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On a problem of sums of mixed powers

by

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1. Introduction. The pruning technique means that if major arcs are too long and numerous for a straightforword treatment of the generating function we prune them. R. C. Vaughan [6], [7] first introduced one sort of pruning technique; some of those techniqus were extended by J. Brüdern [1], [2], [3]. Now, further improving the pruning technique yields the following.

Let $R_{b,c}(n)$ denote the number of representations of n as the sum of one square, four cubes, one bth power and one cth power of natural numbers. C. Hooley [5] got an asymptotic formula for $R_{3,5}(n)$ in 1981, and from J. Brüdern's work [3] one can easily deduce that, for all sufficiently large n,

$$R_{3,k}(n) = \frac{\Gamma(3/2)\Gamma(4/3)^5\Gamma((k+1)/k)}{\Gamma(1/2+5/3+1/k)}\mathfrak{S}_{3,k}(n)n^{7/6+1/k} + O(n^{7/6+1/k-2^{1-k}/k+\varepsilon}), \tag{1.1}$$

where

$$\mathfrak{S}_{b,c}(n) = \sum_{q=1}^{\infty} q^{-7} \sum_{\substack{a=1\\(a,q)=1}}^{q} S_2(q,a)S_3^4(q,a)S_b(q,a)S_c(q,a)e(-an/q) \tag{1.2}$$

with

$$S_k(q,a) = \sum_{a=1}^{q} e(ar^k/q). \tag{1.3}$$

Here and throughout ε is a sufficiently small positive number not necessarily the same in different formulae.

Furthermore, we can establish the asymptotic formulas for $R_{4,k}(n)$ $(4 \leqslant k \leqslant 6)$ and give lower estimates of the expected order of magnitude for $R_{4,k}(n)$ $(7 \leqslant k \leqslant 17)$, $R_{5,j}(n)$ $(5 \leqslant j \leqslant 9)$ and $R_{6,l}(n)$ $(6 \leqslant l \leqslant 7)$.

THEOREM 1. *For all sufficiently large n, we have*

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(1.4) $$R_{4,4}(n) = \frac{\Gamma(3/2)\Gamma(4/3)^4\Gamma(5/4)^2}{\Gamma(1/2+4/3+1/2)}\mathfrak{S}_{4,4}(n)n^{4/3}+O(n^{95/72+\varepsilon}),$$

(1.5) $$R_{4,5}(n) = \frac{\Gamma(3/2)\Gamma(4/3)^4\Gamma(5/4)\Gamma(6/5)}{\Gamma(1/2+4/3+1/4+1/5)}\mathfrak{S}_{4,5}(n)n^{77/60}+O(n^{1847/1440+\varepsilon}),$$

(1.6) $$R_{4,6}(n) = \frac{\Gamma(3/2)\Gamma(4/3)^4\Gamma(5/4)\Gamma(7/6)}{\Gamma(1/2+4/3+1/4+1/6)}\mathfrak{S}_{4,6}(n)n^{5/4}+O(n^{1799/1440+\varepsilon}).$$

THEOREM 2. *If* $b = 4, 7 \leqslant c \leqslant 17$, *or* $b = 5, 5 \leqslant c \leqslant 9$, *or* $b = 6, 6 \leqslant c \leqslant 7$, *we have*

(1.7) $$R_{b,c}(n) \gg n^{5/6+1/b+1/c}$$

for all sufficiently large n.

In the present paper, we only prove (1.6) and (1.7) for the case $b = 4$ and $c = 17$. The other results can be deduced similarly.

2. Notation and auxiliary results. Let

(2.1) $$P_k = n^{1/k},$$

(2.2) $$f_k(\alpha) = \sum_{x \leqslant P_k} e(x^k\alpha).$$

For $S = n^{\varrho}$ $(0 < \varrho \leqslant 1/2)$ and $1 \leqslant a \leqslant q \leqslant S$ with $(a,q) = 1$, let $\mathfrak{M}(q,a)$ denote the set of real numbers α with $|\alpha-a/q| \leqslant S/(qn)$, let $\mathfrak{M}$ denote their union, and note that the $\mathfrak{M}(q,a)$ are pairwise disjoint and contained in $(n^{-1}S, n^{-1}S+1]$. Let $\mathfrak{m} = (n^{-1}S, n^{-1}S+1]\backslash\mathfrak{M}$. By the standard method of estimating generating functions on minor arcs, we have

(2.3) $$f_2(\alpha) \ll n^{1/2+\varepsilon}S^{-1/2}$$

uniformly in $\alpha \in \mathfrak{m}$.

Further, put

(2.4) $$v_k(\beta) = \int_0^{P_k} e(\beta t^k)dt,$$

(2.5) $$V_k(\alpha, q, a) = q^{-1}S_k(q,a)v_k(\alpha-a/q),$$

and define $V_k(\alpha)$, $\Delta_k(\alpha)$ on $\mathfrak{M}$ by

(2.6) $$V_k(\alpha) = V_k(\alpha, q, a) \quad (\alpha \in \mathfrak{M}(q,a)),$$

(2.7) $$\Delta_k(\alpha) = f_k(\alpha) - V_k(\alpha) \quad (\alpha \in \mathfrak{M}).$$

Observe that, by Theorem 2 of Vaughan [9], we have

(2.8) $$\Delta_k(\alpha) \ll q^{1/2+\varepsilon}(1+n|\alpha-a/q|)^{1/2}.$$

Next, for $T = n^{\sigma}$ $(0 < \sigma < \varrho)$ and $1 \leqslant a \leqslant q \leqslant T$ with $(a,q) = 1$, let $\mathfrak{N}(q,a)$ denote the interval $\{\alpha\colon |\alpha-a/q| \leqslant T/(qn)\}$. The $\mathfrak{N}(q,a)$ are pairwise disjoint and contained in $\mathfrak{M}$. Let $\mathfrak{N}$ denote their union.

LEMMA 2.1. *We have*

(2.9) $$f_k(\alpha) \ll (n/T)^{1/k} + S^{1/2+\varepsilon},$$

uniformly in $\alpha \in \mathfrak{M}\backslash\mathfrak{N}$.

Proof. By (2.7) and (2.8), for $\alpha \in \mathfrak{M}$, we have

$$f_k(\alpha) \ll |V_k(\alpha)| + q^{1/2+\varepsilon}(1+n|\alpha-a/q|)^{1/2} \ll |V_k(\alpha)| + S^{1/2+\varepsilon}.$$

And by (2.5), (2.4) and Theorem 4.2, and Lemma 2.8 of Vaughan [10],

(2.10) $$V_k(\alpha) \ll q^{-1/k}\min(n^{1/k}, |\alpha-a/q|^{-1/k}).$$

Since $\alpha \in \mathfrak{M}\backslash\mathfrak{N}$, we have $q > T$ or $|\alpha-a/q| > T/(qn)$, that is,

$$|V_k(\alpha)| \ll (n/T)^{1/k}.$$

The proof is complete.

LEMMA 2.2. *Let* $G\colon \mathfrak{M} \to \boldsymbol{C}$ *be a function satisfying*

(2.11) $$G(\alpha) \ll q^{-2}(1+n|\alpha-a/q|)^{-2} \quad \text{for} \quad \alpha \in \mathfrak{M}(q,a)$$

and let a real function

(2.12) $$\Phi(\alpha) = \sum_{|h| \leqslant U} \eta(h)e(h\alpha)$$

satisfy

(2.13) $$\Phi(\alpha) \geqslant 0, \quad \eta(h) \geqslant 0$$

and $\log U \ll \log n$. *Then*

(2.14) $$\int_{\mathfrak{M}\backslash\mathfrak{N}} G(\alpha)\Phi(\alpha)d\alpha \ll \big(\eta(0) + T^{-1}\sum_{h \neq 0}\eta(h)\big)n^{-1+\varepsilon}.$$

COROLLARY. *For all positive integers* $k \geqslant 2$, *we have*

(2.15) $$\int_{\mathfrak{M}\backslash\mathfrak{N}} |V_k(\alpha)|^{2k}\Phi(\alpha)d\alpha \ll \big(\eta(0) + T^{-1}\sum_{h \neq 0}\eta(h)\big)n^{1+\varepsilon}.$$

Proof. By (2.11) and (2.12),

(2.16) $$\int_{\mathfrak{M}\backslash\mathfrak{N}} G(\alpha)\Phi(\alpha)d\alpha \ll \sum_{q \leqslant S} q^{-2} \sum_{\substack{a=1\\(a,q)=1}}^{q} \int_{\mathfrak{N}(q)} (1+n|\beta|)^{-2}\Phi(a/q+\beta)d\beta$$

$$= \sum_{q \leqslant S} q^{-2} \sum_{|h| \leqslant U} \eta(h)\Big(\sum_{\substack{a=1\\(a,q)=1}}^{q} e(ah/q)\Big) \int_{\mathfrak{N}(q)} (1+n|\beta|)^{-2}e(h\beta)d\beta,$$

where $\mathfrak{N}(q) = \{\beta\colon |\beta| \leqslant 1/2\}$ if $T < q \leqslant S$ and $\mathfrak{N}(q) = \{\beta\colon T/(qn) < |\beta| \leqslant 1/2\}$ if $q \leqslant T$.

Since

$$\int_{\mathfrak{N}(q)} (1+n|\beta|)^{-2} e(h\beta)\,d\beta \ll n^{-1}\min(1,q/T)$$

and

$$\sum_{\substack{a=1\\(a,q)=1}}^{q} e(au/q) = \sum_{d|(q,u)} \mu(q/d)d,$$

the terms in (2.16) with $h=0$ contribute

$$\ll \eta(0)\sum_{q\le S} q^{-1}n^{-1}\min(1,q/T) \ll \eta(0)n^{-1+\varepsilon}$$

and those with $h\neq 0$ contribute

$$\ll n^{-1+\varepsilon}\sum_{\substack{|h|\le U\\h\neq 0}}\eta(h)\sum_{d|h}\sum_{\substack{q\le S\\d|q}} d/q^2\min(1,q/T)$$

$$\ll n^{-1+\varepsilon}\sum_{\substack{|h|\le U\\h\neq 0}}\eta(h)\sum_{d|h}\Big(\sum_{r\le T/d}1/(Tr)+\sum_{r>T/d}1/(r^2d)\Big) \ll n^{-1+\varepsilon}T^{-1}\sum_{\substack{|h|\le U\\h\neq 0}}\eta(h).$$

The assertion now follows from (2.14) and (2.10).

LEMMA 2.3. *Suppose that* $t\ge\max(4,k+1)$. *Then*

(2.17) $$\int_{\mathfrak{N}}|V_k(\alpha)|^t\,d\alpha \ll n^{t/k-1+\varepsilon}.$$

Proof. By (2.5), (2.4) and Lemma 2.8 of Vaughan [10], for $t\ge k+1$, we have

$$\int_{\mathfrak{N}}|V_k(\alpha)|^t\,d\alpha \le \sum_{q\le T}\sum_{\substack{a=1\\(a,q)=1}}^{q}|q^{-1}S_k(q,a)|^t\int_{|\beta|\le 1/2}|v_k(\beta)|^t\,d\beta$$

$$\ll \sum_{q\le T}\sum_{\substack{a=1\\(a,q)=1}}^{q}|q^{-1}S_k(q,a)|^t\int_{|\beta|\le 1/2}\min(n^{t/k},|\beta|^{-t/k})\,d\beta$$

$$\ll n^{t/k-1}\sum_{q\le T}\sum_{\substack{a=1\\(a,q)=1}}^{q}|q^{-1}S_k(q,a)|^t,$$

also, by Lemma 4.9 of Vaughan [10], for $t\ge\max(4,k+1)$,

$$\sum_{q\le T}\sum_{\substack{a=1\\(a,q)=1}}^{q}|q^{-1}S_k(q,a)|^t \ll n^{\varepsilon},$$

therefore (2.17) is proved.

LEMMA 2.4. *Let* $2\le k_1\le k_2\le\ldots\le k_s$ *satisfy*

(2.18) $$\sum_{i=j+1}^{s}1/k_i\le 1/k_j,\qquad j=1,2,\ldots,s-1.$$

Then

(2.19) $$\int_0^1\Big|\prod_{i=1}^{s}f_{k_i}(\alpha)\Big|^2\,d\alpha \ll n^{(1/k_1+1/k_2+\ldots+1/k_s)+\varepsilon}.$$

3. The case $b=4$ **and** $c=6$. Let

(3.1) $$Q_1=n^{173/360},\quad Q_2=n^{4/9},\quad Q_3=n^{5/12},$$

(3.2) $$Q_4=n^{1/3},\quad Q_5=n^{1/4},\quad \tau_1=Q_1n^{-1}.$$

For $1\le a\le q\le Q_j$ with $(a,q)=1$, let $\mathfrak{M}_j(q,a)$ denote the set of real numbers α with $|\alpha-a/q|\le Q_j/(qn)$, and let $\mathfrak{M}_j$ denote their union. Note that the $\mathfrak{M}_j(q,a)$ are pairwise disjoint and $\mathfrak{M}_j\subset\mathfrak{M}_{j-1}$ $(2\le j\le 5)$, $\mathfrak{M}_1\subset(\tau_1,\tau_1+1]$. Furthermore, let $\mathfrak{m}_1=(\tau_1,\tau_1+1]\backslash\mathfrak{M}_1$.

Obviously, we have

(3.3) $$R_{4,6}(n)=\int_{\tau_1}^{\tau_1+1}f_2(\alpha)f_3^4(\alpha)f_4(\alpha)f_6(\alpha)e(-n\alpha)\,d\alpha.$$

LEMMA 3.1. *We have*

(3.4) $$I_1=\int_{\mathfrak{m}_1}f_2(\alpha)f_3^4(\alpha)f_4(\alpha)f_6(\alpha)e(-n\alpha)\,d\alpha \ll n^{1799/1440+\varepsilon},$$

(3.5) $$I_2=\int_{\mathfrak{M}_1}\Delta_2(\alpha)f_3^4(\alpha)f_4(\alpha)f_6(\alpha)e(-n\alpha)\,d\alpha \ll n^{1799/1440+\varepsilon}.$$

Proof. By (2.3), (3.1), Lemma 2.4, Hölder's inequality and Hua's inequality we have

$$I_1\ll n^{1/2+\varepsilon}Q_1^{-1/2}\Big(\int_0^1|f_4(\alpha)|^8d\alpha\Big)^{1/8}\Big(\int_0^1|f_3(\alpha)f_6^2(\alpha)|^2d\alpha\Big)^{1/4}\Big(\int_0^1|f_3(\alpha)|^4d\alpha\Big)^{3/8}\Big(\int_0^1|f_3(\alpha)|^8d\alpha\Big)^{1/4}$$

$$\ll n^{1/2+5/32+1/6+1/4+5/12+\varepsilon}Q_1^{-1/2}\ll n^{1799/1440+\varepsilon}.$$

And by (2.8) and in the same manner as for I_1 we have

$$I_2\ll n^{173/720+\varepsilon}\Big(\int_0^1|f_4(\alpha)|^8d\alpha\Big)^{1/8}\Big(\int_0^1|f_3(\alpha)f_6^2(\alpha)|^2d\alpha\Big)^{1/4}\Big(\int_0^1|f_3(\alpha)|^4d\alpha\Big)^{3/8}\Big(\int_0^1|f_3(\alpha)|^8d\alpha\Big)^{1/4}$$

$$\ll n^{1799/1440+\varepsilon}.$$

LEMMA 3.2. *We have*

(3.6) $$I_3=\int_{\mathfrak{M}_1\backslash\mathfrak{M}_2}V_2(\alpha)f_3^4(\alpha)f_4(\alpha)f_6(\alpha)e(-n\alpha)\,d\alpha \ll n^{1799/1440+\varepsilon},$$

(3.7) $$I_4=\int_{\mathfrak{M}_2\backslash\mathfrak{M}_3}V_2(\alpha)f_3^4(\alpha)f_4(\alpha)f_6(\alpha)e(-n\alpha)\,d\alpha \ll n^{1799/1440+\varepsilon},$$

(3.8) $$I_5=\int_{\mathfrak{M}_3\backslash\mathfrak{M}_4}V_2(\alpha)f_3^4(\alpha)f_4(\alpha)f_6(\alpha)e(-n\alpha)\,d\alpha \ll n^{1799/1440+\varepsilon}.$$

Proof. By Lemma 2.1,

$$f_3(\alpha) \ll n^{173/720+\varepsilon} \tag{3.9}$$

uniformly in $\alpha \in \mathfrak{M}_1 \backslash \mathfrak{M}_2$ and

$$f_3(\alpha) \ll n^{2/9} \tag{3.10}$$

uniformly in $\alpha \in \mathfrak{M}_2 \backslash \mathfrak{M}_3$ and $\alpha \in \mathfrak{M}_3 \backslash \mathfrak{M}_4$, therefore by (2.15), (3.1), (3.2), Lemma 2.4, Hölder's inequality and Hua's inequality we have

$$I_3 \ll (n^{173/720+\varepsilon})^{3/2} \Big(\int_{\mathfrak{M}_1 \backslash \mathfrak{M}_2} |V_2(\alpha)|^4 |f_4(\alpha)|^4 d\alpha \Big)^{1/4}$$
$$\times \Big(\int_0^1 |f_3(\alpha)|^2 |f_6(\alpha)|^4 d\alpha \Big)^{1/4} \Big(\int_0^1 |f_3(\alpha)|^4 d\alpha \Big)^{1/2}$$
$$\ll n^{173/480+\varepsilon} (n^{1+1/2+\varepsilon} + n^{2+\varepsilon} Q_2^{-1})^{1/4} n^{1/6+1/3+\varepsilon} \ll n^{1799/1440+\varepsilon},$$
$$I_4 \ll (n^{2/9})^{3/2} \Big(\int_{\mathfrak{M}_2 \backslash \mathfrak{M}_3} |V_2(\alpha)|^4 |f_4(\alpha)|^4 d\alpha \Big)^{1/4}$$
$$\times \Big(\int_0^1 |f_3(\alpha) f_6^2(\alpha)|^2 d\alpha \Big)^{1/4} \Big(\int_0^1 |f_3(\alpha)|^4 d\alpha \Big)^{1/2}$$
$$\ll n^{1/3} (n^{1+1/2+\varepsilon} + n^{2+\varepsilon} Q_3^{-1})^{1/4} n^{1/2+\varepsilon} \ll n^{1799/1440+\varepsilon}$$

and by Lemma 2 of Brüdern [3],

$$I_5 \ll (n^{2/9})^2 \Big(\int_{\mathfrak{M}_3} |V_2(\alpha)|^2 |f_4(\alpha) f_6(\alpha)|^2 d\alpha \Big)^{1/2} \Big(\int_0^1 |f_3(\alpha)|^4 d\alpha \Big)^{1/2}$$
$$\ll n^{4/9} (n^{5/12+1/4+1/6+\varepsilon} + n^{1/2+1/3+\varepsilon})^{1/2} n^{1/3+\varepsilon} \ll n^{1799/1440+\varepsilon}.$$

LEMMA 3.3. *We have*

$$I_6 = \Big(\int_{\mathfrak{M}_4} V_2(\alpha) (f_3^4(\alpha) - V_3^4(\alpha)) f_4(\alpha) f_6(\alpha) e(-n\alpha) d\alpha \ll n^{1799/1440+\varepsilon}, \tag{3.11}$$

$$I_7 = \int_{\mathfrak{M}_4 \backslash \mathfrak{M}_5} V_2(\alpha) V_3^4(\alpha) f_4(\alpha) f_6(\alpha) e(-n\alpha) d\alpha \ll n^{1799/1440+\varepsilon}, \tag{3.12}$$

$$I_8 = \int_{\mathfrak{M}_5} V_2(\alpha) V_3^4(\alpha) \Delta_4(\alpha) f_6(\alpha) e(-n\alpha) d\alpha \ll n^{1799/1440+\varepsilon}, \tag{3.13}$$

$$I_9 = \int_{\mathfrak{M}_5} V_2(\alpha) V_3^4(\alpha) V_4(\alpha) \Delta_6(\alpha) e(-n\alpha) d\alpha \ll n^{1799/1440+\varepsilon}. \tag{3.14}$$

Proof. By (2.7),

$$f_3^4(\alpha) - V_3^4(\alpha) \ll |\Delta_3(\alpha)| (|V_3(\alpha)|^3 + |\Delta_3(\alpha)|^3),$$

hence by (2.8), (3.2), Lemma 2 of Brüdern [3], Lemma 2.3, Hölder's inequality and Hua's inequality we have

$$I_6 \ll n^{1/6+\varepsilon} \int_{\mathfrak{M}_4} |V_2(\alpha)| |V_3(\alpha)|^3 |f_4(\alpha)| |f_6(\alpha)| d\alpha + n^{2/3+\varepsilon} \int_{\mathfrak{M}_4} |V_2(\alpha)| |f_4(\alpha)| |f_6(\alpha)| d\alpha$$

$$\ll n^{1/6+\varepsilon} \Big(\int_{\mathfrak{M}_4} |V_2(\alpha)|^2 |f_4(\alpha) f_6(\alpha)|^2 d\alpha \Big)^{1/2} \Big(\int_{\mathfrak{M}_4} |V_3(\alpha)|^6 d\alpha \Big)^{1/2}$$
$$+ n^{2/3+\varepsilon} \Big(\int_{\mathfrak{M}_4} |V_2(\alpha)|^4 d\alpha \Big)^{1/4} \Big(\int_0^1 |f_6(\alpha)|^4 d\alpha \Big)^{1/4} \Big(\int_0^1 |f_4(\alpha)|^2 d\alpha \Big)^{1/2}$$
$$\ll n^{1/6+\varepsilon} n^{1/4+1/6+\varepsilon} n^{1/2+\varepsilon} + n^{2/3+\varepsilon} n^{1/4+\varepsilon} n^{1/12+\varepsilon} n^{1/8+\varepsilon} \ll n^{1799/1440+\varepsilon}.$$

By (2.15), (3.2), Lemma 2.3 and Hölder's inequality we have

$$I_7 \ll \Big(\int_{\mathfrak{M}_4 \backslash \mathfrak{M}_5} |V_2(\alpha)|^4 |f_4(\alpha)|^4 d\alpha \Big)^{1/4} \Big(\int_{\mathfrak{M}_4 \backslash \mathfrak{M}_5} |V_3(\alpha)|^6 |f_6(\alpha)|^2 d\alpha \Big)^{1/2} \Big(\int_{\mathfrak{M}_4} |V_3(\alpha)|^4 d\alpha \Big)^{1/4}$$
$$\ll (n^{3/2+\varepsilon} + n^{2+\varepsilon} Q_5^{-1})^{1/4} (n^{7/6+\varepsilon} + n^{4/3+\varepsilon} Q_5^{-1})^{1/2} n^{1/12+\varepsilon}$$
$$\ll n^{7/16+\varepsilon} n^{7/12+\varepsilon} n^{1/12+\varepsilon} \ll n^{1799/1440+\varepsilon}.$$

And by (2.8), Lemma 2.3, Lemma 2 of Brüdern [3] and Hölder's inequality we have

$$I_8 \ll n^{1/8+\varepsilon} \Big(\int_{\mathfrak{M}_5} |V_2(\alpha)|^2 |f_6(\alpha)|^2 d\alpha \Big)^{1/2} \Big(\int_{\mathfrak{M}_5} |V_3(\alpha)|^8 d\alpha \Big)^{1/2}$$
$$\ll n^{1/8+\varepsilon} (n^{1/4+1/6+\varepsilon} + n^{1/3+\varepsilon})^{1/2} n^{5/6+\varepsilon} \ll n^{1799/1440+\varepsilon},$$
$$I_9 \ll n^{1/8+\varepsilon} \Big(\int_{\mathfrak{M}_5} |V_2(\alpha)|^4 d\alpha \Big)^{1/4} \Big(\int_{\mathfrak{M}_5} |V_4(\alpha)|^8 d\alpha \Big)^{1/8} \Big(\int_{\mathfrak{M}_5} |V_3(\alpha)|^4 d\alpha \Big)^{1/4} \Big(\int_{\mathfrak{M}_5} |V_3(\alpha)|^8 d\alpha \Big)^{3/8}$$
$$\ll n^{1799/1440+\varepsilon}.$$

By a standard argument one can deduce

$$\int_{\mathfrak{M}_5} V_2(\alpha) V_3^4(\alpha) V_4(\alpha) V_6(\alpha) e(-n\alpha) d\alpha$$
$$= \frac{\Gamma(3/2)\Gamma(4/3)^4 \Gamma(5/4)\Gamma(7/6)}{\Gamma(1/2+4/3+1/4+1/6)} \mathfrak{S}_{4,6}(n) n^{5/4} + O(n^{1799/1440+\varepsilon}).$$

Therefore (1.6) is proved.

4. The case $b = 4$ and $c = 17$. Throughout δ, η_0, η, λ and ν denote sufficiently small but fixed positive numbers, and let $\eta < \eta_0 \delta$,

$$X = P_3^{1/8} = n^{1/24}, \qquad W = P_3^{\eta}, \tag{4.1}$$

$$\mathscr{A}(P_3, W) = \{n\colon\ n \leqslant P_3, p \mid n \Rightarrow p \leqslant W\}, \tag{4.2}$$

$$f_3(\alpha; m) = \sum_{\substack{x \leqslant P_3 \\ (x,m)=1}} e(\alpha x^3), \tag{4.3}$$

$$g_3(\alpha; m) = \sum_{x \in \mathscr{A}(P_3/m, W)} e(\alpha x^3). \tag{4.4}$$

We define

$$F(\alpha) = \sum_{X < p_1 \leqslant XW} \sum_{X < p_2 \leqslant XW} f_3(\alpha; p_1 p_2) g_3(p_1^3 \alpha; p_1) g_3(p_2^3 \alpha; p_2), \tag{4.5}$$

$$R^*_{4,17}(n) = \int_0^1 f_2(\alpha)f_3(\alpha)f_4(\alpha)f_{17}(\alpha)F(\alpha)e(-n\alpha)d\alpha. \tag{4.6}$$

Clearly, $R^*_{4,17}(\alpha)$ is the number of representations of n in the form

$$x_1^2+x_2^3+x_3^4+x_4^{17}+y^3+p_1^3z_1^3+p_2^3z_2^3$$

with $X < p_1,\ p_2 \leqslant XW$, $(y, p_1p_2) = 1$ and $z_i \in \mathscr{A}(P_3/p_i, W)$. Since no number not exceeding $n^{1/3}$ has more than 7 prime divisors exceeding $n^{1/24}$, it follows that

$$R_{4,17}(n) \geqslant (1/49)R^*_{4,17}(n). \tag{4.7}$$

We now show that

$$R^*_{4,17}(n) \gg n^{233/204}. \tag{4.8}$$

Let

$$R_1 = n^{15/34+4\delta},\quad R_2 = n^{1/3},\quad R_3 = n^{6/25}, \tag{4.9}$$

$$R_4 = n^{1/24},\quad R_5 = (\log n)^{\lambda},\quad \tau_2 = n^{-1}R_1. \tag{4.10}$$

For $1 \leqslant a \leqslant q \leqslant R_j$ with $(a,q) = 1$, let $\mathfrak{R}_j(q,a)$ denote the set of real numbers α with $|\alpha - a/q| < R_j/(qn)$, and let $\mathfrak{R}_j$ denote their union. Note that the $\mathfrak{R}_j(q,a)$ are pairwise disjoint and $\mathfrak{R}_j \subset \mathfrak{R}_{j-1}$ $(2 \leqslant j \leqslant 5)$, $\mathfrak{R}_1 \subset (\tau_2, \tau_2+1]$. Further, let $\mathfrak{n}_1 = (\tau_2, \tau_2+1]\backslash\mathfrak{R}_1$.

LEMMA 4.1. *We have*

$$J_1 = \int_{\mathfrak{n}_1} f_2(\alpha)f_3(\alpha)f_4(\alpha)f_{17}(\alpha)F(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}, \tag{4.11}$$

$$J_2 = \int_{\mathfrak{R}_1} \Delta_2(\alpha)f_3(\alpha)f_4(\alpha)f_{17}(\alpha)F(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}. \tag{4.12}$$

Proof. By Lemmas 3.7, 4.4 of Vaughan [11] and (4.1) we have, for $i = 1, 2$,

$$\begin{aligned}\sum_{X<p_i\leqslant XW}\int_0^1 |f_3(\alpha;p_1p_2)|^2|g_3(p_1^3\alpha;p_i)|^4 d\alpha &\leqslant T_3(P_3, W, 1/8)\\ &\ll P_3^{3+\varepsilon}X^{-1}W + P_3^{7/6+\varepsilon}X^{-3/2}WS_3(P_3/X, W)^{2/3}\\ &\ll P_3^{3+\varepsilon}X^{-1}W + P_3^{7/6+\varepsilon}X^{-3/2}W((P_3/X)^{13/4+\delta})^{2/3} \ll P_3^{3+\delta+\varepsilon}X^{-1},\end{aligned}$$

hence by Cauchy's inequality and Schwarz's inequality

$$\int_0^1 |F(\alpha)|^2 d\alpha \ll X^2W^3P_3^{3+\delta+\varepsilon} \ll X^2n^{1+3\delta/2}. \tag{4.13}$$

And by (2.3), (4.9), Lemma 2.4 and (4.13) we have

$$\begin{aligned}J_1 &\ll n^{1/2+\varepsilon}R_1^{-1/2}\Big(\int_0^1 |f_3(\alpha)f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/2}\Big(\int_0^1 |F(\alpha)|^2 d\alpha\Big)^{1/2}\\ &\ll n^{1/2+\varepsilon}R_1^{-1/2}n^{1/6+1/8+1/34+\varepsilon}Xn^{1/2+3\delta/4} \ll n^{233/204-\delta},\end{aligned}$$

also by (2.8) and $R_1^{1/2+\varepsilon} \ll n^{1/2+\varepsilon}R_1^{-1/2}$ we have

$$J_2 \ll R_1^{1/2+\varepsilon}\Big(\int_0^1 |f_3(\alpha)f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/2}\Big(\int_0^1 |F(\alpha)|^2 d\alpha\Big)^{1/2} \ll n^{233/204-\delta}.$$

LEMMA 4.2. *We have*

$$J_3 = \int_{\mathfrak{R}_1\backslash\mathfrak{R}_2} V_2(\alpha)f_3(\alpha)f_4(\alpha)f_{17}(\alpha)F(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}, \tag{4.14}$$

$$J_4 = \int_{\mathfrak{R}_2} V_2(\alpha)\Delta_3(\alpha)f_4(\alpha)f_{17}(\alpha)F(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}. \tag{4.15}$$

Proof. By Lemma 2.1 we have

$$f_3(\alpha) \ll (n^{2/3})^{1/3} + n^{15/68+2\delta+\varepsilon} \ll n^{2/9}$$

uniformly for $\alpha \in \mathfrak{R}_1\backslash\mathfrak{R}_2$, hence by (2.15), Lemma 2.4, (4.13) and Hölder's inequality we obtain

$$\begin{aligned}J_3 &\ll (n^{2/9})^{1/2}\Big(\int_{\mathfrak{R}_1\backslash\mathfrak{R}_2} |V_2(\alpha)|^4|f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/4}\\ &\qquad\times\Big(\int_0^1 |f_3(\alpha)f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/4}\Big(\int_0^1 |F(\alpha)|^2 d\alpha\Big)^{1/2}\\ &\ll n^{1/9}(n^{1+1/4+1/17+\varepsilon} + n^{1+1/2+2/17+\varepsilon}R_2^{-1})^{1/4}\\ &\qquad\times n^{1/12+1/16+1/68+\varepsilon}n^{1/2+3\delta/4}X \ll n^{233/204-\delta}.\end{aligned}$$

And by (2.8), Lemma 2 of Brüdern [3] and (4.13) we have

$$\begin{aligned}J_4 &\ll n^{1/6+\varepsilon}\Big(\int_{\mathfrak{R}_2} |V_2(\alpha)|^2|f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/2}\Big(\int_0^1 |F(\alpha)|^2 d\alpha\Big)^{1/2}\\ &\ll n^{1/6+\varepsilon}(n^{1/3+1/4+1/17+\varepsilon} + n^{1/2+2/17+\varepsilon})^{1/2}n^{13/24+3\delta/4} \ll n^{233/204-\delta}.\end{aligned}$$

LEMMA 4.3. *We have*

$$J_5 = \int_{\mathfrak{R}_2\backslash\mathfrak{R}_3} V_2(\alpha)V_3(\alpha)f_4(\alpha)f_{17}(\alpha)F(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}, \tag{4.16}$$

$$J_6 = \int_{\mathfrak{R}_3} V_2(\alpha)V_3(\alpha)\Delta_4(\alpha)f_{17}(\alpha)F(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}. \tag{4.17}$$

Proof. By (2.15), (4.13), Lemma 2.4 and Hölder's inequality we have

$$\begin{aligned}J_5 &\ll \Big(\int_{\mathfrak{R}_2\backslash\mathfrak{R}_3} |V_2(\alpha)|^4|f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/4}\Big(\int_0^1 |f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/12}\\ &\qquad\times\Big(\int_{\mathfrak{R}_2\backslash\mathfrak{R}_3} |V_3(\alpha)|^6|f_4(\alpha)f_{17}(\alpha)|^2 d\alpha\Big)^{1/6}\Big(\int_0^1 |F(\alpha)|^2 d\alpha\Big)^{1/2}\\ &\ll (n^{1+1/4+1/17+\varepsilon} + R_3^{-1}n^{1+1/2+2/17+\varepsilon})^{5/12}(n^{1/4+1/17+\varepsilon})^{1/12}n^{13/24+3\delta/4}\\ &\ll n^{233/204-\delta}.\end{aligned}$$

And by (2.8), (4.13), Lemma 2 of Brüdern [3], Lemma 2.3 and Hölder's inequality we have

$$J_6 \ll n^{3/25+\varepsilon}\Big(\int_{\mathfrak{R}_3} |V_2(\alpha)|^2 |f_{17}(\alpha)|^4 d\alpha\Big)^{1/4}\Big(\int_{\mathfrak{R}_3} |V_2(\alpha)|^4 d\alpha\Big)^{1/8}$$

$$\times\Big(\int_{\mathfrak{R}_3} |V_3(\alpha)|^8 d\alpha\Big)^{1/8}\Big(\int_0^1 |F(\alpha)|^2 d\alpha\Big)^{1/2}$$

$$\ll n^{3/25+\varepsilon}(n^{6/25+2/17+\varepsilon}+n^{4/17+\varepsilon})^{1/4} n^{1/8+5/24+\varepsilon} n^{13/24+3\delta/4} \ll n^{233/204-\delta}.$$

Now let

$$S_3(q,a;m) = \sum_{\substack{x=1 \\ (x,m)=1}}^{qm} e(ax^3/q), \tag{4.18}$$

$$V_3(\alpha;m) = V_3(\alpha,q,a;m) = q^{-1}m^{-1}S_3(q,a;m)v_3(\alpha-a/q) \quad (\alpha\in\mathfrak{R}_3(q,a)), \tag{4.19}$$

$$\psi(\alpha) = \left(\frac{n/q}{1+n|\alpha-a/q|}\right)^{1/3}. \tag{4.20}$$

By §6 of Vaughan [7] we have

$$\Delta_3(\alpha;m) = f_3(\alpha;m) - V_3(\alpha,q,a;m) \ll q^{1/2+\varepsilon}d(m)(1+n|\alpha-a/q|)^{1/2}, \tag{4.21}$$

$$V_3(\alpha;m) \ll \psi(\alpha) \tag{4.22}$$

and for $q \leqslant W$ and $j = 1,2$ we define

$$w(\beta;p_j) = \sum_{W^3<m\leqslant n/p_j^3} \frac{1}{3}m^{-2/3}\varrho\left(\frac{\log m}{3\log W}\right)e(p_j^3\beta m), \tag{4.23}$$

where $\varrho(x)$ is Dickman's function and

$$W_3(\alpha;p_j) = W_3(\alpha,q,a;p_j) = q^{-1}S_3(q,p_j^3a)w(\alpha-a/q;p_j). \tag{4.24}$$

In the same manner as in the proof of Lemma 5.4 of Vaughan [11] we have

$$g_3(p_j^3\alpha;p_j) = W_3(\alpha;p_j)+E_3(\alpha;p_j), \tag{4.25}$$

$$E_3(\alpha;p_j) \ll \frac{qP_3/p_j}{\log P_3}(1+n|\alpha-a/q|), \tag{4.26}$$

$$\sum_{X<p_j\leqslant XW} E_3(\alpha;p_j) \ll \frac{qP_3}{\log P_3}(1+n|\alpha-a/q|). \tag{4.27}$$

Since $p_j > P_3^{1/8} > W$ we may have $S_3(q,p_j^3a) = S_3(q,a)$, hence by Lemma 5.4 of Vaughan [11]

$$W_3(\alpha;p_j) \ll q^{-1/3}\min(P_3/p_j,(p_j^3|\alpha-a/q|)^{-1/3}), \tag{4.28}$$

$$\sum_{X<p_j\leqslant XW} W_3(\alpha;p_j) \ll \psi(\alpha). \tag{4.29}$$

Furthermore, we define

$$F^*(\alpha) = \sum_{X<p_1\leqslant XW}\ \sum_{X<p_2\leqslant XW} V_3(\alpha;p_1p_2)g_3(p_1^3\alpha;p_1)g_3(p_2^3\alpha;p_2), \tag{4.30}$$

$$V(\alpha) = \sum_{X<p_1\leqslant XW}\ \sum_{X<p_2\leqslant XW} V_3(\alpha;p_1p_2)W_3(\alpha;p_1)W_3(\alpha;p_2). \tag{4.31}$$

LEMMA 4.4. *We have*

$$J_7 = \int_{\mathfrak{R}_3} V_2(\alpha)V_3(\alpha)V_4(\alpha)f_{17}(\alpha)(F(\alpha)-F^*(\alpha))e(-n\alpha)d\alpha \ll n^{233/204-\delta}, \tag{4.32}$$

$$J_8 = \int_{\mathfrak{R}_3\setminus\mathfrak{R}_4} V_2(\alpha)V_3(\alpha)V_4(\alpha)f_{17}(\alpha)F^*(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}, \tag{4.33}$$

$$J_9 = \int_{\mathfrak{R}_4} V_2(\alpha)V_3(\alpha)V_4(\alpha)\Delta_{17}(\alpha)F^*(\alpha)e(-n\alpha)d\alpha \ll n^{233/204-\delta}. \tag{4.34}$$

Proof. By (4.21), for $\alpha\in\mathfrak{R}_3$ we have

$$F(\alpha)-F^*(\alpha) \ll n^{3/25+\varepsilon}\Big(\sum_{X<p\leqslant XW} |g_3(p^3\alpha;p)|\Big)^2$$

and by Hölder's inequality and Hua's inequality

$$\int_0^1\Big(\sum_{X<p\leqslant XW} |g_3(p^3\alpha;p)|\Big)^4 d\alpha \leqslant (XW)^3 \sum_{X<p\leqslant XW}\int_0^1 |g_3(p^3\alpha;p)|^4 d\alpha \tag{4.35}$$

$$\ll (XW)^3 \sum_{X<p\leqslant XW} (P_3/p)^{2+\varepsilon} \ll n^{3/4+\eta+\varepsilon},$$

hence by (4.35), Lemma 2.3 and Hölder's inequality we obtain

$$J_7 \ll n^{3/25+\varepsilon}n^{1/17}\Big(\int_{\mathfrak{R}_3} |V_2(\alpha)|^4 d\alpha\Big)^{1/4}\Big(\int_{\mathfrak{R}_3} |V_3(\alpha)|^8 d\alpha\Big)^{1/8}$$

$$\times\Big(\int_{\mathfrak{R}_3} |V_4(\alpha)|^8 d\alpha\Big)^{1/8}\Big(\int_0^1\Big(\sum_{X<p\leqslant XW} |g_3(p^3\alpha;p)|\Big)^4 d\alpha\Big)^{1/2}$$

$$\ll n^{3/25+\varepsilon}n^{1/17}n^{1/4+\varepsilon}n^{5/24+\varepsilon}n^{1/8+\varepsilon}n^{3/8+\eta/2+\varepsilon} \ll n^{233/204-\delta}.$$

Moreover, by (4.22), Lemma 4.6 of Vaughan [10], Lemma 2 of Brüdern [3] and Hölder's inequality

$$\int_{\mathfrak{R}_3} |V_3(\alpha)||F^*(\alpha)|^2 d\alpha \ll (XW)^3 \sum_{X<p\leqslant XW}\int_{\mathfrak{R}_3} |V_3(\alpha)|\psi(\alpha)^2|g_3(p^3\alpha;p)|^4 d\alpha \tag{4.36}$$

$$\ll (XW)^3 \sum_{X<p\leqslant XW} (n^{6/25+\varepsilon}(P_3/p)^2+(P_3/p)^{4+\varepsilon})$$

$$\ll (XW)^3(n^{6/25+\varepsilon}n^{2/3}X^{-1}+n^{4/3+\varepsilon}X^{-3}) \ll n^{4/3+\eta+\varepsilon}.$$

Therefore, by (2.15), (4.36), Lemma 2.3 and Hölder's inequality we have

$$J_8 \ll \Big(\int_{\mathfrak{R}_3\setminus\mathfrak{R}_4} |V_2(\alpha)|^4|f_{17}(\alpha)|^4 d\alpha\Big)^{1/4}\Big(\int_{\mathfrak{R}_3} |V_3(\alpha)|^4 d\alpha\Big)^{1/8}$$

$$\times\Big(\int_{\mathfrak{R}_3} |V_4(\alpha)|^8 d\alpha\Big)^{1/8}\Big(\int_{\mathfrak{R}_3} |V_3(\alpha)||F^*(\alpha)|^2 d\alpha\Big)^{1/2}$$

$$\ll (n^{1+2/17+\varepsilon}+R_4^{-1}n^{1+4/17+\varepsilon})^{1/4}n^{1/6+\varepsilon}n^{2/3+\eta/2+\varepsilon} \ll n^{233/204-\delta}.$$

Also, by (2.8), (4.36), Lemma 2.3 and Hölder's inequality we have

$$J_9 \ll n^{1/48+\varepsilon}\Big(\int_{\mathfrak{R}_4} |V_2(\alpha)|^4 d\alpha\Big)^{1/4}\Big(\int_{\mathfrak{R}_4} |V_3(\alpha)|^4 d\alpha\Big)^{1/8}$$
$$\times \Big(\int_{\mathfrak{R}_4} |V_4(\alpha)|^8 d\alpha\Big)^{1/8}\Big(\int_{\mathfrak{R}_4} |V_3(\alpha)|\,|F^*(\alpha)|^2 d\alpha\Big)^{1/2}$$
$$\ll n^{1/48+\varepsilon} n^{5/12+\varepsilon} n^{2/3+\eta/2+\varepsilon} \ll n^{233/204-\delta}.$$

LEMMA 4.5. *We have*

$$(4.37)\qquad J_{10} = \int_{\mathfrak{R}_4\setminus\mathfrak{R}_5} V_2(\alpha)V_3(\alpha)V_4(\alpha)V_{17}(\alpha)F^*(\alpha)e(-n\alpha)d\alpha \ll n^{233/204}(\log n)^{-\nu}.$$

$$(4.38)\qquad J_{11} = \int_{\mathfrak{R}_5} V_2(\alpha)V_3(\alpha)V_4(\alpha)V_{17}(\alpha)\big(F^*(\alpha)-V(\alpha)\big)e(-n\alpha)d\alpha$$
$$\ll n^{233/204}(\log n)^{-\nu}.$$

Proof. Let

$$S(q) = \sum_{\substack{a=1\\(a,q)=1}}^{q} q^{-5}|S_2(q,a)S_3(q,a)^2S_4(q,a)S_{17}(q,a)|.$$

By Theorem 4.2 of Vaughan [10], $q^{1/17}S(q) \ll q^{-5/12}$. Thus $\sum_{h=3}^{\infty} p^{h/17}S(p^h) \ll p^{-5/4}$. And by Lemmas 4.3 and 4.4 of Vaughan [10], for $p > 17$ we have $p^{1/17}S(p) \ll p^{-5/4}$ and $p^{2/17}S(p^2) \ll p^{-5/4}$. Moreover, $q^{1/17}S(q)$ is a multiplicative function of q. Therefore, there is an absolute constant C such that

$$(4.39)\qquad \sum_{q\le R_4} q^{1/17}S(q) \ll \prod_{p\le R_4}(1+Cp^{-5/4}) \ll 1.$$

For $q \le n^{1/24} < p_j$ $(j=1,2)$, $S_3(q,a;p_1p_2) = \varphi(p_1p_2)S_3(q,a)$, therefore by (4.19)

$$(4.40)\qquad |V_3(\alpha;p_1p_2)| \le |V_3(\alpha)|.$$

Noting that

$$\sum_{X<p\le XW} |g_3(p^3\alpha;p)| \le \sum_{X<p\le XW}(P_3/p) \ll P_3,$$

we can obtain

$$J_{10} \ll \int_{\mathfrak{R}_4\setminus\mathfrak{R}_5} |V_2(\alpha)|\,|V_3(\alpha)|^2|V_4(\alpha)|\,|V_{17}(\alpha)|\Big(\sum_{X<p\le XW}|g_3(p^3\alpha;p)|\Big)^2 d\alpha$$
$$\ll P_3^2 \int_{\mathfrak{R}_4\setminus\mathfrak{R}_5} |V_2(\alpha)|\,|V_3(\alpha)|^2\,|V_4(\alpha)|\,|V_{17}(\alpha)|d\alpha$$
$$\ll n^{233/204}\sum_{q\le R_4} S(q)\min\big(1,(q/R_5)^{5/12+1/17}\big)$$
$$\le n^{233/204}\Big(R_5^{-1/17}\sum_{q\le R_5} q^{1/17}S(q) + \sum_{R_5<q\le R_4} S(q)\Big) \ll n^{233/204}R_5^{-1/17}.$$

Now choose ν with

$$(4.41)\qquad \nu < \min(\lambda/17, 1-\lambda).$$

This establishes (4.37).

By (4.30), (4.31), (4.25) and (4.40)

$$F^*(\alpha)-V(\alpha) \ll |V_3(\alpha)|\Big(\sum_{X<p\le XW}|W_3(\alpha;p)|\sum_{X<p\le XW}|E_3(\alpha;p)| + \Big(\sum_{X<p\le XW}|E_3(\alpha;p)|\Big)^2\Big)$$

thus by (4.26), (4.29), (4.39) and (4.41) we have

$$J_{11} \ll \big(R_5P_3^2/(\log n) + R_5^2P_3^2/(\log n)^2\big)\int_{\mathfrak{R}_5} |V_2(\alpha)|\,|V_3(\alpha)|^2\,|V_4(\alpha)|\,|V_{17}(\alpha)|d\alpha$$
$$\ll n^{2/3+5/12+1/17}R_5/(\log n) \ll n^{233/204}(\log n)^{-\nu}.$$

We further define

$$(4.42)\qquad V_k^*(\alpha) = q^{-1}S_k(q,a)\sum_{m\le n}(1/k)m^{1/k-1}e\big(m(\alpha-a/q)\big).$$

Then for $\alpha \in \mathfrak{R}_5$ we have

$$V_k(\alpha) = V_k^*(\alpha) + O\big(q^{-1/k}(1+n|\alpha-a/q|)\big) = V_k^*(\alpha) + O(R_5q^{-1-1/k}),$$

therefore

$$V_2(\alpha)\big(V_3(\alpha)\big)^2V_4(\alpha)V_{17}(\alpha) - V_2^*(\alpha)\big(V_3^*(\alpha)\big)^2V_4^*(\alpha)V_{17}^*(\alpha)$$
$$\ll R_5q^{-1-1/2-2/3-1/4-1/17}\min\big(n^{1/2+2/3+1/4}, |\alpha-a/q|^{-1/2-2/3-1/4}\big),$$

so that by (4.19) and (4.29) we have

$$\int_{\mathfrak{R}_5} V_2(\alpha)V_3(\alpha)V_4(\alpha)V_{17}(\alpha)V(\alpha)e(-n\alpha)d\alpha$$
$$= \int_{\mathfrak{R}_5} V_2^*(\alpha)\big(V_3^*(\alpha)\big)^2V_4^*(\alpha)V_{17}^*(\alpha)\sum_{X<p_1\le XW}\ \sum_{X<p_2\le XW}\frac{\varphi(p_1p_2)}{p_1p_2}W_3(\alpha;p_1)$$
$$\times W_3(\alpha;p_2)e(-n\alpha)d\alpha + O(R_5n^{13/12})$$
$$= I + O(n^{233/204}(\log n)^{-\nu}),\quad \text{say}.$$

By imitating the usual method of estimation on the major arcs, we obtain

$$I = \mathfrak{S}_{4,17}(n)J(n) + O\big(n^{233/204}(\log n)^{-\nu}\big),$$

where

$$J(n) = \sum_{p_1}\sum_{p_2}\sum_{y_1}\sum_{y_2}\sum_{y_3}\sum_{y_4}\sum_{y_5}\sum_{x_1}\sum_{x_2}\frac{1}{2}\Big(\frac{1}{3}\Big)^4\frac{1}{20}\frac{\varphi(p_1p_2)}{p_1p_2}y_1^{1/2-1}$$
$$\times (y_2y_3x_1x_2)^{1/3-1}y_4^{1/4-1}y_5^{1/17-1}\varrho_1\varrho_2$$

with

$$\varrho_j = \varrho((\log x_j)/(3\log W)),$$

and the multiple sum is over $p_1, p_2, y_1, y_2, y_3, y_4, y_5, x_1, x_2$ satisfying

$$y_j \leqslant n \quad (1 \leqslant j \leqslant 5), \qquad X < p_1, p_2 \leqslant XW, \qquad W^3 < x_j \leqslant n/p_j^3 \quad (j = 1, 2),$$

$$y_1 + y_2 + y_3 + y_4 + y_5 + x_1 p_1^3 + x_2 p_2^3 = n.$$

A simple counting argument combined with the fact that

$$\varrho((\log x_j)/(3\log W)) \gg 1$$

for

$$W^3 < x_j \leqslant n/p_j^3 < n^{7/8} \quad \text{and} \quad \sum_{X < p_1 \leqslant XW} \sum_{X < p_2 \leqslant XW} \varphi(p_1 p_2)/(p_1 p_2)^2 \gg 1$$

establishes

$$J(n) \gg n^{233/204}.$$

Moreover, $\mathfrak{S}_{4,17}(n) \gg 1$, therefore the proof of (4.8) is complete.

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