J < 2A$, this implies that

$$\sum_{j=1}^{J} x_j a_j = s \quad \text{and} \quad \sum_{j=1}^{K} y_j b_j = r \tag{6}$$

or

$$\sum_{j=1}^{J} x_j a_j = A+s \quad \text{and} \quad \sum_{j=1}^{K} y_j b_j = r-1. \tag{7}$$

Since there are at most g_1 $\bar{x}\in C(h,J)$ and at most g_2 $\bar{y}\in C(h,K)$ such that (6) is satisfied, there are at most $g_1 g_2$ $(\bar{x},\bar{y})\in D_k$ such that (6) is satisfied. Similarly, there are at most $g_1 g_2$ $(\bar{x},\bar{y})\in D_k$ such that (7) is satisfied.

We now show that (6) can be satisfied only if $r \neq 1$ and (7) can be satisfied only if $r = 1$. This then proves that $|D_k| \leqslant g_1 g_2$. First, if (6) is satisfied, then $\sum_{j=1}^{K} y_j b_j = r$. Since $b_j > 1$ for all j, this implies that $r = 0$ or $r > 1$, i.e. $r \neq 1$. Next, suppose that (7) is satisfied. Then

$$a_J \sum_{j=1}^{J} x_j \geqslant \sum_{j=1}^{J} x_j a_j = A+s \geqslant A > (h-1)a_J,$$

i.e. $\sum_{j=1}^{J} x_j > h-1$, and so $\sum_{j=1}^{J} x_j = h$. Hence $y_j = 0$ for $1 \leqslant j \leqslant K$. This implies that

$$rA+s = \sum_{j=1}^{J} x_j a_j + \sum_{j=1}^{K} y_j A b_j = \sum_{j=1}^{J} x_j a_j = A+s$$

and hence $r = 1$.

To get building blocks for Construction 2 we need good short sequences. We first determine the optimal $B(g,h,1)$- and $B(g,h,2)$-sequences. Next we construct some good $B(g,h,3)$-sequences.

THEOREM 2. *For $g \geqslant h+1$ we have $N(g,h,1) = 0$, and $(0,0)$ is the optimal $B(g,h,1)$-sequence.*

For $g \leqslant h$ we have $N(g,h,1) = 1$, and $(0,1)$ is the optimal $B(g,h,1)$-sequence.

Proof. Trivial.

THEOREM 3. *For $g \geqslant (h+2)(h+1)/2$ we have $N(g,h,2) = 0$, and $(0,0,0)$ is the optimal $B(g,h,2)$-sequence.*

For $h < g < (h+2)(h+1)/2$ we have $N(g,h,2) = 1$, and $(0,0,1)$, $(0,1,1)$ are the optimal $B(g,h,2)$-sequences.

For $g \leqslant h$ we have $N(g,h,2) = \lceil (h+1)/g \rceil$, and the optimal $B(g,h,2)$-sequences are $(0, a, \lceil (h+1)/g \rceil)$ for all a such that $1 \leqslant a < \lceil (h+1)/g \rceil$ and $\gcd(a, \lceil (h+1)/g \rceil) = 1$.

Proof. The part with $g > h$ is trivial. Consider $g \leqslant h$. First, we show that $N(g,h,2) > h/g$. Consider $(0,a,b)$ where $0 \leqslant a \leqslant b \leqslant h/g$. For $0 \leqslant \alpha \leqslant g$ we have $\alpha b \geqslant 0$, $ga - \alpha a \geqslant 0$,

$$\alpha b + (ga - \alpha a) = \alpha(b-a) + ga \leqslant g(b-a) + ga = gb \leqslant h,$$

and $(\alpha b)a + (ga - \alpha a)b = gab$. Hence $(\alpha b, ga - \alpha a)\in D_{gab}$ for $0 \leqslant \alpha \leqslant g$ and so $|D_{gab}| \geqslant g+1$. Therefore $(0,a,b)$ is not a $B(g,h,2)$-sequence. This proves that $N(g,h,2) > h/g$, i.e. $N(g,h,2) \geqslant \lceil (h+1)/g \rceil$.

Next, consider $(0,a,b)$ where $b = \lceil (h+1)/g \rceil$ and $\gcd(a,b) = 1$. Consider D_k for some k such that D_k is non-empty. Let $(y_1, y_2)\in D_k$ be such that $y_1 \leqslant x_1$ for all $(x_1, x_2)\in D_k$. Let $(x_1,x_2)\in D_k$.

Then

$$x_1 a + x_2 b = y_1 a + y_2 b. \tag{8}$$

In particular $x_1 a \equiv y_1 a \pmod{b}$. Hence $x_1 \equiv y_1 \pmod{b}$, and so $x_1 = y_1 + \alpha b$, where $\alpha \geqslant 0$ by the minimality of (y_1, y_2). Substituting in (8), we get $x_2 = y_2 - \alpha a$. Further, $h \geqslant x_1 \geqslant \alpha b$, and so $\alpha \leqslant h/b < g$. Hence $D_k \subseteq \{(y_1+\alpha b, y_2 - \alpha a)|\ 0 \leqslant \alpha < g\}$ and so $|D_k| \leqslant g$.

Finally, consider $(0,a,b)$ where $b = \lceil (h+1)/g \rceil$ and $d = \gcd(a,b) > 1$. Similarly to the first part of the proof we get

$$\left(\alpha\frac{b}{d}, g\frac{a}{d} - \alpha\frac{a}{d}\right)\in D_{g\frac{a}{d}\frac{b}{d}}$$

for $0 \leqslant \alpha \leqslant g$. Hence $(0,a,b)$ is not a $B(g,h,2)$-sequence.

CONSTRUCTION 3. *The following sequences are $B(1,h,3)$-sequences.*

(i) *If h is even, $1 \leqslant a \leqslant h$, and $\gcd(a, h+1) = 1$:*

$$\left(0, a, \frac{h(h+1)}{2}+a, \frac{h(h+1)}{2}+h+1\right);$$

(ii) *If h is odd*:

$$\left(0,1,\frac{h(h+1)}{2}+1,\frac{h(h+1)}{2}+h+2\right),$$

$$\left(0,h+1,\frac{h(h+1)}{2}+h+1,\frac{h(h+1)}{2}+h+2\right);$$

(iii) *If $h \equiv 3 \pmod 4$*:

$$\left(0,\frac{h-1}{2},\frac{(h+1)^2}{2},\frac{h(h+1)}{2}+h+2\right),$$

$$\left(0,\frac{h+3}{2},\frac{(h+1)^2}{2}+2,\frac{h(h+1)}{2}+h+2\right).$$

Proof. To prove (i), suppose (x_1,x_2,x_3), $(y_1,y_2,y_3)\in C(h,3)$ and

(9) $$x_1a+x_2\left(\frac{h(h+1)}{2}+a\right)+x_3\left(\frac{h(h+1)}{2}+h+1\right) = y_1a+y_2\left(\frac{h(h+1)}{2}+a\right)+y_3\left(\frac{h(h+1)}{2}+h+1\right).$$

We have to show that $(x_1,x_2,x_3)=(y_1,y_2,y_3)$. From (9) we get

$$x_1a+x_2a \equiv y_1a+y_2a \pmod{h+1}.$$

Since $\gcd(a,h+1)=1$, we get $x_1+x_2 \equiv y_1+y_2 \pmod{h+1}$ and so

(10) $$x_1+x_2 = y_1+y_2.$$

Combining (9) and (10) we get

(11) $$x_2\frac{h}{2}+x_3\left(\frac{h}{2}+1\right) = y_2\frac{h}{2}+y_3\left(\frac{h}{2}+1\right).$$

In particular

(12) $$x_2 \equiv y_2 \pmod{(h/2)+1}.$$

Without loss of generality we may assume that $x_2 \geqslant y_2$. Suppose that $x_2 > y_2$. Since $x_2 \leqslant h$, (12) implies that $x_2 = y_2+h/2+1$. By (10), $y_1 = x_1+h/2+1$, and by (11), $y_3 = x_3+h/2$. Hence $h \geqslant y_1+y_3 = x_1+x_3+h+1 \geqslant h+1$, a contradiction. Hence $x_2=y_2$. By (10) and (11), $x_1=y_1$ and $x_3=y_3$. This proves (i).

To prove (ii) and (iii) we first note that each one of the sequences in (ii) is obtained from the other by Construction 1 and similarly for (iii). Therefore, it is sufficient to show that one of (ii) and one of (iii) is a $B(1,h,3)$-sequence.

We prove that the second of the two sequences in (ii) is a $B(1,h,3)$-sequence. Suppose that (x_1,x_2,x_3), $(y_1,y_2,y_3)\in C(h,3)$ and

(13) $$x_1(h+1)+x_2\left(\frac{h(h+1)}{2}+h+1\right)+x_3\left(\frac{h(h+1)}{2}+h+2\right) = y_1(h+1)+y_2\left(\frac{h(h+1)}{2}+h+1\right)+y_3\left(\frac{h(h+1)}{2}+h+2\right).$$

Without loss of generality we may assume that $x_3 \geqslant y_3$. From (13) we get

(14) $$x_3 \equiv y_3 \pmod{(h+1)/2}.$$

We consider two cases.

Case I, $x_3=y_3$. From (13) we get

(15) $$2x_1+x_2(h+2) = 2y_1+y_2(h+2).$$

Hence $2x_1 \equiv 2y_1 \pmod{h+2}$ and so $x_1 \equiv y_1 \pmod{h+2}$. This implies that $x_1=y_1$. By (15), $x_2=y_2$.

Case II, $x_3=y_3+(h+1)/2$. We will show that this is not possible. From (13) we get

(16) $$2x_1+x_2(h+2)+h(h+1)/2+h+2 = 2y_1+y_2(h+2).$$

In particular $2x_1+1 \equiv 2y_1 \pmod{h+2}$ and so $x_1 \equiv y_1+(h+1)/2 \pmod{h+2}$. There are now two subcases:

Case II (i), $x_1=y_1+(h+1)/2$. Then $h \geqslant x_1+x_3 = y_1+y_3+h+1 \geqslant h+1$, a contradiction.

Case (II) (ii), $y_1=x_1+(h+3)/2$. Then (16) gives $y_2=x_2+(h-1)/2$. Hence $h \geqslant y_1+y_2 = x_1+x_2+h+1 \geqslant h+1$, again a contradiction.

Finally, consider the first of the two sequences in (iii). Suppose that (x_1,x_2,x_3), $(y_1,y_2,y_3)\in C(h,3)$ and

(17) $$x_1\frac{h-1}{2}+x_2\frac{(h+1)^2}{2}+x_3\left(\frac{h(h+1)}{2}+h+2\right) = y_1\frac{h-1}{2}+y_2\frac{(h+1)^2}{2}+y_3\left(\frac{h(h+1)}{2}+h+2\right).$$

Then

$$x_1\frac{h-1}{2}+x_3\frac{h+3}{2} \equiv y_1\frac{h-1}{2}+y_3\frac{h+3}{2}\left(\mathrm{mod}\,\frac{(h+1)^2}{2}\right).$$

Without loss of generality we may assume that

$$x_1\frac{h-1}{2}+x_3\frac{h+3}{2} \geqslant y_1\frac{h-1}{2}+y_3\frac{h+3}{2}.$$

Since

$$0 \leqslant x_1 \frac{h-1}{2} + x_3 \frac{h+3}{2} \leqslant h \frac{h+3}{2} < 2 \frac{(h+1)^2}{2}$$

there are again two cases to consider.

Case I, $x_1 \frac{h-1}{2} + x_3 \frac{h+3}{2} = y_1 \frac{h-1}{2} + y_3 \frac{h+3}{2}$. In this case,

$$-2x_1 \equiv -2y_1 \left(\bmod \frac{h+3}{2}\right).$$

Since $(h+3)/2$ is odd, this implies that $x_1 \equiv y_1 \pmod{(h+3)/2}$. Without loss of generality we may assume that $x_1 \geqslant y_1$.

Case I (i), $x_1 = y_1$. Then $x_3 = y_3$, and so (17) implies $x_2 = y_2$.

Case I (ii), $x_1 = y_1 + (h+3)/2$. Then $y_3 = x_3 + (h-1)/2$. From (17) we get $x_2 = y_2 + (h-1)/2$. Hence, $x_1 + x_2 = y_1 + y_2 + h + 1 \geqslant h+1$, a contradiction.

Case II, $x_1 \frac{h-1}{2} + x_3 \frac{h+3}{2} = y_1 \frac{h-1}{2} + y_3 \frac{h+3}{2} + \frac{(h+1)^2}{2}$. Then

$$(x_1 + x_3) \frac{h+3}{2} \geqslant x_1 \frac{h-1}{2} + x_3 \frac{h+3}{2}$$

$$= y_1 \frac{h-1}{2} + y_3 \frac{h+3}{2} + \frac{(h+1)^2}{2} \geqslant \frac{(h+1)^2}{2} = (h-1)\frac{h+3}{2} + 2.$$

Hence $x_1 + x_3 > h-1$ and so $x_1 + x_3 = h$. This implies that

$$2x_3 = \frac{3h+1}{2} + y_1 \frac{h-1}{2} + y_3 \frac{h+3}{2}. \tag{18}$$

Hence $2x_3 \equiv 2 + 2y_3 \pmod{(h-1)/2}$ and so $x_3 \equiv 1 + y_3 \pmod{(h-1)/2}$. From (18) we get $x_3 \geqslant (3h+1)/4$ and so $x_3 = 1 + y_3 + (h-1)/2$. Combining this with (18) we get

$$0 = \frac{h-1}{2}(1 + y_1 + y_3) \geqslant \frac{h-1}{2},$$

a contradiction.

THEOREM 4. *For all h we have*

$$N(1, h, 3) \leqslant \frac{h(h+1)}{2} + 2 \left\lfloor \frac{h+1}{2} \right\rfloor + 1.$$

In particular $5/12 \leqslant \underline{c}(1,3) \leqslant \bar{c}(1,3) \leqslant 1/2$.

Proof. The theorem follows directly from Construction 3 and (3).

A direct search has shown that for $1 < h \leqslant 24$ we have

$$N(1, h, 3) = \frac{h(h+1)}{2} + 2 \left\lfloor \frac{h+1}{2} \right\rfloor + 1$$

and Construction 3 gives *all* the optimal $B(1, h, 3)$-sequences. Whether this is true for all $h > 1$ is an open question.

CONSTRUCTION 4. *If $g \geqslant 2$ and $m \geqslant 1$, then*

$$(0, m, (g+1)m^2 + gm, (g+1)m^2 + (g+1)m + 1)$$

is a $B(g, 2mg, 3)$-sequence.

Proof. We order the elements of D_k as follows: $(y_1, y_2, y_3) < (x_1, x_2, x_3)$ if

$$y_1 m + y_3(m+1) < x_1 m + x_3(m+1)$$

or if

$$y_1 m + y_3(m+1) = x_1 m + x_3(m+1) \quad \text{and} \quad y_1 < x_1.$$

Let (y_1, y_2, y_3) be the minimal element of D_k under this ordering. Let

$$y_i = \varepsilon_i m + \delta_i \quad \text{for } i = 1, 2, 3 \quad \text{where } 0 \leqslant \delta_i < m.$$

Let $(x_1, x_2, x_3) \in D_k$. Then

$$x_1 m + x_2((g+1)m^2 + gm) + x_3((g+1)m^2 + (g+1)m + 1) \tag{19}$$
$$= y_1 m + y_2((g+1)m^2 + gm) + y_3((g+1)m^2 + (g+1)m + 1).$$

In particular

$$x_1 m + x_3(m+1) \equiv y_1 m + y_3(m+1) \pmod{(g+1)m^2 + gm}.$$

Hence

$$x_1 m + x_3(m+1) = y_1 m + y_3(m+1) + \alpha((g+1)m^2 + gm) \tag{20}$$

where $\alpha \geqslant 0$ by the minimality of (y_1, y_2, y_2). Further

$$\alpha((g+1)m^2 + gm) \leqslant x_1 m + x_3(m+1) \leqslant 2mg(m+1) < 2((g+1)m^2 + gm).$$

Hence, $\alpha = 0$ or $\alpha = 1$. Let

$$n_\alpha = \left|\{(x_1, x_2, x_3) \in D_k \mid x_1 m + x_3(m+1) = y_1 m + y_3(m+1) + \alpha((g+1)m^2 + gm)\}\right|.$$

We have to show that $n_0 + n_1 \leqslant g$.

First, we consider the case $\alpha = 0$. By (20),

$$x_1 \equiv y_1 \pmod{m+1}.$$

Hence $x_1 = y_1 + \beta(m+1)$, where

$$\beta \geqslant 0 \tag{21}$$

by the minimality of (y_1, y_2, y_3). Substituting in (20) and (19) we get

(22) $$x_1 = y_1 + \beta(m+1),$$

(23) $$x_2 = y_2 + \beta m,$$

(24) $$x_3 = y_3 - \beta m.$$

In particular, $0 \leqslant x_3 = y_3 - \beta m = (\varepsilon_3 - \beta)m + \delta_3$. Hence

(25) $$\beta \leqslant \varepsilon_3.$$

Further,

$$2mg \geqslant x_1 + x_2 = \varepsilon_1 m + \delta_1 + \varepsilon_2 m + \delta_2 + \beta(2m+1).$$

Rearranging, we get

(26) $$\beta \leqslant \frac{(2g-\varepsilon_1-\varepsilon_2)m-(\delta_1+\delta_2)}{2m+1}.$$

Define ϕ by

(27) $$\phi = 1 \quad \text{if } \varepsilon_3 \leqslant \frac{(2g-\varepsilon_1-\varepsilon_2)m-(\delta_1+\delta_2)}{2m+1},$$

(28) $$\phi = 0 \quad \text{otherwise}.$$

Combining (21) and (25)–(28) we get

(29) $$n_0 \leqslant \varepsilon_3 + \phi.$$

Next, consider the case $\alpha = 1$. Similarly to the case $\alpha = 0$ we get

(30) $$x_1 = y_1 - 1 + \gamma(m+1),$$

(31) $$x_2 = y_2 - (g+1)m - 1 + \gamma m,$$

(32) $$x_3 = y_3 + (g+1)m - \gamma m.$$

We have $0 \leqslant x_1 = \varepsilon_1 m + \delta_1 - 1 + \gamma(m+1)$. Hence

(33) $$\gamma \geqslant (1-\varepsilon_1 m - \delta_1)/(m+1).$$

Further, $0 \leqslant x_2 = (\varepsilon_2 - g - 1 + \gamma)m + \delta_2 - 1$ and so

(34) $$\gamma \geqslant g + 1 - \varepsilon_2 + \frac{1}{m}(1-\delta_2).$$

Define ψ by

(35) $$\psi = 1 \quad \text{if } \delta_2 = 0,$$

(36) $$\psi = 0 \quad \text{if } \delta_2 > 0.$$

Then, by (34),

(37) $$\gamma \geqslant g + 1 - \varepsilon_2 + \psi.$$

Next, $2mg \geqslant x_1 + x_2 + x_3 = \varepsilon_1 m + \delta_1 + \varepsilon_2 m + \delta_2 + \varepsilon_3 m + \delta_3 - 2 + \gamma(m+1)$. Rearranging, we get

(38) $$\gamma \leqslant 2g - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \frac{1}{m+1}(2g - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta_1 + \delta_2 + \delta_3 - 2).$$

We note that

$$2gm \geqslant y_1 + y_2 + y_3 = (\varepsilon_1 + \varepsilon_2 + \varepsilon_3)m + \delta_1 + \delta_2 + \delta_3.$$

Hence

(39) $$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 \leqslant 2g$$

and

(40) $$\varepsilon_1 + \varepsilon_2 + \varepsilon_3 < 2g \quad \text{if } \delta_1 + \delta_2 + \delta_3 > 0.$$

Define χ by

(41) $$\chi = 1 \quad \text{if } 2g - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta_1 + \delta_2 + \delta_3 > 2,$$

(42) $$\chi = -1 \quad \text{if } m = 1,\ \delta_1 = \delta_2 = \delta_3 = 0, \text{ and } \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 2g,$$

(43) $$\chi = 0 \quad \text{otherwise}.$$

Then, by (38),

(44) $$\gamma \leqslant 2g - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \chi.$$

Combining (37) and (44) we get

(45) $$n_1 \leqslant 1 + (2g - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 - \chi) - (g + 1 - \varepsilon_2 + \psi)$$

(46) $$= g - \varepsilon_1 - \varepsilon_3 - \chi - \psi.$$

Further combining (29) and (46) we get

(47) $$n_0 + n_1 \leqslant g + \phi - \varepsilon_1 - \chi - \psi.$$

If $\phi - \varepsilon_1 - \chi - \psi \leqslant 0$ we are finished. Suppose that $\phi - \varepsilon_1 - \chi - \psi > 0$. This is possible only if

(48) $$\phi = 1, \quad \varepsilon_1 = 0, \quad \chi = -1, \quad \psi = 1$$

or

(49) $$\phi = 1, \quad \varepsilon_1 = \chi = \psi = 0.$$

In the first case $\delta_1 = \delta_3 = 0$, $\delta_2 = 0 = 1 - \psi$, $\varepsilon_2 + \varepsilon_3 = 2g = 2g - 1 + \psi$. In the second case, $\delta_2 > 0$, and, by (40), $\varepsilon_2 + \varepsilon_3 \leqslant 2g - 1$. Since $2g - \varepsilon_1 - \varepsilon_2 - \varepsilon_3 + \delta_1 + \delta_2 + \delta_3 \leqslant 2$, this implies that $\delta_1 = \delta_3 = 0$, $\delta_2 = 1 = 1 - \psi$, $\varepsilon_2 + \varepsilon_3 = 2g - 1 = 2g - 1 + \psi$. Hence, in both cases,

(50) $$\varepsilon_1 = 0, \quad \varepsilon_2 + \varepsilon_3 = 2g - 1 + \psi, \quad \delta_1 = \delta_3 = 0, \quad \delta_2 = 1 - \psi.$$

Since $\phi = 1$ we have, by (27),

$$(2m+1)\varepsilon_3 \leqslant 2mg - m\varepsilon_2 - 1 + \psi$$
$$= 2mg - m(2g-1+\psi-\varepsilon_3) - 1 + \psi.$$

Rearranging we get $(m+1)\varepsilon_3 \leqslant (m-1)(1-\psi)$. Hence $\varepsilon_3 = 0$ and so by (29)

(51) $$n_0 \leqslant 1.$$

Next, since $\varepsilon_1 = \delta_1 = 0$, we get, by (33),

(52) $$\gamma \geqslant 1.$$

Further, substituting (50) in (38) we get

(53) $$\gamma \leqslant 1 - \psi + 2\psi/(m+1) \leqslant 1.$$

Combining this with (52) we get $n_1 \leqslant 1$. Hence, by (51),

$$n_0 + n_1 \leqslant 2 \leqslant g.$$

This completes the proof of Construction 4. From the construction we get

THEOREM 5. *For $g \geqslant 2$ and all h we have*

$$N(g,h,3) \leqslant (g+1)(m^2+m)+1$$

where $m = \lceil h/(2g) \rceil$. In particular

$$\frac{1}{6g} \leqslant \underline{c}(g,3) \leqslant \bar{c}(g,3) \leqslant \frac{1}{4g}\cdot\frac{g+1}{g}.$$

Remark. From Theorems 4 and 5 we get $\bar{c}(g,3) < (1/g)\underline{c}(1,3)$ for $g > 6$.

For $g = 2$ Construction 4 gives the sequence $(0, m, 3m^2+2m, 3m^2+3m+1)$. For $m \leqslant 10$ this is optimal, and this and the one obtained from it by Construction 1 are the only optimal $B(2,2m,3)$-sequences. We conjecture that this is true for all m. For $g \geqslant 3$ it appears that the sequence in Construction 4 is not optimal. E.g. for $1 \leqslant m \leqslant 4$ an optimal $B(6,12m,3)$-sequence is $(0, m, 7m^2+3m-1, 7m^2+4m)$, and so $N(6,12m,3) = 7m^2+4m$, whereas Construction 4 gives $(0, m, 7m^2+6m, 7m^2+7m+1)$.

Based on the limited numerical data available, it appears that $N(g,2mg,3) \sim (g+1)m^2$ when g is fixed and $m \to \infty$.

For $m = 1$, Construction 4 gives the $B(g,2g,3)$-sequence $(0,1,2g+1,2g+3)$. However, we can prove that $(0,1,g+3,g+5)$ is a $B(g,2g,3)$-sequence for all g. Hence $N(g,2g,3) \leqslant g+5$. Moreover, for $g \leqslant 50$, $N(g,2g,3) = g+5$, and we conjecture that this is true in general.

THEOREM 6. *For all $g \geqslant 1$, $h > 1$, J, K we have*

$$N(g,h,J+K) \leqslant ((h-1)N(g,h,J)+1)N(1,h,K)$$

and $\bar{c}(g,J+K) \leqslant \bar{c}(g,J)\bar{c}(1,K)$.

Proof. Let $(0, a_1, \ldots, a_J)$ be a $B(g,h,J)$-sequence with $a_J = N(g,h,J)$ and $(0, b_1, \ldots, b_K)$ a $B(1,h,K)$-sequence with $b_K = N(1,h,K)$. We cannot have $b_1 = 1$ and $b_J - b_{J-1} = 1$ simultaneously since this would imply

$$1\cdot b_1 + 1\cdot b_{J-1} = 1\cdot b_J$$

which is impossible for a $B(1,h,J)$-sequence. Therefore, Construction 1 shows that we may assume that $b_1 > 1$. From Construction 2, with $A = (h-1)a_J+1$, it follows that

$$N(g,h,J+K) \leqslant ((h-1)N(g,h,J)+1)N(1,h,K).$$

In particular

$$N(g,h,J+K) \leqslant hN(g,h,J)N(1,h,K),$$

which implies that

$$\limsup_{h\to\infty} \frac{N(g,h,J+K)}{h^{J+K-1}} \leqslant \limsup_{h\to\infty} \frac{N(g,h,J)}{h^{J-1}} \limsup_{h\to\infty} \frac{N(1,h,K)}{h^{K-1}},$$

i.e. $\bar{c}(g,J+K) \leqslant \bar{c}(g,J)\bar{c}(1,K)$.

THEOREM 7. *For all $g_1 \geqslant 1$, $g_2 > 1$, $h > 1$, J we have*

$$N(g_1g_2,h,J+3) \leqslant ((h-1)N(g_1,h,J)+1)((g_2+1)(m^2+m)+1)$$

where $m = \lceil h/(2g) \rceil$.

Further

$$\bar{c}(g_1g_2,J+3) \leqslant \bar{c}(g_1,J)\frac{g_2+1}{4g_2^2}.$$

Proof. Combining Constructions 1 and 4 we get the $B(g,2gm,3)$-sequence

$$(0, m+1, (g+1)m^2+gm+1, (g+1)(m^2+m)+1).$$

This is in particular a $B(g,h,3)$-sequence, and $m+1 > 1$. The Theorem follows from Construction 2.

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Polynomials whose powers are sparse

by

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Erdös [Erd] defines $Q(N)$ as the least possible number of nonzero coefficients ("the number of terms") in the square of a polynomial $f(x)$ with exactly N nonzero real coefficients. Erdös proves the existence of positive constants C_1, C_2 such that

$$Q(N) < C_1 N^{1-C_2}.$$

Verdenius [Ver] extends this result in two directions. He works with *complete* polynomials f, that is,

$$f(x) = \sum_{i=0}^{N-1} d_i x^i, \quad d_i \neq 0, \quad 0 \leqslant i \leqslant N-1.$$

He also establishes a similar inequality for cubes. Letting $Q_k(N)$ denote the least possible number of terms in the kth power of a complete real polynomial of degree $N-1$, Verdenius gives positive constants $C_{1,2}$, $C_{1,3}$ such that for any integer $N \geqslant 1$,

$$Q_2(N) < C_{1,2} N^{0.81071\ldots}, \quad Q_3(N) < C_{1,3} N^{0.99934\ldots}.$$

In the present note we extend this result to kth powers for each integer $k \geqslant 2$. Our main theorem is:

THEOREM 1. *Given an integer $k \geqslant 2$, there are positive constants $C_{1,k}$, $C_{2,k}$ such that for any integer $N \geqslant 1$,*

$$Q_k(N) < C_{1,k} N^{1-C_{2,k}}.$$

Remark. Schinzel [Sch] has studied a similar problem for fields of prime characteristic p. For any integer k not a power of p, he obtains polynomials with arbitrarily many terms, whose kth power has at most $2k$ terms. He also obtains lower bounds.

Two consequences of Theorem 1:

THEOREM 2. *Given an integer $k \geqslant 2$, there are positive constants $C_{1,j,k}$, $C_{2,j,k}$, $2 \leqslant j \leqslant k$, such that for any integer $N \geqslant 1$ there is a complete polynomial $f(x) \in \mathbf{R}[x]$ of degree $N-1$ such that the number of terms in each power $f^j(x)$, $2 \leqslant j \leqslant k$, is bounded by $C_{1,j,k} N^{1-C_{2,j,k}}$.*