that is, since n is prime, $H' = \{f^c\}_{c\in(\mathbf{Z}/n\mathbf{Z})^*}$. Then, by (5.4.1),

$$\mathrm{Tr}_{F/\mathbf{Q}}(W'(k, f)) = \sum_{g\in H'} W'(k, g),$$

and Theorem (5.2) enables us to write

$$|\mathrm{Tr}_{F/\mathbf{Q}}(W'(k, f))| = \frac{(r-1)(2g_X-2)+\sum_{c\in(\mathbf{Z}/n\mathbf{Z})^*}\sum_{u\in U'(f^c)}\deg u}{2}[2\sqrt{q}]$$

$$= (n-1)\frac{C(f)}{2}[2\sqrt{q}].$$

Since n is prime and $f \notin k^* K^{*n}$, hence $r = n$.

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Received on 28.6.1990 (2057)

ACTA ARITHMETICA
LIX.3 (1991)

Symmetric Diophantine systems

by

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1. Introduction. In this paper we will consider symmetric Diophantine systems in $2n$ independent variables x_j, $j = 1, 2, \ldots, n$, and y_j, $j = 1, 2, \ldots, n$, consisting of a set of simultaneous Diophantine equations of the type

$$(1.1) \qquad f_i(x_1, x_2, \ldots, x_n) = f_i(y_1, y_2, \ldots, y_n), \quad i = 1, 2, \ldots, m,$$

where $f_i(x_1, x_2, \ldots, x_n)$ [written briefly as $f_i(x_j)$ or simply as f_i] is, for each i, a homogeneous form in the variables $x_1, x_2, \ldots, x_n$. We shall describe a method that can be applied to solve several such Diophantine systems. The solutions obtained are parametric but, unless otherwise stated, are not necessarily complete. We shall use L's, Q's, C's and F's to denote linear, quadratic, cubic and quartic forms. We shall first solve under quite general conditions the following Diophantine systems in $2n$ variables x_j, y_j, $j = 1, 2, \ldots, n$:

$$(\mathrm{I}) \qquad \begin{cases} L_i(x_j) = L_i(y_j), & i = 1, 2, \ldots, n-1, \\ Q_i(x_j) = Q_i(y_j), & i = 1, 2, \ldots, n-1. \end{cases}$$

$$(\mathrm{II}) \qquad \begin{cases} L_i(x_j) = L_i(y_j), & i = 1, 2, \ldots, n-2, \\ C(x_j) = C(y_j). \end{cases}$$

A particular case of interest is the system

$$\begin{cases} L(x_1, x_2, x_3) = L(y_1, y_2, y_3), \\ C(x_1, x_2, x_3) = C(y_1, y_2, y_3). \end{cases}$$

$$(\mathrm{III}) \qquad \begin{cases} Q_i(x_j) = Q_i(y_j), & i = 1, 2, \ldots, n-2, \\ C(x_j) = C(y_j). \end{cases}$$

A particular case of interest is the system

$$\begin{cases} Q(x_1, x_2, x_3) = Q(y_1, y_2, y_3), \\ C(x_1, x_2, x_3) = C(y_1, y_2, y_3). \end{cases}$$

$$(\mathrm{IV}) \qquad \begin{cases} L_i(x_j) = L_i(y_j), & i = 1, 2, \ldots, n-2, \\ Q_i(x_j) = Q_i(y_j), & i = 1, 2, \ldots, k < n/2, \\ C(x_j) = C(y_j). \end{cases}$$

A particular case of interest is the system

$$\begin{cases} L(x_1, x_2, x_3) = L(y_1, y_2, y_3), \\ Q(x_1, x_2, x_3) = Q(y_1, y_2, y_3), \\ C(x_1, x_2, x_3) = C(y_1, y_2, y_3). \end{cases}$$

$$\text{(V)} \qquad \begin{cases} L(x_j) = L(y_j), \\ Q_i(x_j) = Q_i(y_j), \quad i = 1, 2, \ldots, n-3, \\ F(x_j) = F(y_j). \end{cases}$$

Next, we shall obtain parametric solutions of the following specific Diophantine systems:

$$\text{(VI)} \qquad \sum_{j=1}^{3} x_j^r = \sum_{j=1}^{3} y_j^r, \quad r = 1, 3.$$

$$\text{(VII)} \qquad \begin{cases} x_1+x_2+x_3 = y_1+y_2+y_3, \\ x_1x_2x_3 = y_1y_2y_3. \end{cases}$$

$$\text{(VIII)} \qquad \begin{cases} x_1+x_2+x_3 = y_1+y_2+y_3, \\ x_1^3+x_2^3+x_3^3 = y_1^3+y_2^3+y_3^3, \\ x_1x_2x_3 = y_1y_2y_3. \end{cases}$$

$$\text{(IX)} \qquad \sum_{j=1}^{3} x_j^r = \sum_{j=1}^{3} y_j^r, \quad r = 2, 3.$$

$$\text{(X)} \qquad \begin{cases} x_1^2+x_2^2+x_3^2 = y_1^2+y_2^2+y_3^2, \\ x_1x_2x_3 = y_1y_2y_3. \end{cases}$$

$$\text{(XI)} \qquad \sum_{j=1}^{4} x_j^r = \sum_{j=1}^{4} y_j^r, \quad r = 1, 2, 3, 5.$$

$$\text{(XII)} \qquad \sum_{j=1}^{5} x_j^r = \sum_{j=1}^{5} y_j^r, \quad r = 1, 2, 3, 4.$$

$$\text{(XIII)} \qquad \sum_{j=1}^{7} x_j^r = \sum_{j=1}^{7} y_j^r, \quad r = 1, 2, 3, 4, 5, 6.$$

$$\text{(XIV)} \qquad \sum_{j=1}^{3} x_j^r = \sum_{j=1}^{3} y_j^r, \quad r = 1, 3, 4.$$

Finally, we shall obtain a parametric solution in Gaussian integers of the system

$$\text{(XV)} \qquad \sum_{j=1}^{3} x_j^r = \sum_{j=1}^{3} y_j^r, \quad r = 1, 3, 7.$$

All the results obtained in this paper are new. The Diophantine systems (I)–(V), (VIII), (IX), (XIV) and (XV) have not been considered earlier. The solutions for systems (I), (VI) and (VII) given in this paper are complete while the solution for the system (XII) has more parameters than the solutions obtained earlier.

2. The general method applied in this paper for solving systems of the type (1.1) consists in writing $x_j = a_j\theta+\alpha_j$, $j = 1, 2, \ldots, n$, and $y_j = b_j\theta+\alpha_j$, $j = 1, 2, \ldots, n$, and expressing each equation of the system in the form $\sum_r c_r(a_j, b_j, \alpha_j)\theta^r = 0$. We then equate to zero the coefficients of θ^r for all values of r, for all except one of the equations. In this remaining equation we equate to zero all the coefficients of θ^r except for two consecutive values of r. The resulting equations in a_j, b_j, α_j are solved for these variables. With these values of a_j, b_j, α_j all except one of the equations are satisfied for all values of θ. The last equation reduces to a linear equation in θ and can accordingly be solved. This value of θ provides us with a rational solution of our Diophantine system. As all of the equations of the system are homogeneous, solutions in integers can be obtained by multiplying each of the x_j, y_j already obtained by a suitable constant.

We now consider, one by one, each of the systems mentioned above.

3. *The system*

$$\text{(I)} \quad \begin{cases} L_i(x_j) = L_i(y_j), \quad i = 1, 2, \ldots, n-1, & (3.1) \\ Q_i(x_j) = Q_i(y_j), \quad i = 1, 2, \ldots, n-1. & (3.2) \end{cases}$$

While the general method can be applied to this system, there exists a simpler solution in this case. We re-write the equations (3.1) as

$$L_i(x_j-y_j) = 0, \quad i = 1, 2, \ldots, n-1.$$

Considering these as $n-1$ linear equations in variables x_j-y_j, $j = 1, 2, \ldots, n$, we obtain a non-trivial solution of the type

$$\frac{x_1-y_1}{\lambda_1} = \frac{x_2-y_2}{\lambda_2} = \frac{x_3-y_3}{\lambda_3} = \ldots = \frac{x_n-y_n}{\lambda_n}. \tag{3.3}$$

We note that

$$x_j^2-y_j^2 = (x_j-y_j)(x_j+y_j),$$

$$2(x_jx_k-y_jy_k) = (x_j-y_j)(x_k+y_k)+(x_j+y_j)(x_k-y_k).$$

Thus, using the relations (3.3), the equation $Q_i(x_j)-Q_i(y_j) = 0$ reduces to a linear equation. Hence the equations (3.2) reduce to a set of $n-1$ linear equations and accordingly the system (I) reduces to a set of $2n-2$ linear equations in $2n$ variables x_j, y_j. The complete solution of the system (I) is now readily obtained.

4. *The Diophantine system*

$$\text{(II)}\quad \begin{cases} L_i(x_j) = L_i(y_j), & i = 1, 2, \ldots, n-2, \quad (4.1) \\ C(x_j) = C(y_j). & (4.2) \end{cases}$$

We substitute $x_j = a_j\theta + \alpha_j$, $y_j = b_j\theta + \alpha_j$, $j = 1, 2, \ldots, n$. Then (4.1) reduces to

$$L_i(a_j - b_j) = 0, \quad i = 1, 2, \ldots, n-2, \tag{4.3}$$

and we impose the further condition

$$\sum_j \left(\frac{\partial C}{\partial x_j}\right)_{x_j = \alpha_j} (a_j - b_j) = 0. \tag{4.4}$$

As these are $n-1$ equations in n independent variables $a_j - b_j$, we have a non-zero solution for $a_j - b_j$. With these values of a_j, b_j, the equation (4.2) reduces to

$$\theta\{C(a_1, \ldots, a_n) - C(b_1, \ldots, b_n)\} + \frac{1}{2}\left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^2 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^2\right\} C\right]_{x_j = \alpha_j} = 0. \tag{4.5}$$

We assume that $C(a_j) \neq C(b_j)$, for otherwise we already have a solution of the system (II). Thus (4.5) can be solved for θ and we get the following parametric solution of the system (II). For $u = 1, 2, \ldots, n$,

$$x_u = -a_u\left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^2 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^2\right\} C\right]_{x_j = \alpha_j} + 2\alpha_u\{C(a_1, \ldots, a_n) - C(b_1, \ldots, b_n)\}, \tag{4.6}$$

$$y_u = -b_u\left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^2 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^2\right\} C\right]_{x_j = \alpha_j} + 2\alpha_u\{C(a_1, \ldots, a_n) - C(b_1, \ldots, b_n)\} \tag{4.7}$$

where the a_j, b_j satisfy the linear equations (4.3) and (4.4). The solution is non-trivial if

$$\left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^2 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^2\right\} C\right]_{x_j = \alpha_j} \neq 0. \tag{4.8}$$

We can choose a_j, b_j, α_j to satisfy a system of additional $n-1$ linear non-symmetric equations in x_j, y_j. For, any such equation $L(x_j) = L'(y_j)$ reduces to

$$\{L(a_j) - L'(b_j)\}\theta + L(\alpha_j) - L'(\alpha_j) = 0$$

and will hold for all values of θ if we choose a_j, b_j, α_j such that

$$L(a_j) = L'(b_j) \quad \text{and} \quad L(\alpha_j) = L'(\alpha_j).$$

As there are only $n-1$ equations, non-zero solutions for a_j, b_j, α_j are always possible while a_j, b_j simultaneously satisfy the equations (4.3) and (4.4). These values of a_j, b_j, α_j then lead to a solution of the system

$$\begin{aligned} L_i(x_j) &= L_i(y_j), & i &= 1, 2, \ldots, n-2, \\ L_i(x_j) &= L'_i(y_j), & i &= n-1, n, \ldots, 2n-3, \\ C(x_j) &= C(y_j) \end{aligned}$$

where $L_i \neq L'_i$ for $i = n-1, n, \ldots, 2n-3$.

When we take $n = 3$, the system (II) reduces to the system

$$\text{(II')}\quad \begin{cases} L(x_1, x_2, x_3) = L(y_1, y_2, y_3), & (4.9) \\ C(x_1, x_2, x_3) = C(y_1, y_2, y_3). & (4.10) \end{cases}$$

The seven-parameter solution of (II'), obtained as above, is given by (4.6) and (4.7) — α_j, b_j being independent parameters while the a_j, $j = 1, 2, 3$, are defined by

$$a_j = b_j + t\left\{\frac{\partial L}{\partial x_{j+1}}\frac{\partial C}{\partial x_{j+2}} - \frac{\partial L}{\partial x_{j+2}}\frac{\partial C}{\partial x_{j+1}}\right\}_{x_j = \alpha_j}$$

where x_4, x_5 refer to x_1, x_2 respectively and t is arbitrary.

5. *The Diophantine system*

$$\text{(III)}\quad \begin{cases} Q_i(x_j) = Q_i(y_j), & i = 1, 2, \ldots, n-2, \quad (5.1) \\ C(x_j) = C(y_j). & (5.2) \end{cases}$$

With the usual substitutions $x_j = a_j\theta + \alpha_j$, $y_j = b_j\theta + \alpha_j$, $j = 1, 2, \ldots, n$, we find that (5.1) reduces to

$$\theta\{Q_i(a_1, \ldots, a_n) - Q_i(b_1, \ldots, b_n)\} + \sum_j \left(\frac{\partial Q_i}{\partial x_j}\right)_{x_j = \alpha_j} (a_j - b_j) = 0, \quad i = 1, 2, \ldots, n-2, \tag{5.3}$$

and (5.2) gives

$$\theta^2\{C(a_1, \ldots, a_n) - C(b_1, \ldots, b_n)\} + \frac{\theta}{2}\left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^2 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^2\right\} C\right]_{x_j = \alpha_j} + \sum_j \left(\frac{\partial C}{\partial x_j}\right)_{x_j = \alpha_j} (a_j - b_j) = 0. \tag{5.4}$$

Now we impose the conditions

$$\sum_j \left(\frac{\partial Q_i}{\partial x_j}\right)_{x_j=\alpha_j} a_j = \sum_j \left(\frac{\partial Q_i}{\partial x_j}\right)_{x_j=\alpha_j} b_j, \quad i = 1, 2, \ldots, n-2, \tag{5.5}$$

$$\sum_j \left(\frac{\partial C}{\partial x_j}\right)_{x_j=\alpha_j} a_j = \sum_j \left(\frac{\partial C}{\partial x_j}\right)_{x_j=\alpha_j} b_j, \tag{5.6}$$

$$Q_i(a_j) = Q_i(b_j), \quad i = 1, 2, \ldots, n-2. \tag{5.7}$$

The equations (5.5), (5.6) and (5.7) in $2n$ variables a_j, b_j constitute a subsystem of the system (I) considered earlier and we will get a parametric solution for a_j, b_j. With these values of a_j, b_j the equations (5.1) hold identically. We also presume that $C(a_j) \neq C(b_j)$ for otherwise a_j, b_j already constitute a solution of the system (III). Finally, we solve (5.4) for θ and hence obtain a parametric solution of the system (III). The solution is given by the expressions (4.6) for x_u and (4.7) for y_u, $u = 1, 2, \ldots, n$, where the a_j, b_j are chosen so as to satisfy the conditions (5.5), (5.6) and (5.7). As before, the solution is non-trivial if the condition (4.8) holds.

When we take $n = 3$, the system (III) reduces to the system

$$(\mathrm{III}') \quad \begin{cases} Q(x_1, x_2, x_3) = Q(y_1, y_2, y_3), & (5.8) \\ C(x_1, x_2, x_3) = C(y_1, y_2, y_3). & (5.9) \end{cases}$$

As in the general case, we must choose a_j, b_j, $j = 1, 2, 3$, to satisfy the equations

$$\sum_j \left(\frac{\partial Q}{\partial x_j}\right)_{x_j=\alpha_j} (a_j - b_j) = 0, \tag{5.10}$$

$$\sum_j \left(\frac{\partial C}{\partial x_j}\right)_{x_j=\alpha_j} (a_j - b_j) = 0, \tag{5.11}$$

$$Q(a_j) = Q(b_j). \tag{5.12}$$

Now (5.10) and (5.11) give

$$a_j = b_j + t\lambda_j, \quad j = 1, 2, 3, \tag{5.13}$$

where

$$\lambda_j = \left\{\frac{\partial Q}{\partial x_{j+1}} \frac{\partial C}{\partial x_{j+2}} - \frac{\partial Q}{\partial x_{j+2}} \frac{\partial C}{\partial x_{j+1}}\right\}_{x_j=\alpha_j}, \quad j = 1, 2, 3. \tag{5.14}$$

In the relation (5.14), x_4 refers to x_1 and x_5 refers to x_2. Using (5.13), we find that

$$Q(a_1, a_2, a_3) = Q(b_1, b_2, b_3) + t \sum_{j=1}^{3} \lambda_j \left(\frac{\partial Q}{\partial x_j}\right)_{x_j=b_j} + t^2 Q(\lambda_1, \lambda_2, \lambda_3).$$

Hence the condition (5.12) will also be satisfied if we take

$$t = -\{Q(\lambda_1, \lambda_2, \lambda_3)\}^{-1} \sum_{j=1}^{3} \lambda_j \left(\frac{\partial Q}{\partial x_j}\right)_{x_j=b_j}. \tag{5.15}$$

Thus, when the λ_j are defined by (5.14), t by (5.15) and the a_j, $j = 1, 2, 3$, by (5.13), a solution of the system (III′) is given by the relations (4.6) and (4.7) in terms of the parameters α_j and b_j, $j = 1, 2, 3$.

6. *The Diophantine system*

$$(\mathrm{IV}) \quad \begin{cases} L_i(x_j) = L_i(y_j), & i = 1, 2, \ldots, n-2, & (6.1) \\ Q_i(x_j) = Q_i(y_j), & i = 1, 2, \ldots, k < n/2, & (6.2) \\ C(x_j) = C(y_j). & & (6.3) \end{cases}$$

With the usual substitutions $x_j = a_j\theta + \alpha_j$, $y_j = b_j\theta + \alpha_j$, (6.1) reduces to

$$L_i(a_j - b_j) = 0, \quad i = 1, 2, \ldots, n-2, \tag{6.4}$$

while (6.2) will hold for all values of θ if

$$\sum_j \left(\frac{\partial Q_i}{\partial x_j}\right)_{x_j=\alpha_j} (a_j - b_j) = 0, \quad i = 1, 2, \ldots, k, \tag{6.5}$$

$$Q_i(a_j) = Q_i(b_j), \quad i = 1, 2, \ldots, k. \tag{6.6}$$

It is easily proved that when $k < n/2$ and $\partial Q_i/\partial x_j \neq 0$ for each i, j, it is possible to choose α_j such that each of the equations (6.5) is a linear combination of the equations (6.4). With these values of α_j we solve the equations (6.4) along with

$$\sum_j \left(\frac{\partial C}{\partial x_j}\right)_{x_j=\alpha_j} (a_j - b_j) = 0 \tag{6.7}$$

and the equations (6.6). The equations (6.4), (6.6) and (6.7) constitute a subsystem of the system (I) and, in general, we will get a parametric solution for a_j, b_j. With these values of a_j, b_j, α_j, (6.1) and (6.2) hold identically and we finally solve (6.3) for θ. The solution of the system (IV) is then given by the relations (4.6) and (4.7) where the a_j, b_j, α_j are as determined above. As before, the solution is non-trivial if the condition (4.8) holds.

It is noteworthy that when $n = 3$, we find that, in general, the system

$$(\mathrm{IV}') \quad \begin{cases} L(x_1, x_2, x_3) = L(y_1, y_2, y_3), & (6.8.) \\ Q(x_1, x_2, x_3) = Q(y_1, y_2, y_3), & (6.9) \\ C(x_1, x_2, x_3) = C(y_1, y_2, y_3) & (6.10) \end{cases}$$

has a non-trivial solution. In this case the α_j are determined by the three linear equations

(6.11) $$\left(\frac{\partial Q}{\partial x_j}\right)_{x_j=\alpha_j} = \frac{\partial L}{\partial x_j}, \quad j = 1, 2, 3.$$

Also, we get

(6.12) $$a_j = b_j + t\lambda_j, \quad j = 1, 2, 3,$$

where

(6.13) $$\lambda_j = \left\{\frac{\partial L}{\partial x_{j+1}}\frac{\partial C}{\partial x_{j+2}} - \frac{\partial L}{\partial x_{j+2}}\frac{\partial C}{\partial x_{j+1}}\right\}_{x_j=\alpha_j}, \quad j = 1, 2, 3,$$

with x_4, x_5 being taken as x_1, x_2 respectively and

(6.14) $$t = -\{Q(\lambda_1, \lambda_2, \lambda_3)\}^{-1} \sum_{j=1}^{3} \lambda_j \left(\frac{\partial Q}{\partial x_j}\right)_{x_j=b_j}.$$

Then the relations (4.6) and (4.7) give a parametric solution for the system (IV′). A notable exception is the system

(6.15) $$\sum_{j=1}^{3} A_j x_j^r = \sum_{j=1}^{3} A_j y_j^r, \quad r = 1, 2, 3,$$

for which the above method does not yield a non-trivial solution for then the condition (4.8) does not hold.

7. *The Diophantine system*

$$\text{(V)} \quad \begin{cases} L(x_j) = L(y_j), & (7.1) \\ Q_i(x_j) = Q_i(y_j), \quad i = 1, 2, \ldots, n-3, & (7.2) \\ F(x_j) = F(y_j). & (7.3) \end{cases}$$

We write $x_j = a_j\theta + \alpha_j$, $y_j = b_j\theta + \alpha_j$ and impose the following conditions on a_j, b_j:

(7.4) $$L(a_j - b_j) = 0,$$

(7.5) $$\sum_j \left(\frac{\partial Q_i}{\partial x_j}\right)_{x_j=\alpha_j} (a_j - b_j) = 0, \quad i = 1, 2, \ldots, n-3,$$

(7.6) $$\sum_j \left(\frac{\partial F}{\partial x_j}\right)_{x_j=\alpha_j} (a_j - b_j) = 0,$$

(7.7) $$Q_i(a_j) - Q_i(b_j) = 0, \quad i = 1, 2, \ldots, n-3,$$

(7.8) $$\left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^2 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^2\right\} F\right]_{x_j=\alpha_j} = 0.$$

The above equations constitute a subsystem of the system (I) and hence, a non-trivial solution for a_j, b_j can be obtained in terms of the parameters α_j. With these values of a_j, b_j, the equations (7.1) and (7.2) hold identically while (7.3) can be solved for θ to obtain a parametric solution of the system (V). The solution is given by

$$x_u = -a_u \left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^3 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^3\right\} F\right]_{x_j=\alpha_j} + 6\alpha_u \{F(a_1, \ldots, a_n) - F(b_1, \ldots, b_n)\},$$

$$y_u = -b_u \left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^3 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^3\right\} F\right]_{x_j=\alpha_j} + 6\alpha_u \{F(a_1, \ldots, a_n) - F(b_1, \ldots, b_n)\},$$

for $u = 1, 2, \ldots, n$ with the a_j, b_j as determined above. The solution is non-trivial if

$$\left[\left\{\left(\sum_j a_j \frac{\partial}{\partial x_j}\right)^3 - \left(\sum_j b_j \frac{\partial}{\partial x_j}\right)^3\right\} F\right]_{x_j=\alpha_j} \neq 0.$$

8. *The Diophantine system*

$$\text{(VI)} \quad \begin{cases} x_1 + x_2 + x_3 = y_1 + y_2 + y_3, \\ x_1^3 + x_2^3 + x_3^3 = y_1^3 + y_2^3 + y_3^3. \end{cases}$$

This is a particular case of the system (II′) and the solution, obtained similarly, may be stated as:

(8.1) $$x_u = \varrho\left\{-3a_u \sum_{j=1}^{3} \alpha_j(a_j^2 - b_j^2) + \alpha_u \sum_{j=1}^{3} (a_j^3 - b_j^3)\right\}, \quad u = 1, 2, 3,$$

(8.2) $$y_u = \varrho\left\{-3b_u \sum_{j=1}^{3} \alpha_j(a_j^2 - b_j^2) + \alpha_u \sum_{j=1}^{3} (a_j^3 - b_j^3)\right\}, \quad u = 1, 2, 3,$$

where

(8.3) $$\begin{aligned} a_1 &= b_1 + t(\alpha_2^2 - \alpha_3^2), \\ a_2 &= b_2 + t(\alpha_3^2 - \alpha_1^2), \\ a_3 &= b_3 + t(\alpha_1^2 - \alpha_2^2) \end{aligned}$$

and ϱ, t, α_j, b_j, $j = 1, 2, 3$, are arbitrary parameters.

We now establish that the solution given above is complete. Let X_j, $j = 1, 2, 3$, and Y_j, $j = 1, 2, 3$, be any non-trivial solution of (VI), i.e. $X_j \neq Y_j$ for any j. We shall show that there exist α_j, b_j, t, ϱ such that (8.1) and (8.2) yield the solution X_j, Y_j, $j = 1, 2, 3$. We first choose α_j such that

$$\alpha_1^2(X_1-Y_1)+\alpha_2^2(X_2-Y_2)+\alpha_3^2(X_3-Y_3) = 0 \tag{8.4}$$

where

$$\alpha_1 \neq \pm\alpha_2, \quad \alpha_2 \neq \pm\alpha_3, \quad \alpha_3 \neq \pm\alpha_1 \quad \text{and} \quad \sum_j (X_j^2-Y_j^2)\alpha_j \neq 0.$$

As $\alpha_1 = \alpha_2 = \alpha_3 = 1$ is a solution of (8.4) we can easily find a parametric solution of (8.4) and a suitable choice of parameters gives the desired solution.

With these values of α_j, we choose

$$b_j = Y_j - \alpha_j, \quad j = 1, 2, 3. \tag{8.5}$$

We know that

$$(X_1-Y_1)+(X_2-Y_2)+(X_3-Y_3) = 0. \tag{8.6}$$

Now (8.4) and (8.6) give

$$\frac{X_1-Y_1}{\alpha_2^2-\alpha_3^2} = \frac{X_2-Y_2}{\alpha_3^2-\alpha_1^2} = \frac{X_3-Y_3}{\alpha_1^2-\alpha_2^2}. \tag{8.7}$$

We choose $t = (X_1-Y_1)/(\alpha_2^2-\alpha_3^2)$ and the a_j are then defined by (8.3). Now (8.3) and (8.7) give

$$X_j - a_j = Y_j - b_j = \alpha_j, \quad j = 1, 2, 3. \tag{8.8}$$

Hence

$$X_j = a_j + \alpha_j, \quad j = 1, 2, 3. \tag{8.9}$$

We also know that $\sum_1^3 X_j^3 = \sum_1^3 Y_j^3$. Using (8.5) and (8.9), this gives the relation

$$\sum_j (a_j^3-b_j^3)+3\sum_j (a_j^2-b_j^2)\alpha_j = 0.$$

Hence (8.1), (8.2) give us for $u = 1, 2, 3$,

$$x_u = \varrho(a_u+\alpha_u)\sum_j (a_j^3-b_j^3),$$

$$y_u = \varrho(b_u+\alpha_u)\sum_j (a_j^3-b_j^3).$$

Now

$$\sum_j (a_j^3-b_j^3) = -3\sum_j (X_j^2-Y_j^2)\alpha_j \neq 0$$

and taking $\varrho = \{\sum_j (a_j^3-b_j^3)\}^{-1}$, we get

$$x_j = X_j, \quad y_j = Y_j, \quad j = 1, 2, 3.$$

Thus, the solution given by (8.1) and (8.2) is a complete solution of the system (VI). In the solution obtained above, we may take $\alpha_3 = 0$, $b_3 = 0$, to get a solution of the system

$$\sum_{j=1}^{3} x_j^r = \sum_{j=1}^{2} y_j^r \tag{8.10}$$

in terms of the parameters ϱ, t, α_j, b_j, $j = 1, 2$.

A two-parameter solution of the system (VI) has earlier been given by Gerardin (quoted by Dickson [3, pp. 565 and 713]) and later Bremner [1] has given a complete solution in four parameters. The advantage of the present solution is that, apart from being complete, it can also be made to satisfy two additional linear asymmetric conditions as discussed in the case of the general system (II).

9. *The Diophantine system*

$$\text{(VII)} \quad \begin{cases} x_1+x_2+x_3 = y_1+y_2+y_3, \\ x_1x_2x_3 = y_1y_2y_3. \end{cases}$$

This is also a particular case of the system (II′) and the solution, obtained as before, is as follows:

$$x_j = \varrho[-a_j\{\alpha_1(a_2a_3-b_2b_3)+\alpha_2(a_3a_1-b_3b_1)+\alpha_3(a_1a_2-b_1b_2)\} \\ +\alpha_j(a_1a_2a_3-b_1b_2b_3)],$$

$$y_j = \varrho[-b_j\{\alpha_1(a_2a_3-b_2b_3)+\alpha_2(a_3a_1-b_3b_1)+\alpha_3(a_1a_2-b_1b_2)\} \\ +\alpha_j(a_1a_2a_3-b_1b_2b_3)]$$

for $j = 1, 2, 3$ where $a_1 = b_1+t\alpha_1(\alpha_2-\alpha_3)$, $a_2 = b_2+t\alpha_2(\alpha_3-\alpha_1)$, $a_3 = b_3+t\alpha_3(\alpha_1-\alpha_2)$ and ϱ, t, α_j, b_j, $j = 1, 2, 3$, are arbitrary parameters.

As in the case of the system (VI), the solution is complete and the proof being similar is omitted.

10. *The Diophantine system*

$$\text{(VIII)} \quad \begin{cases} x_1+x_2+x_3 = y_1+y_2+y_3, & (10.1) \\ x_1^3+x_2^3+x_3^3 = y_1^3+y_2^3+y_3^3, & (10.2) \\ x_1x_2x_3 = y_1y_2y_3. & (10.3) \end{cases}$$

In view of the identity

$$x_1^3+x_2^3+x_3^3-3x_1x_2x_3 = (x_1+x_2+x_3)(x_1^2+x_2^2+x_3^2-x_1x_2-x_2x_3-x_3x_1)$$

Whenever (10.1) and (10.2) hold together with the condition

$$x_1+x_2+x_3 = 0,$$

the equation (10.3) will also be satisfied. Accordingly a solution of the system (VIII) is given by (8.1), (8.2), (8.3) in terms of parameters ϱ, t, α_j, b_j where α_j, b_j satisfy the conditions

$$\alpha_1+\alpha_2+\alpha_3=0, \quad b_1+b_2+b_3=0.$$

Another solution of this system can similarly be derived from the solution of the system (VII).

11. *The Diophantine system*

$$\text{(IX)} \quad \begin{cases} x_1^2+x_2^2+x_3^2=y_1^2+y_2^2+y_3^2, & (11.1)\\ x_1^3+x_2^3+x_3^3=y_1^3+y_2^3+y_3^3. & (11.2)\end{cases}$$

This is a particular case of the system (III′) and its solution, obtained as before, may be written in terms of six parameters α_j, b_j, $j=1,2,3$, as follows:

$$x_u=-3a_u\sum_{j=1}^{3}\alpha_j(a_j^2-b_j^2)+\alpha_u\sum_{j=1}^{3}(a_j^3-b_j^3), \quad u=1,2,3, \tag{11.3}$$

$$y_u=-3b_u\sum_{j=1}^{3}\alpha_j(a_j^2-b_j^2)+\alpha_u\sum_{j=1}^{3}(a_j^3-b_j^3), \quad u=1,2,3, \tag{11.4}$$

where

$$a_1=b_1+t\alpha_2\alpha_3(\alpha_2-\alpha_3), \quad a_2=b_2+t\alpha_3\alpha_1(\alpha_3-\alpha_1), \quad a_3=b_3+t\alpha_1\alpha_2(\alpha_1-\alpha_2)$$

and

$$t=-2\{\sum\alpha_1^2\alpha_2^2(\alpha_1-\alpha_2)^2\}^{-1}\{b_1\alpha_2\alpha_3(\alpha_2-\alpha_3)+b_2\alpha_3\alpha_1(\alpha_3-\alpha_1)$$
$$+b_3\alpha_1\alpha_2(\alpha_1-\alpha_2)\}.$$

A numerical example obtained by taking $\alpha_1=2$, $\alpha_2=3$, $\alpha_3=4$, $b_1=69$, $b_2=73$, $b_3=93$ is as follows:

$$2812^2+10154^2+17499^2=6700^2+7131^2+17930^2,$$

$$2812^3+10154^3+17499^3=6700^3+7131^3+17930^3.$$

No solutions of this system have been obtained earlier.

12. *The Diophantine system*

$$\text{(X)} \quad \begin{cases} x_1^2+x_2^2+x_3^2=y_1^2+y_2^2+y_3^2,\\ \quad x_1x_2x_3=y_1y_2y_3.\end{cases}$$

This is also a particular case of the system (III′) and its solution, obtained as before, may be written in terms of six parameters α_j, b_j, $j=1,2,3$,

as follows:

$$x_j=-a_j\{\alpha_1(a_2a_3-b_2b_3)+\alpha_2(a_3a_1-b_3b_1)+\alpha_3(a_1a_2-b_1b_2)\}$$
$$+\alpha_j(a_1a_2a_3-b_1b_2b_3), \tag{12.1}$$

$$y_j=-b_j\{\alpha_1(a_2a_3-b_2b_3)+\alpha_2(a_3a_1-b_3b_1)+\alpha_3(a_1a_2-b_1b_2)\}$$
$$+\alpha_j(a_1a_2a_3-b_1b_2b_3), \tag{12.2}$$

for $j=1,2,3$ where

$$a_1=b_1+t\alpha_1(\alpha_2^2-\alpha_3^2), \quad a_2=b_2+t\alpha_2(\alpha_3^2-\alpha_1^2), \quad a_3=b_3+t\alpha_3(\alpha_1^2-\alpha_2^2) \tag{12.3}$$

and

$$t=-2\{\sum\alpha_1^2(\alpha_2^2-\alpha_3^2)^2\}^{-1}\{b_1\alpha_1(\alpha_2^2-\alpha_3^2)+b_2\alpha_2(\alpha_3^2-\alpha_1^2)$$
$$+b_3\alpha_3(\alpha_1^2-\alpha_2^2)\}. \tag{12.4}$$

A six-parameter solution of the system (X) has earlier been given by Gloden [5, pp. 36–37].

13. *The Diophantine system*

$$\text{(XI)} \quad \sum_{j=1}^{4}x_j^r=\sum_{j=1}^{4}y_j^r, \quad r=1,2,3,5.$$

We write

$$x_1=X_1+X_2+X_3, \quad x_2=X_1-X_2-X_3,$$
$$x_3=-X_1+X_2-X_3, \quad x_4=-X_1-X_2+X_3.$$

Then we have the identities

$$\sum_{j=1}^{4}x_j=0, \quad \sum_{j=1}^{4}x_j^2=4(X_1^2+X_2^2+X_3^2),$$

$$\sum_{j=1}^{4}x_j^3=24X_1X_2X_3, \quad \sum_{j=1}^{4}x_j^5=80X_1X_2X_3(X_1^2+X_2^2+X_3^2).$$

Thus, if we also write $y_1=Y_1+Y_2+Y_3$, $y_2=Y_1-Y_2-Y_3$, $y_3=-Y_1+Y_2-Y_3$, $y_4=-Y_1-Y_2+Y_3$, then in view of the above identities, the system (XI) will be satisfied if

$$X_1^2+X_2^2+X_3^2=Y_1^2+Y_2^2+Y_3^2, \tag{13.1}$$

$$X_1X_2X_3=Y_1Y_2X_3. \tag{13.2}$$

A six-parameter solution of this system has been obtained above and hence a six-parameter solution of (XI) follows immediately.

We now impose the further condition

$$-Y_1-Y_2+Y_3=0. \tag{13.3}$$

This is possible by choice of α_j, b_j in the solution of the system (X) given by (12.3) and (12.4). In fact, the solution of (13.1), (13.2), (13.3) is given by

$$(13.4)\quad X_j = -a_j\{\alpha_1(a_2a_3-b_2b_3)+\alpha_2(a_3a_1-b_3b_1)+\alpha_3(a_1a_2-b_1b_2)\} + \alpha_j(a_1a_2a_3-b_1b_2b_3),$$

$$(13.5)\quad Y_j = -b_j\{\alpha_1(a_2a_3-b_2b_3)+\alpha_2(a_3a_1-b_3b_1)+\alpha_3(a_1a_2-b_1b_2)\} + \alpha_j(a_1a_2a_3-b_1b_2b_3)$$

for $j = 1, 2, 3$ where $\alpha_3 = \alpha_1+\alpha_2$, $b_3 = b_1+b_2$ and the a_j are defined by (12.3) and (12.4). Thus, we have obtained a four-parameter solution of the system

$$\sum_{j=1}^{4} x_j^r = \sum_{j=1}^{3} y_j^r \quad r = 1, 2, 3, 5.$$

The system (XI) has been considered earlier by Gloden [5, pp. 42–43] who also obtained a six-parameter solution.

14. *The Diophantine system*

$$\text{(XII)}\quad \sum_{j=1}^{5} x_j^r = \sum_{j=1}^{5} y_j^r, \quad r = 1, 2, 3, 4.$$

This is the well-known Tarry–Escott problem of degree 4. From the relations

$$\sum_{j=1}^{3} X_j^r = \sum_{j=1}^{2} Y_j^r, \quad r = 1, 3,$$

we derive the relations

$$\sum_{j=1}^{3} X_j^r + \sum_{j=1}^{2} (-Y_j)^r = \sum_{j=1}^{3} (-X_j)^r + \sum_{j=1}^{2} Y_j^r, \quad r = 1, 2, 3, 4,$$

and it follows from a well-known theorem [4, p. 614] relating to the Tarry–Escott problem that

$$\sum_{j=1}^{3} (mX_j+d)^r + \sum_{j=1}^{2} (-mY_j+d)^r = \sum_{j=1}^{3} (-mX_j+d)^r + \sum_{j=1}^{2} (mY_j+d)^r, \quad r = 1, 2, 3, 4$$

where m and d are arbitrary.

The parametric solution of (8.10) obtained earlier thus leads to a 7-parameter solution of the system (XII).

Earlier, Chernick [2, p. 629] has given a two-parameter solution and Gloden [5, pp. 41–42] a three-parameter solution of the system (XII).

15. *The Diophantine system*

$$\text{(XIII)}\quad \sum_{j=1}^{7} x_j^r = \sum_{j=1}^{7} y_j^r, \quad r = 1, 2, 3, 4, 5, 6.$$

Proceeding as in the previous section, the 4-parameter solution of the system

$$\sum_{j=1}^{4} x_j^r = \sum_{j=1}^{3} y_j^r, \quad r = 1, 2, 3, 5,$$

obtained earlier, leads to a six-parameter solution of the system (XIII). Gloden [5, p. 43] has also given a 6-parameter solution of (XIII), different from the present solution.

16. *The Diophantine system*

$$\text{(XIV)}\quad \begin{cases} x_1+x_2+x_3 = y_1+y_2+y_3, & (16.1)\\ x_1^3+x_2^3+x_3^3 = y_1^3+y_2^3+y_3^3, & (16.2)\\ x_1^4+x_2^4+x_3^4 = y_1^4+y_2^4+y_3^4. & (16.3)\end{cases}$$

We write $x_j = a_j\theta+\alpha_j$, $y_j = a_{j+1}\theta+\alpha_j$, $j = 1, 2, 3$, with $a_4 = a_1$ and impose the following conditions on a_j, α_j:

$$(16.4)\quad (a_1-a_2)\alpha_1^2+(a_2-a_3)\alpha_2^2+(a_3-a_1)\alpha_3^2 = 0,$$

$$(16.5)\quad (a_1-a_2)\alpha_1^3+(a_2-a_3)\alpha_2^3+(a_3-a_1)\alpha_3^3 = 0,$$

$$(16.6)\quad (a_1^2-a_2^2)\alpha_1+(a_2^2-a_3^2)\alpha_2+(a_3^2-a_1^2)\alpha_3 = 0.$$

We choose α_j such that

$$(16.7)\quad \alpha_1\alpha_2+\alpha_2\alpha_3+\alpha_3\alpha_1 = 0.$$

Then (16.4) and (16.5) become identical while (16.6) reduces to a linear equation in a_1, a_2, a_3. We thus get the following solution for a_j:

$$(16.8)\quad \begin{cases} a_1 = \alpha_1(\alpha_2^2-\alpha_3^2)^2-\alpha_2(\alpha_3^2-\alpha_1^2)(\alpha_1^2-2\alpha_2^2+\alpha_3^2)-\alpha_3(\alpha_1^2-\alpha_2^2)^2,\\ a_2 = \alpha_2(\alpha_3^2-\alpha_1^2)^2-\alpha_3(\alpha_1^2-\alpha_2^2)(\alpha_2^2-2\alpha_3^2+\alpha_1^2)-\alpha_1(\alpha_2^2-\alpha_3^2)^2,\\ a_3 = \alpha_3(\alpha_1^2-\alpha_2^2)^2-\alpha_1(\alpha_2^2-\alpha_3^2)(\alpha_3^2-2\alpha_1^2+\alpha_2^2)-\alpha_2(\alpha_3^2-\alpha_1^2)^2.\end{cases}$$

When α_j, a_j are determined by (16.7) and (16.8), the resulting x_j, y_j satisfy (16.1) and (16.2) for all values of θ, while (16.3) reduces to a linear equation in θ. This last equation is solved for θ and we thus get the following solution of the system (XIV):

$$x_u = 2\alpha_u\sum_{j=1}^{3}(a_j^3-a_{j+1}^3)\alpha_j - 3a_u\sum_{j=1}^{3}(a_j^2-a_{j+1}^2)\alpha_j^2, \quad u = 1, 2, 3,$$

$$y_u = 2\alpha_u\sum_{j=1}^{3}(a_j^3-a_{j+1}^3)\alpha_j - 3a_{u+1}\sum_{j=1}^{3}(a_j^2-a_{j+1}^2)\alpha_j^2, \quad u = 1, 2, 3,$$

where α_j satisfy (16.7) and a_j are defined by (16.8).

Taking $\alpha_1 = 6$, $\alpha_2 = 3$, $\alpha_3 = -2$, we get the numerical solution

$$5658^r + 5583^r + (-3254)^r = 6738^r + (-1329)^r + 2578^r$$

for $r = 1, 3, 4$.

No solutions of this system have been obtained earlier.

17. The system

$$\text{(XV)} \qquad \sum_{j=1}^{3} x_j^r = \sum_{j=1}^{3} y_j^r, \quad r = 1, 3, 7.$$

We write $x_1 = X_1 - X_2 - X_3$, $x_2 = -X_1 + X_2 - X_3$, $x_3 = -X_1 - X_2 + X_3$, $x_4 = X_1 + X_2 + X_3$, $y_1 = Y_1 - Y_2 - Y_3$, $y_2 = -Y_1 + Y_2 - Y_3$, $y_3 = -Y_1 - Y_2 + Y_3$, $y_4 = Y_1 + Y_2 + Y_3$. Then we have the identities

$$\sum_{j=1}^{4} x_j = 0, \qquad \sum_{j=1}^{4} x_j^3 = 24X_1X_2X_3,$$

$$\sum_{j=1}^{4} x_j^7 = 56X_1X_2X_3\{3(X_1^4 + X_2^4 + X_3^4) + 10(X_1^2X_2^2 + X_2^2X_3^2 + X_3^2X_1^2)\}.$$

Hence we will get a solution of the system (XV) if we can find X_j, Y_j, $j = 1, 2, 3$, such that

$$\text{(17.1)} \qquad X_1 + X_2 + X_3 = Y_1 + Y_2 + Y_3,$$

$$\text{(17.2)} \qquad X_1X_2X_3 = Y_1Y_2Y_3,$$

$$\text{(17.3)} \qquad 3(X_1^4 + X_2^4 + X_3^4) + 10(X_1^2X_2^2 + X_2^2X_3^2 + X_3^2X_1^2) = 3(Y_1^4 + Y_2^4 + Y_3^4) + 10(Y_1^2Y_2^2 + Y_2^2Y_3^2 + Y_3^2Y_1^2).$$

The system of equations (17.1), (17.2), (17.3) is solved just as the system (XIV) by substituting $X_j = a_j\theta + \alpha_j$, $Y_j = a_{j+1}\theta + \alpha_j$, $j = 1, 2, 3$, and imposing the following conditions on a_j:

$$\text{(17.4)} \qquad (a_1 - a_2)\alpha_2\alpha_3 + (a_2 - a_3)\alpha_3\alpha_1 + (a_3 - a_1)\alpha_1\alpha_2 = 0,$$

$$\text{(17.5)} \qquad (a_1 - a_2)\{3\alpha_1^3 + 5\alpha_1(\alpha_2^2 + \alpha_3^2)\} + (a_2 - a_3)\{3\alpha_2^3 + 5\alpha_2(\alpha_3^2 + \alpha_1^2)\} + (a_3 - a_1)\{3\alpha_3^3 + 5\alpha_3(\alpha_1^2 + \alpha_2^2)\} = 0,$$

$$\text{(17.6)} \qquad (a_1 - a_2)a_3\alpha_1 + (a_2 - a_3)a_1\alpha_2 + (a_3 - a_1)a_2\alpha_3 = 0.$$

Now (17.4) and (17.5) will be identical if

$$\begin{vmatrix} 3\alpha_1^3 + 5\alpha_1(\alpha_2^2 + \alpha_3^2) & 3\alpha_2^3 + 5\alpha_2(\alpha_3^2 + \alpha_1^2) & 3\alpha_3^3 + 5\alpha_3(\alpha_1^2 + \alpha_2^2) \\ \alpha_2\alpha_3 & \alpha_3\alpha_1 & \alpha_1\alpha_2 \\ 1 & 1 & 1 \end{vmatrix} = 0$$

or, if

$$\text{(17.7)} \qquad (3\alpha_1 - \alpha_2 - \alpha_3)^2 + 2(2\alpha_2 - \alpha_3)^2 + 6\alpha_3^2 = 0.$$

This has no solution in rational integers but solutions in Gaussian integers do exist. Using (17.4), the equation (17.6) reduces to a linear equation in a_j and the solution of (17.4), (17.5), (17.6) is given by

$$a_1 = \alpha_1\alpha_3, \quad a_2 = \alpha_1\alpha_2, \quad a_3 = \alpha_2\alpha_3$$

where the α_j satisfy (17.7).

With these values of a_j, α_j, the equations (17.1), (17.2) hold identically for all values of θ while (17.3) reduces to a linear equation in θ. This gives a non-zero solution of θ and we may thus obtain a parametric solution of the system given by (17.1), (17.2), (17.3). This eventually leads to a parametric solution in Gaussian integers of the system (XV).

As an example, taking $\alpha_1 = 5 + 3i$, $\alpha_2 = 3 + 3i$, $\alpha_3 = 6i$, we get the following solution:

$$(201 - 243i)^r + (135 + 285i)^r + (132 - 16i)^r = (265 + 285i)^r + (327 - 211i)^r + (-124 - 48i)^r$$

for $r = 1, 3, 7$.

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Received on 17.7.1990
and in revised form on 5.11.1990 (2062)