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On almost even arithmetical functions via orthonormal systems on Vilenkin groups

by

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1. Introduction, preliminaries. Let $k \in \mathbb{N}$. An arithmetical function $f : \mathbb{N} \to \mathbb{C}$ is called *even* $\bmod k$ if $f((n,k)) = f(n)$ for all $n \in \mathbb{N}$. Define $D := \{f \mid f : \mathbb{N} \to \mathbb{C}\}$, $D^b := \{f \in D, f \text{ bounded}\}$, $\mathcal{B}_k := \{f \in D \mid f \text{ even } \bmod k\}$, $\mathcal{B} := \bigcup_{k\in\mathbb{N}} \mathcal{B}_k$. Then $\mathcal{B}$ is the $\mathbb{C}$-algebra (with pointwise addition and multiplication) of even functions, and D^b with the "uniform norm" $\|f\|_u := \sup_{n\in\mathbb{N}} |f(n)|$ is a complex Banach algebra. The limit (if it exists)

$$M(f) := \lim_{x\to\infty} x^{-1} \sum_{1\le j\le x} f(j)$$

is called the *mean value* of f.

If $f : \mathbb{N} \to \mathbb{R}$, then

$$\overline{M}(f) := \limsup_{x\to\infty} x^{-1} \sum_{1\le j\le x} f(j), \quad \underline{M}(f) := \liminf_{x\to\infty} x^{-1} \sum_{1\le j\le x} f(j)$$

are called the *upper* and the *lower* mean values of f, respectively.

From now on throughout this paper we suppose that $1 \le q < \infty$. Then the upper mean value gives rise to a seminorm $\|f\|_q := \{\overline{M}(|f|^q)\}^{1/q}$ on the linear space $\{f \in D \mid \|f\|_q < \infty\}$. We denote by $\mathcal{B}^q$ the closure of $\mathcal{B}$ in $\{f \in D \mid \|f\|_q < \infty\}$ relative to the topology defined by $\|\cdot\|_q$. Functions in $\mathcal{B}^q$ are called $\mathcal{B}^q$ *almost even arithmetical functions.* Denote by $\mathcal{B}^u$ the closure of $\mathcal{B}$ in D^b relative to the topology defined by $\|\cdot\|_u$. Every $f \in \mathcal{B}^u$ is called a *uniform almost even arithmetical function.*

Let $m := (m_0, m_1, \ldots)$ denote a sequence of positive integers not less than 2. Denote by $Z_{m_j} := \{0, 1, \ldots, m_j - 1\}$ the additive group of integers modulo m_j ($j \in \mathbb{N}_0$). Define the group G_m as the cartesian product of the discrete cyclic groups Z_{m_j},

$$G_m := \bigtimes_{j=0}^{\infty} Z_{m_j}.$$

The elements of G_m can be represented by sequences $x := (x_0, x_1, \ldots)$ ($x_j \in Z_{m_j}$). It is easy to give a base for the neighborhoods of G_m:

$$I_0(x) := G_m,$$
$$I_n(x) := \{y \in G_m \mid y_0 = x_0, \ldots, y_{n-1} = x_{n-1}\}$$

for $x \in G_m$, $n \in \mathbb{N}$. Define $I_n := I_n(0)$ for $n \in \mathbb{N}_0$. Then I_n is a subgroup of G_m ($n \in \mathbb{N}_0$). Moreover, G_m is a compact zero-dimensional abelian group. The direct product μ of the measures

$$\mu_k(\{j\}) := m_k^{-1} \quad (j \in Z_{m_k}, k \in \mathbb{N}_0)$$

is the Haar measure on G_m with $\mu(G_m) = 1$.

Define the *generalized powers* by $M_0 := 1$, $M_{k+1} := m_k M_k$ ($k \in \mathbb{N}_0$). Then every nonnegative integer n can be uniquely expressed as $\sum_{j=0}^{\infty} n_j M_j$, where $n_j \in Z_{m_j}$ ($j \in \mathbb{N}_0$) and only a finite number of n_j's differ from zero.

Define on G_m the *generalized Rademacher functions* in the following way:

$$r_k(x) := \exp(2\pi i x_k / m_k) \quad (i := (-1)^{1/2},\ x \in G_m,\ k \in \mathbb{N}_0).$$

It is known that the functions

$$\psi_n := \prod_{k=0}^{\infty} r_k^{n_k} \quad (n \in \mathbb{N}_0)$$

on G_m are elements of the character group of G_m, and all the elements of the character group are of this form. The system ($\psi_n \mid n \in \mathbb{N}_0$) is called a *Vilenkin system* and G_m a *Vilenkin group.* For more details on Vilenkin analysis see e.g. [1, 14, 17, 18].

Let $\mathcal{A}_n$ ($n \in \mathbb{N}_0$) be the σ-algebra generated by the cosets $I_n(z)$ ($z \in G_m$). Let α_j^k, α_n ($k, j, n \in \mathbb{N}$) be functions satisfying the following conditions:

(i) $\alpha_j^k : G_m \to \mathbb{C}$ is $\mathcal{A}_j$-measurable ($k, j \in \mathbb{N}_0$),

(ii) $|\alpha_j^k| = \alpha_0^k = \alpha_k^0 = 1$ ($k, j \in \mathbb{N}_0$),

(iii) $\alpha_n := \prod_{j=0}^{\infty} \alpha_j^{j(n)}$ ($n \in \mathbb{N}_0,\ j(n) := \sum_{k=j}^{\infty} n_k M_k$).

Let $\phi_n := \psi_n \alpha_n$ ($n \in \mathbb{N}_0$). A function system $\{\phi_n \mid n \in \mathbb{N}_0\}$ of this type is called a $\psi\alpha$ *system* on the Vilenkin group G_m. We can identify G_m with the unit interval $[0, 1)$ by associating with each $(x_0, x_1, \ldots) = x \in G_m$ the point $\sum_{j=0}^{\infty} x_j M_{j+1}^{-1} \in [0, 1)$.

If we disregard the countable set of m-rationals,

$$\mathbb{Q}_m \cap (0, 1) \quad (\mathbb{Q}_m := \{x \in G_m \mid \exists j \in \mathbb{N}_0 : \forall k \geq j,\ k \in \mathbb{N}_0,\ x_k = 0\}),$$

then this mapping is one-one and measure preserving. Every $y \in \mathbb{Q}_m \cap (0, 1)$ has a duplicate in G_m, one of them has a finite and the other an infinite representation of the form $y = \sum y_j M_{j+1}^{-1}$.

Now we introduce a special kind of $\psi\alpha$ systems on the Vilenkin group G_m (which is identified with $[0,1)$ in the way described above).

If $n \in \mathbb{N}_0$, $n = \sum_{j=0}^{\infty} n_j M_j$, then let

$$\check{n} := \sum_{j=0}^{\infty} n_j M_{j+1}^{-1} \in [0,1)$$

(of course only a finite number of n_j's are not zero).

If $x \in G_m$, $A \in \mathbb{N}_0$, then let $\sigma_A x := \sum_{j=0}^{A} x_j M_j \in \mathbb{N}_0$. Suppose that $n < M_{A+1}$ for some $A \in \mathbb{N}_0$. Then $\check{n} = \sum_{j=0}^{A} n_j M_{j+1}^{-1}$. Then the function $\mathbb{N}_0 \ni k \to \exp(2\pi i \check{n} \sigma_{A+k} x)$ $(n, A \in \mathbb{N}_0, x \in G_m$ fixed) is constant, because

$$\begin{aligned}
&\exp(2\pi i \check{n} \sigma_{A+k} x) \\
&\quad = \exp\Big\{2\pi i\Big(\frac{n_0}{M_1} + \frac{n_1}{M_2} + \ldots + \frac{n_A}{M_{A+1}}\Big) \\
&\qquad \times (x_0 + x_1 M_1 + \ldots + x_A M_A + x_{A+1} M_{A+1} + \ldots + x_{A+k} M_{A+k})\Big\} \\
&\quad = \exp\Big(2\pi i\Big(\frac{n_0}{M_1} + \ldots + \frac{n_A}{M_{A+1}}\Big)(x_0 + x_1 M_1 + \ldots + x_A M_A)\Big) \\
&\quad = \exp(2\pi i \check{n} \sigma_A x).
\end{aligned}$$

Thus the definition

$$\kappa_n(x) := \lim_{k\to\infty} \exp(2\pi i \check{n} \sigma_k x) \qquad (n \in \mathbb{N}_0, x \in G_m)$$

makes sense.

$\kappa_n(x)$ can also be denoted as $\exp(2\pi i \check{n} \sigma x)$. The system $\{\kappa_n \mid n \in \mathbb{N}_0\}$ on G_m is a $\psi\alpha$ system.

Indeed, suppose that $M_A \le n < M_{A+1}$, $x \in G_m$, $1 \le A$. Then

$$\begin{aligned}
\kappa_n(x) &= \exp\left(2\pi i\left(\frac{n_0}{m_0} + \ldots + \frac{n_A}{m_0 \ldots m_A}\right) x_0\right) \\
&\qquad \times \exp\left(2\pi i\left(\frac{n_1}{m_1} + \ldots + \frac{n_A}{m_1 \ldots m_A}\right) x_1\right) \ldots \exp\left(2\pi i \frac{n_A}{m_A} x_A\right) \\
&= r_0^{n_0}(x) \ldots r_A^{n_A}(x) \prod_{j=0}^{A-1} \exp\left(2\pi i\left(\frac{n_{j+1}}{m_j m_{j+1}} + \ldots + \frac{n_A}{m_j \ldots m_A}\right)\right).
\end{aligned}$$

That is, in this case

$$\alpha_j^{j(n)} = \exp\left(2\pi i x_{j-1}\left(\frac{n_j}{m_{j-1} m_j} + \ldots + \frac{n_A}{m_{j-1} \ldots m_A}\right)\right) \qquad (1 \le j \in \mathbb{N}).$$

Of course if $A = 0$, then the product $\prod_{j=0}^{A-1}$ is equal to 1, and for $n < M_0$, i.e. $n = 0$, $\kappa_0(x) = 1 = \psi_0(x)\alpha_0(x)$. We have proved that $\{\kappa_n \mid n \in \mathbb{N}_0\}$ is a $\psi\alpha$ system on G_m.

The Fourier coefficients of $f \in L^1(G_m)$ with respect to the $\psi\alpha$ system κ are defined by

$$\hat{f}(k) = \hat{f}^\kappa(k) := \int_{G_m} f(x)\overline{\kappa}_k(x)\, d\mu(x) \qquad (k \in \mathbb{N}_0).$$

The Dirichlet kernels are given by

$$D_n(x,y) = D_n^\kappa(x,y) := \sum_{j=0}^{n-1} \kappa_j(x)\overline{\kappa}_j(y) \qquad (x, y \in G_m,\ n \in \mathbb{N}).$$

The nth partial sum of the Fourier series of $f \in L^1(G_m)$ (with respect to the $\psi\alpha$ system κ) is

$$S_n f(x) = S_n^\kappa f(x) := \sum_{j=0}^{n-1} \hat{f}^\kappa(j)\kappa_j(x) \qquad (x \in G_m, n \in \mathbb{N}).$$

We give some examples of Vilenkin groups G_m.

If each m_j $(j \in \mathbb{N}_0)$ equals 2, then G_m is called the *Walsh–Paley group*. The character system of this special Vilenkin group is the set of Walsh functions. The Walsh functions have three most studied enumerations, namely the original Walsh, the Walsh–Kaczmarz and the Walsh–Paley one. The last one coincides with the ordering used in this paper. For more details on Walsh functions see e.g. the recent book of F. Schipp, W. R. Wade, P. Simon and J. Pál ([14]).

If the sequence m is bounded, then G_m is called a *bounded Vilenkin group*. Most of the results on the Walsh–Paley group also hold for bounded Vilenkin groups. But if the sequence m is not bounded, then the situation changes. There are many theorems which hold on bounded Vilenkin groups but fail to hold on unbounded ones. For more details on Vilenkin groups see [1].

Define the dyadic addition of $k, n \in \mathbb{N}_0$ as follows:

$$k \oplus n := \sum_{j=0}^{\infty} ((k_j + n_j) \mod m_j) M_j.$$

Since (κ_n) is a $\psi\alpha$ system, Theorems 1, 2, 3 below are direct applications of similar ones in [4].

THEOREM 1. *The system* $\{\kappa_n\}_{n\geq 0}$ *is orthonormal on* G_m, *that is*,

$$\int_{G_m} \kappa_k(x)\overline{\kappa}_n(x)\, d\mu(x) = \delta_{k,n} \qquad \text{(the Kronecker delta)},$$

$k, n \in \mathbb{N}_0$, *and complete in* $L^1(G_m)$.

Theorem 2. *If $t \in \mathbb{N}_0$ and $x, y \in G_m$, then*

$$D_{M_t}(x,y) = \begin{cases} 0 & \text{if } x \notin I_t(y), \\ M_t & \text{if } x \in I_t(y). \end{cases}$$

Theorem 3. *If $f \in L^q(G_m)$ $(q \geq 1)$ and $n \in \mathbb{N}_0$, then*

$$\Big(\int\limits_{G_m} |S_{M_n} f|^q \Big)^{1/q} \leq A_q \Big(\int\limits_{G_m} |f|^q \Big)^{1/q} =: A_q \|f\|_{L^q},$$

where the constant A_q does not depend on f.

Theorem 4 can be proved by a slight modification of F. Schipp's method [12].

Theorem 4. *If $f \in L^q(G_m)$ $(q > 1)$ and $n \in \mathbb{N}$, then*

$$\|S_n f\|_{L^q} \leq A_q \|f\|_{L^q}$$

for some A_q depending only on q.

Next we deal with the relation between almost even arithmetical functions and Vilenkin analysis. John Knopfmacher has also been concerned with Fourier analysis of arithmetical functions; it is worthwhile to compare his theory and the analytical methods on Vilenkin groups (see [9] and [10]).

2. Results on $\mathcal{B}^q$ and $\mathcal{B}^u$. From now on throughout this paper the following condition will hold for the sequence $\{m_j\}$:

For all $k \in \mathbb{N}$ there exists an $n = n(k) \in \mathbb{N}$ such that $k \mid M_n$.

A Vilenkin group G_m generated by a sequence m of this kind is called *R (Ramanujan)–Vilenkin.*

Theorem 5. *If $f \in \mathcal{B}^u$, then there exists a unique continuous $f^* : G_m \to \mathbb{C}$ such that $f^*(\check{n}) = f(n)$ for all $n \in \mathbb{N}$ and $M(f) = \int_{G_m} f^* \, d\mu$.*

Theorem 6 (compare Knopfmacher [9]). *If $f \in \mathcal{B}^q$ $(q \geq 1)$, then there exists an $f^* : G_m \to \mathbb{C}$ such that*

$$\|f^*\|_{L^q} = \|f\|_q, \qquad f_n \overset{\|\cdot\|_q}{\longrightarrow} f \Leftrightarrow f_n^* \overset{\|\cdot\|_{L^q}}{\longrightarrow} f^*.$$

f^* is unique (in the sense of equality μ-almost everywhere).

Theorem 7. *If $f, g \in \mathcal{B}^q$ $(q \geq 1)$ and $|g| < c$, then $fg \in \mathcal{B}^q$ and $(fg)^* = f^* g^*$ μ-a.e.*

Theorem 8. *If $n \in \mathbb{N}_0$ and $g(j) = \exp(2\pi i \check{n} j)$ $(j \in \mathbb{N})$, then*

$$g^*(x) = \exp(2\pi i \check{n} \sigma x) = \kappa_n(x) \qquad (x \in G_m).$$

Theorem 9. *If $f \in \mathcal{B}^q$ $(q \geq 1)$ and $k \in \mathbb{N}_0$, then*

$$M(fe^{-2\pi i\check{k}}) = \int\limits_{G_m} f^*(x)\overline{\kappa}_k(x)\,d\mu(x) = (\hat{f}^*)(k).$$

The *Ramanujan sum* c_r is defined as

$$c_r(n) := \sum_{\substack{a=1\\(a,r)=1}}^{r} \exp(2\pi i(a/r)n) \qquad (r, n \in \mathbb{N}).$$

If $r \mid k$, then $c_r \in \mathcal{B}_k$. Cohen [2] and later Schwarz and Spilker [15] proved that $f \in \mathcal{B}_k$ implies

$$f = \sum_{r|k} \alpha_r c_r, \qquad \alpha_r = \varphi^{-1}(r)k^{-1}\sum_{n=1}^{k} f(n)c_r(n),$$

where the coefficients α_r are uniquely determined and φ is the Euler function. Define

$$\mathcal{L}^q(G_m) := \{f \in L^q(G_m) \mid \text{there exists a } g \in \mathcal{B}^q \text{ such that } g^* = f \ \mu\text{-a.e.}\} \quad (q \geq 1),$$

$$\hat{g}^{\mathrm{R}}(r) := \varphi^{-1/2}(r)\int\limits_{G_m} g^*\overline{c}_r^*, \qquad K_k g := \sum_{r|k} \varphi^{-1}(r)M(g\overline{c}_r)c_r$$

$(r, k \in \mathbb{N},\ g \in \mathcal{B}^q,\ q \geq 1)$.

It is not difficult to prove that on each R–Vilenkin group G_m the set of m-rationals $\mathbb{Q}_m$ equals the set of “ordinary” rationals $\mathbb{Q}$. This yields

Proposition 10.

$$c_r^* = \sum_{\substack{a=1\\(a,r)=1}}^{r} \kappa_{(a/r)^\vee} \qquad (1 < r \in \mathbb{N},\ c_1^* = \kappa_0)$$

on R–Vilenkin groups.

Proposition 11. *Let $f^* \in \mathcal{L}^q(G_m)$ $(q \geq 1)$. Then each member of the set $\{\hat{f}^*(n) \mid \check{n} = a/r,\ (a,r) = 1,\ a \in \{1, \ldots, r\}\}$ equals $\hat{f}^{\mathrm{R}}(r)\varphi^{-1/2}(r)$.*

Corollaries 12 and 13 below are obvious consequences of Theorems 1, 9 and Propositions 10, 11.

Corollary 12. *$\{\varphi^{-1/2}(r)c_r\}_{r\geq 1}$ is orthonormal and complete in $\mathcal{B}^1$.*

Corollary 13. *If $f \in \mathcal{B}^q$ $(q \geq 1)$ and $M(f\overline{c}_r) = 0$ for every $r \in \mathbb{N}$, then $\|f\|_q = 0$.*

In 1976 Schwarz and Spilker [16] proved Corollary 13 in the case of $q = 2$ and in the case of $q = 1$ for bounded f. In 1988 Hildebrand, Schwarz and Spilker [8] proved Theorem 16 in the case of $q = 2$ and noticed that the theorem also holds for $q = 1$ and $f \in \mathcal{B}^1$ bounded (unpublished). I have been informed by K. H. Indlekofer that Theorem 16 is already known in the general case, but it does not seem to be published yet.

LEMMA 14. *If $f \in \mathcal{B}^q$ $(q \geq 1)$ and $s \in \mathbb{N}$, then*

$$\sum_{r|s} \hat{f}^{\mathrm{R}} \varphi^{-1/2}(r) c_r^* = S_{M_t} f^*$$

on some R–Vilenkin group, $s = M_t$.

Let $s : \mathbb{N} \to \mathbb{N}$ be a sequence of natural numbers. Consider the condition

(1) For each $k \in \mathbb{N}$ there exists an $n = n(k)$ such that $k \mid s(n')$ for all $n' \geq n$.

THEOREM 15. *If $f \in \mathcal{B}^u$ and the sequence $s : \mathbb{N} \to \mathbb{N}$ satisfies condition* (1), *then $K_s f(n)$ converges to $f(n)$, uniformly in n.*

THEOREM 16. *If $f \in \mathcal{B}^q$ $(q \geq 1)$ and the sequence $s : \mathbb{N} \to \mathbb{N}$ satisfies condition* (1), *then $K_s f$ $\|\cdot\|_q$-converges to f.*

We now define the modulus of continuity of arithmetical functions. The origin of the definition is in Vilenkin analysis.

DEFINITION 17. Let $f \in D$. The $\|\cdot\|_u$-*modulus of continuity* and $\|\cdot\|_q$-*modulus of continuity* of f $(q \geq 1)$ are defined by

$$\omega_n^q(f) := \sup_{p \in \mathbb{N}} \|f(\cdot \oplus pM_n) - f(\cdot)\|_q,$$
$$\omega_n(f) := \sup_{p \in \mathbb{N}} \sup_{j \in \mathbb{N}} |f(j \oplus pM_n) - f(j)|,$$

where $n \in \mathbb{N}_0$ and G_m is some fixed R–Vilenkin group.

We define the corresponding best approximation of f by trigonometric polynomials as follows:

$$E_n^q(f) := \inf_{\{c_k\}} \Big\| f - \sum_{k=0}^{n-1} c_k e^{2\pi i \check{k}} \Big\|_q \quad \text{for } \|f\|_q < \infty,$$
$$E_n(f) := \inf_{\{c_k\}} \Big\| f - \sum_{k=0}^{n-1} c_k e^{2\pi i \check{k}} \Big\|_u \quad \text{for } \|f\|_u < \infty,$$

where $c_k \in \mathbb{C}$, $k, n \in \mathbb{N}_0$.

The following theorems show that these definitions are not unnatural.

THEOREM 18. *If $f \in \mathcal{B}^q$ $(q \geq 1)$ (resp. $f \in \mathcal{B}^u$), then*

(2) $$\omega_n^q(f) \text{ (resp. } \omega_n(f)) \downarrow 0 \quad \text{for all R–Vilenkin groups } G_m,$$

(3) $$|\{M(fe^{-2\pi i a/r}) \mid (a,r) = 1,\ a \in \{1, \ldots, r\}\}| = 1.$$

Theorem 19. *Let $f\in D$. If there exists an R–Vilenkin group G_m such that $\omega_n(f)\downarrow 0$, then f is a $\|\cdot\|_u$-periodic arithmetical function* (1). *If* (3) *holds, then $f\in\mathcal{B}^u$.*

Theorem 20. *Let $f\in D$. If there exists an R–Vilenkin group G_m such that*

$$(4)\qquad \|f_n\|_1\to 0,\qquad \textit{where } f_n(j):=\|f(j)-M(f(j\oplus\cdot M_n))\|_q^q\quad (q\ge 1),$$

then f is a $\|\cdot\|_q$-periodic arithmetical function. If (3) *holds, then $f\in\mathcal{B}^q$. If $f\in\mathcal{B}^q$, then $\|f_n\|_1\downarrow 0$ on each R–Vilenkin G_m.*

Theorem 21. *If $f\in\mathcal{B}^q$ $(q\ge 1)$ (resp. $f\in\mathcal{B}^u$), then*

$$E^q_{M_n}(f)\le\|K_{M_n}f-f\|_q\le\omega^q_n(f)\le 2E^q_{M_n}(f),$$
$$(\textit{resp. } E_{M_n}(f)\le\|K_{M_n}f-f\|_u\le\omega_n(f)\le 2E_{M_n}(f)),$$

where G_m is any fixed R–Vilenkin group.

Corollary 22. *Let $r\in\mathbb{N}$, $f\in\mathcal{B}^q$ $(q\ge 1)$ (resp. $f\in\mathcal{B}^u$). Then for all $a,b\in\mathcal{B}_r$,*

$$(5)\qquad \|K_rf-f\|_q\le 2\|a-f\|_q\qquad (\textit{resp. } \|K_rf-f\|_u\le 2\|b-f\|_u).$$

In the case of $q=2$, Corollary 22 with constant 1 is proved in [8] by Hildebrand, Schwarz and Spilker. Their method does not seem to work in the general case. It is also possible that (5) does not hold without the constant 2 for all q.

The following theorem for $C(G_m)$ and $L^q(G_m)$ $(q=1,2)$ is proved by Rubinshteĭn [11] and for arbitrary $q\ge 1$ by Fridli [3].

Theorem 23. *Let G_m be an R–Vilenkin group and let $z_n\downarrow 0$, $q\ge 1$. There exists an $f\in\mathcal{B}^q$ and also a $g\in\mathcal{B}^u$ for which $\omega^q_n(f)=\omega_n(g)=z_n$ for each $n\in\mathbb{N}_0$.*

Denote by

$$S^{\mathrm{R}}_n(f):=\sum_{r=1}^n\hat f^{\mathrm{R}}(r)c_r\varphi^{-1/2}(r)\qquad (f\in\mathcal{B}^1)$$

the nth partial sum of the Ramanujan series of f. Theorem 24 is proved for limit periodic arithmetical functions in [5], and our version is a trivial consequence.

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(1) The Banach space of $\|\cdot\|_x$-periodic ($x=q$ or $x=u$) arithmetical functions is the closure of B ($B:=\bigcup_{k\in\mathbb{N}}B_k$, B_k is the set of $\bmod k$ periodic arithmetical functions) in $\{f\in D\mid \|f\|_x<\infty\}$ with respect to $\|\cdot\|_x$.

Theorem 24. *Let* $f \in \mathcal{B}^q$, $1 \le q \le 2$, $p^{-1} + q^{-1} = 1$, *and let* G_m *be an R–Vilenkin group. If*

$$A := \sum_{k=0}^{\infty} M_k^{1-1/p} m_k \ln m_k \omega_k^q(f) < \infty,$$

then

$$c_q A > \sum_{\alpha \in \mathbb{Q}} |M(fe^{-2\pi i \alpha})|,$$

thus $S_n^{\mathrm{R}} f$ *uniformly converges to* f $(n \to \infty)$.

In [5] Theorem 4 is proved for $\|\cdot\|_q$ -limit periodic arithmetical functions $(q > 1)$, hence it also holds for $f \in \mathcal{B}^q$ $(q > 1)$. An easy consequence is that $f \in \mathcal{B}^q$ $(q > 1)$ implies

$$S_n f \xrightarrow{\|\cdot\|_q} f.$$

This fails to hold for $q = 1$.

Theorem 25. *Let* G_m *be an R–Vilenkin group. There exists an* $f \in \mathcal{B}^1$ *such that* $\sup_{n\in\mathbb{N}} \|S_n f\|_1 = \infty$.

Simon [17] proved that for each Vilenkin group G_m there exists an $F \in L^1(G_m)$ such that $S_n F$ diverges everywhere. Does this hold for $\mathcal{B}^1$ functions?

Most interesting is the case of a $\mathcal{B}^q$ $(q > 1)$ because Hildebrand [7] proved the existence of a $\mathcal{B}^q$ almost even arithmetical function whose Ramanujan expansion converges to plus infinity everywhere. But Gosselin [6] and Schipp [13] proved on bounded Vilenkin groups ($\sup m_s < \infty$, of course in this case G_m is not R–Vilenkin) the μ-almost everywhere convergence of $S_n F$ for $F \in L^q(G_m)$ $(q > 1)$. What can be said of the convergence of $S_n f^*$ in the case of $f \in \mathcal{B}^q$ $(q > 1)$?

The theorem of Gosselin and Schipp is an open question for unbounded G_m groups ($\sup m_s = \infty$, G_m not necessarily R–Vilenkin of course; the origin of this topic is Luzin's conjecture, and Carleson's and Hunt's results), therefore it would be interesting to construct (if possible) a counterexample by Hildebrand's method.

Here we remark that the author proved the existence of a $\|\cdot\|_u$ (uniform) limit periodic arithmetical function such that

$$S_n f^*(\check{j}) = \sum_{k=0}^{n-1} M(fe^{-2\pi i \check{k}}) e^{-2\pi i \check{k} j}$$

diverges for each $j \in \mathbb{N}$ (the proof will be published elsewhere).

3. Proofs

Proof of Theorem 5. For every $0 < \varepsilon$ there exists an $f_\varepsilon \in \mathcal{B}_{k(\varepsilon)}$ such that $\|f - f_\varepsilon\|_u < \varepsilon$. Since f_ε is even, it is easy to see that there exists a unique continuous step function $f^*_\varepsilon : G_m \to \mathbb{C}$ such that $f_\varepsilon(n) = f^*_\varepsilon(\check{n})$ for all $n \in \mathbb{N}$ and $M(f_\varepsilon) = \int_{G_m} f^*_\varepsilon \, d\mu$.

The limit $f^*(x) := \lim_{\varepsilon\to 0} f^*_\varepsilon(x)$ exists for all $x \in G_m$.

Indeed, let $\varepsilon_1, \varepsilon_2 > 0$. Set

$$k^* := \min_{n\in\mathbb{N}}(n \in \mathbb{N} : k \mid M_n).$$

Take an $x \in G_m$. If $x = x_{\varepsilon_1} + x' = x_{\varepsilon_2} + x''$, where $x' \in I_{M_{k^*(\varepsilon_1)}}$ and $x'' \in I_{M_{k^*(\varepsilon_2)}}$, then for the step functions $f^*_{\varepsilon_1}, f^*_{\varepsilon_2}$ we have

$$|f^*_{\varepsilon_1}(x) - f^*_{\varepsilon_2}(x)| = |f_{\varepsilon_1}(\check{x}_{\varepsilon_1}) - f_{\varepsilon_2}(\check{x}_{\varepsilon_2})|.$$

Since $x - x_{\varepsilon_i} \in I_{M_{k^*(\varepsilon_i)}}$ $(i = 1, 2)$, supposing $k^*(\varepsilon_1) \le k^*(\varepsilon_2)$ we find that

$$M_{k^*(\varepsilon_1)} \mid \check{x}_{\varepsilon_1} - \check{x}_{\varepsilon_2},$$

hence

$$f_{\varepsilon_1}(\check{x}_{\varepsilon_1}) = f_{\varepsilon_1}(\check{x}_{\varepsilon_2}).$$

This implies that

$$\begin{aligned}|f^*_{\varepsilon_1}(\check{x}_{\varepsilon_1}) - f^*_{\varepsilon_2}(\check{x}_{\varepsilon_2})| &= |f_{\varepsilon_1}(\check{x}_{\varepsilon_2}) - f_{\varepsilon_2}(\check{x}_{\varepsilon_2})| \\ &\le \|f - f_{\varepsilon_1}\|_u + \|f - f_{\varepsilon_2}\|_u < \varepsilon_1 + \varepsilon_2.\end{aligned}$$

This shows the uniform convergence of the continuous step functions f^*_ε to f^* on the R–Vilenkin group G_m. Since G_m is compact, f^* is also continuous. Since $\{\check{n} \mid n \in \mathbb{N}\}$ is dense in G_m, the unicity of f^* is proved. We have

$$\int_{G_m} f^* \, d\mu = \lim_{\varepsilon\to 0} \int f^*_\varepsilon \, d\mu.$$

That is, $M(f_\varepsilon)$ converges as $\varepsilon \to 0$. We have

$$\begin{aligned}M(f^0_\varepsilon) - \varepsilon &\le \underline{M}(f^0_\varepsilon) - \overline{M}(|f^0 - f^0_\varepsilon|) \le \underline{M}(f^0) \le \overline{M}(f^0) \\ &\le \overline{M}(|f^0 - f^0_\varepsilon|) + \overline{M}(f^0_\varepsilon) \le \varepsilon + M(f^0_\varepsilon)\end{aligned}$$

$(g^0 = \operatorname{Re} g$ or $\operatorname{Im} g$, $g = f, f_\varepsilon)$.

Thus $\int_{G_m} f^*_\varepsilon \, d\mu = M(f_\varepsilon)$ implies that $\int f^* \, d\mu = M(f)$. If $n \in \mathbb{N}$, then

$$f^*(\check{n}) = \lim_{\varepsilon\to 0} f^*_\varepsilon(\check{n}) = \lim_{\varepsilon\to 0} f_\varepsilon(n).$$

The proof of Theorem 5 is complete.

Proof of Theorem 6. For each $\varepsilon > 0$, there exists an $f_\varepsilon \in \mathcal{B}_{k(\varepsilon)}$ such that $\|f - f_\varepsilon\|_q < \varepsilon$. Thus $f^*_{\varepsilon_i}$ $(0 < \varepsilon_i \to 0)$ is a Cauchy sequence in

$L^q(G_m)$. Hence there exists a unique $f^* \in L^q(G_m)$ such that

$$\lim_{\varepsilon_i\to 0} \|f^* - f^*_{\varepsilon_i}\|_{L^q} = \lim_{\varepsilon_i\to 0} \|f - f_{\varepsilon_i}\|_q = 0$$

(uniqueness in the sense of equality μ-a.e.).

The proof of Theorem 6 is complete.

Theorem 7 can be proved by the method of W. Schwarz (Proposition 3.2 in [16]). The proof of Theorem 8 can be found in [5]. Theorem 9 is proved for $\|\cdot\|_q$-limit periodic arithmetical functions ($q \ge 1$) in [5] hence it also holds for $\mathcal{B}^q$ ($q \ge 1$) functions.

P r o o f o f P r o p o s i t i o n 10. If $a/r = 1$ and $(a,r) = 1$, then $a = r = 1$. Since $c_1^* = \kappa_0 = 1$, we can suppose $a/r < 1 < r$. Let n be the least natural number for which $aM_n/r \in \mathbb{N}$. If k is an integer in $[0, M_n)$, then $k = k_{n-1}M_{n-1} + \ldots + k_0M_0$. This gives $\check{k} = M_n^{-1}(k_0m_{n-1}\ldots m_1 + k_1m_{n-1}\ldots m_2 + \ldots + k_{n-1}) =: M_n^{-1}k'$. It is easy to see that k' can be any integer in $[0, M_n)$, hence there exists a unique $k \in [0, M_n)$ such that $k' = aM_n/r$, thus $\check{k} = a/r$. From Theorem 8 it follows that

$$c_r^* = \sum_{\substack{a=1\\(a,r)=1}}^{r} \kappa_{(a/r)^\vee}.$$

The proof of Proposition 10 is complete.

P r o o f o f P r o p o s i t i o n 11. Let $\varepsilon > 0$. There exists an $f_\varepsilon \in \mathcal{B}$ such that

$$\varepsilon > \|f - f_\varepsilon\|_q = \|f^* - f^*_\varepsilon\|_{L^q} \qquad \text{(Theorem 6)}.$$

Since $f_\varepsilon \in \mathcal{B}_j$ for some $j \in \mathbb{N}$, by Cohen's theorem

$$f_\varepsilon = \sum_{r\mid j} \beta_r c_r \varphi^{-1/2}(r).$$

Hence

$$f^*_\varepsilon = \sum_{r\mid j} \beta_r c^*_r \varphi^{-1/2}(r), \qquad \text{where } \beta_r = \hat{f}^{\mathrm{R}}_\varepsilon(r).$$

This obviously gives the proof for the function f_ε. Now,

$$\begin{aligned}|\hat{f}^*(n) - \varphi^{-1/2}(r)\hat{f}^{\mathrm{R}}(r)| &\le |\hat{f}^*(n) - \hat{f}^*_\varepsilon(n)| + |\hat{f}^*_\varepsilon(n) - \hat{f}^{\mathrm{R}}_\varepsilon(r)\varphi^{-1/2}(r)| \\ &\quad + \varphi^{-1/2}(r)|\hat{f}^{\mathrm{R}}_\varepsilon(r) - \hat{f}^{\mathrm{R}}(r)| \\ &\le 2\int_{G_m} |f^* - f^*_\varepsilon|\,d\mu \le 2\|f^* - f^*_\varepsilon\|_{L^q} < 2\varepsilon.\end{aligned}$$

The proof of Proposition 11 is complete.

P r o o f o f L e m m a 14. First we give the construction of the desired R–Vilenkin group. Let $m_0, m_1, \ldots, m_{t-1}$ be integers not less than 2 with

$m_0 m_1 \dots m_{t-1} = s$. The m_i's for $i \ge t$ are defined in such a way that G_m is an R–Vilenkin group. Then obviously

$$\{a/r \,|\, (a,r)=1, a \in \{1,\dots,r\}, r \,|\, s\} = \{b/s \,|\, b \in \{1,\dots,s\}\}. \tag{6}$$

Let b/s belong to the right side of (6), $b \neq s$. We have $b/s = b/M_t$, hence $((b/s)^\vee)' = b$. Since the set of b's is $\{1,\dots,s-1\} = \{1,\dots,M_t-1\}$, the set of $(b/M_t)^\vee = (b/s)^\vee$ is also $\{1,\dots,M_t-1\}$. We have $c_1^* = \kappa_0 = 1$. By the application of Propositions 10, 11 the proof of Lemma 14 is complete.

P r o o f o f T h e o r e m 15. Use the result of Lemma 14 and apply Theorem 2. Let $\varepsilon > 0$ and $f_\varepsilon \in \mathcal{B}_{k(\varepsilon)}$ such that $\|f - f_\varepsilon\|_u < \varepsilon$. By (1) there exists an n_ε such that $k(\varepsilon) \,|\, s(n)$ for $n \ge n_\varepsilon$. Lemma 14 and Theorems 2 and 5 give

$$\begin{aligned} |K_{s(n)} f(j) - f(j)| &= |S_{M_{t(n)}} f^*(\check{j}) - f^*(\check{j})| \\ &= M_{t(n)} \int\limits_{I_{t(n)}(\check{j})} |f^*(x) - f^*(\check{j})| \, d\mu(x). \end{aligned}$$

Since f^* is uniformly continuous on the compact set G_m,

$$\begin{aligned} &\sup_{x \in I_t(\check{j})} |f^*(x) - f^*(\check{j})| \\ &\quad = \sup_{x \in I_t(\check{j}) \cap \mathbb{Q}_m} |f^*(x) - f^*(\check{j})| = \sup_{k \equiv j \,(\mathrm{mod}\, M_{t(n)})} |f(k) - f(j)| \\ &\quad \le 2\|f - f_\varepsilon\|_u + \sup_{k \equiv j \,(\mathrm{mod}\, M_t)} |f_\varepsilon(k) - f_\varepsilon(j)| = 2\|f - f_\varepsilon\|_u < 2\varepsilon. \end{aligned}$$

Thus $|K_{s(n)} f(j) - f(j)| < 2\varepsilon$.

This completes the proof of Theorem 15.

P r o o f o f T h e o r e m 16. Since $f \in \mathcal{B}^q$ $(q \ge 1)$, $f^* \in L^q(G_m)$. Fix an $\varepsilon > 0$. There exists an $f_\varepsilon \in \mathcal{B}_{k(\varepsilon)}$ for which $\|f - f_\varepsilon\|_q < \varepsilon$. (1) implies the existence of an $n_\varepsilon \in \mathbb{N}$ such that $k(\varepsilon) | s(n)$ for all $n_\varepsilon \le n \in \mathbb{N}$. Fix such an $s = s(n)$ and consider the Vilenkin group given by Lemma 14. Cohen's theorem gives

$$f_\varepsilon^* = \sum_{r|k} \beta_r c_r^* \qquad (\beta_r = \varphi^{-1}(r) M(f \bar{c}_r)).$$

Since by Lemma 14, $(K_s g)^* = S_{M_t} g^*$ for all $g \in \mathcal{B}^1$, Theorem 1, Propositions 10, 11 and Lemma 14 give $(K_s f_\varepsilon)^* = S_{M_t} f_\varepsilon^* = f_\varepsilon^*$. This and Theorems 6 and 3 imply

$$\begin{aligned} \|K_s f - f\|_q &= \|S_{M_t} f^* - f^*\|_{L^q} \le \|S_{M_t} f^* - f_\varepsilon^*\|_{L^q} + \|f - f_\varepsilon\|_q \\ &= \|S_{M_t}(f^* - f_\varepsilon^*)\|_{L^q} + \|f - f_\varepsilon\|_q \\ &< (A_q + 1)\|f - f_\varepsilon\|_q < (A_q + 1)\varepsilon. \end{aligned}$$

The proof of Theorem 16 is complete.

P r o o f o f T h e o r e m 18. Theorems 6 and 5 give respectively

$$\omega_n^q(f)=\sup_{y\in I_n}\Big\{\int\limits_{G_m}|f^*(x+y)-f^*(x)|^q\,d\mu(x)\Big\}^{1/q},\tag{7}$$

$$\omega_n(f)=\sup_{y\in I_n}\sup_{x\in G_m}|f^*(x+y)-f^*(x)|.\tag{8}$$

The right side of (7) is the usual $L^q(G_m)$-modulus of continuity of f^*. The right side of (8) is the $C(G_m)$-modulus of continuity of f^*. Thus $\omega_n^q(f)$ (resp. $\omega_n(f)\downarrow 0$, directly from Vilenkin analysis [1]. (3) is easy to verify. The proof is complete.

P r o o f o f T h e o r e m 19. Let $\varepsilon>0$ be fixed. For $n>n_0(\varepsilon)$,

$$\omega_n(f)=\sup_{p\in\mathbb{N}}\sup_{j\in\mathbb{N}}|f(j)-f(j\oplus pM_n)|<\varepsilon,$$

that is,

$$|f(j)-f(j\oplus pM_n)|<\varepsilon\quad\text{ for all } j,p\in\mathbb{N}.$$

Define (with $f(0):=0$)

$$\hat f_c(k):=M_c^{-1}\sum_{a=0}^{M_c-1}f(a)\overline{\kappa}_k(\check a)\quad\text{ and }\quad g(j):=\sum_{k=0}^{M_n-1}\hat f_c(k)\overline{\kappa}_k(\check j)\quad(c\in\mathbb{N}_0).$$

Then g is periodic. Suppose that $c>n$. Then

$$\begin{aligned}|f(j)-g(j)|&=\Big|f(j)-M_c^{-1}\sum_{a=0}^{M_c-1}f(a)\sum_{k=0}^{M_n-1}\overline{\kappa}_k(\check a)\kappa_k(\check j)\Big|\\&=\Big|f(j)-\frac{M_n}{M_c}\sum_{\substack{a=0\\ a\equiv j\ (\mathrm{mod}\ M_n)}}^{M_c-1}f(a)\Big|\qquad\text{(Theorem 2)}\\&\le\frac{M_n}{M_c}\sum_{p=0}^{M_c/M_n-1}\Big|f(j)-f\Big(\sum_{i=0}^{n-1}j_iM_i\oplus pM_n\Big)\Big|<2\varepsilon.\end{aligned}$$

That is, f is uniform limit periodic.

In [5] it is proved for uniform limit periodic arithmetical functions that $S_{M_n}f^*\xrightarrow{\|\cdot\|_u}f$. The rest of the proof of Theorem 19 follows from $S_{M_n}f^*=\sum_{r|M_n}M(f\overline{c}_r)c_r\varphi^{-1}(r)$ as (3) is satisfied for f. The proof is complete.

P r o o f o f T h e o r e m 20. Suppose that $f\in\mathcal{B}^q$. Then

$$\begin{aligned}&\Big\{\int\limits_{G_m}M_n\int\limits_{I_n}|f^*(x+h)-f^*(x)|^q\,d\mu(h)\,d\mu(x)\Big\}^{1/q}\\&\quad=M_n^{1/q}\Big\{\int\limits_{G_m}\int\limits_{I_n}|f^*(x+h)-f^*(x)|^q\,d\mu(h)\,d\mu(x)\Big\}^{1/q}\end{aligned}$$

$$= M_n^{1/q}\Big\{ \int_{I_n}\int_{G_m} |f^*(x+h)-f^*(x)|^q \,d\mu(x)\,d\mu(h)\Big\}^{1/q} \quad \text{(Fubini's theorem)}$$

$$= \Big\{ M_n \int_{I_n}\int_{G_m} |f^*(x+h)-f^*(x)|^q \Big\}^{1/q}$$

$$\le \{\sup_{h\in I_n} \|f^*(x+h)-f^*(x)\|_q^q\}^{1/q} = \omega_n^q(f)\downarrow 0$$

by Theorem 18. Since

$$\|f_n\|_1^{q^{-1}} = \Big\{ \int_{G_m} M_n \int_{I_n} |f^*(x+h)-f^*(x)|^q \,d\mu(h)\,d\mu(x)\Big\}^{1/q},$$

the last statement of Theorem 20 is proved. The other two statements being trivial, the proof is complete.

Corollary 22 is a straightforward consequence of Theorem 21, which can be proved by the application of the similar result for $\psi\alpha$ systems on Vilenkin groups [4].

The proof of Theorem 24 can also be obtained with the help of the similar result for $\psi\alpha$ systems [4], which generalizes the result of Zhantlesov [19] proved in the case of the original Vilenkin system (i.e. $\alpha = 1$).

P r o o f o f T h e o r e m 23. The original idea, concerning Vilenkin (and not necessarily R–Vilenkin) groups comes from Fridli [3] and Rubinshteĭn [11]. They prove the existence of $L^q(G_m)$ and $C(G_m)$ functions whose ω_n^q respectively ω_n modulus of continuity is z_n. In [5] (Theorem 25) it is proved that there exists a $\|\cdot\|_1$-limit periodic arithmetical function $f:\mathbb{N}\to\mathbb{N}$ such that $\omega_n^q(f) = z_n$ for each $n\in\mathbb{N}_0$ and $f^* = F$ μ-almost everywhere on G_m, where $F\in L^q(G_m)$ satisfies the following relation: $\|F_n - F\|_{L^q}\to 0$ $(n\to+\infty)$, where

$$F_n = \sum_{i=0}^{\infty} \alpha_{n,i}\operatorname{char}_{I_i\setminus I_{i+1}}$$

($n\in\mathbb{N}_0$, char_B denotes the characteristic function of the set B, $\alpha_{n,i}$ is some complex number, $n,i\in\mathbb{N}_0$).

Define arithmetical functions

$$f_n := \sum_{i=0}^{\infty} \alpha_{n,i}\beta_i,$$

where

$$\beta_i(j) := \begin{cases} 1 & \text{if } M_i \,|\, j \text{ and } M_{i+1}\nmid j,\\ 0 & \text{otherwise} \quad (\beta_i(0) := 0).\end{cases}$$

It suffices to show that

(9) β_i is an even arithmetical function, and $f_n \in \mathcal{B}^q$,

(10) $$f_n^* = F_n \quad \mu\text{-a.e. on } G_m.$$

Indeed, by (9) and (10) Theorem 6 gives that the arithmetical function f, whose $\|\cdot\|$-modulus of continuity is z, is a $\|\cdot\|_q$-almost even arithmetical function, that is, the proof of Theorem 23 would be complete. Now,

$$\begin{aligned}
&M(\beta_k(\cdot)e^{-2\pi i(a/n)\cdot}) \\
&\quad= \lim_{b\to+\infty} b^{-1}\sum_{j=0}^{b-1}\beta_k(j)e^{-2\pi i(a/n)j} = (M_{k+1}n)^{-1}\sum_{j=0}^{M_{k+1}n}\beta_k(j)e^{-2\pi i(a/n)j} \\
&\quad= (M_{k+1}n)^{-1}\sum_{\substack{j=0\\ M_k\mid j}}^{M_{k+1}n} e^{-2\pi i(a/n)j} - (M_{k+1}n)^{-1}\sum_{\substack{j=0\\ M_{k+1}\mid j}}^{M_{k+1}n} e^{-2\pi i(a/n)j} \\
&\quad=: A_1 - A_2 \qquad (a,n\in\mathbb{N},\ (a,n)=1).
\end{aligned}$$

It is easy to see that

$$A_1 = \begin{cases} M_k^{-1} & \text{if } n \mid M_k, \\ 0 & \text{otherwise;}\end{cases} \qquad A_2 = \begin{cases} M_{k+1}^{-1} & \text{if } n \mid M_{k+1}, \\ 0 & \text{otherwise,}\end{cases}$$

therefore $A_1 - A_2$ does not depend on a.

Hence, as β_k is periodic modulo M_{k+1}, β_k is even. Define

$$f_n^K := \sum_{i=0}^{K}\alpha_{n,i}\beta_i, \qquad F_n^K := \sum_{i=0}^{K}\alpha_{n,i}\mathrm{char}_{I_i\setminus I_{i+1}} \qquad (f_n^K \in D,\ F_n^K : G_m \to \mathbb{C}).$$

We have

$$\|f_n - f_n^K\|_q = \Big\{\sum_{i=K+1}^{\infty}|\alpha_{n,i}|^q(1/M_i - 1/M_{i+1})\Big\}^{1/q} = \|F_n - F_n^K\|_{L^q} \to 0$$

as $K\to\infty$ because $F_n \in L^q(G_m)$.

Since f_n^K is even, f_n is almost even. We have $\|f_n\|_q = \|F_n\|_{L^q}$, consequently $f_n \in \mathcal{B}^q$. Next, $\beta_i^* = \mathrm{char}_{I_i\setminus I_{i+1}}$ μ-almost everywhere on G_m, thus $(f_n^K)^* = F_n^K$ μ-a.e. on G_m for each $k\in\mathbb{N}_0$. This implies that $f_n^* = F_n$ μ-a.e. on G_m, that is, (9) and (10) are proved. Thus the proof of Theorem 23 for the $\|\cdot\|_q$-modulus of continuity is complete. To prove the existence of a uniform almost even arithmetical function whose $\|\cdot\|_u$-modulus of continuity is a given sequence, one can apply the idea of Rubinshteĭn [11] and the methods used above.

P r o o f o f T h e o r e m 25. Define arithmetical functions

$$a_n(j) := \begin{cases} M_{n+1} - M_n & \text{if } M_{n+1} \mid j, \\ -M_n & \text{if } M_n \mid j \text{ and } M_{n+1} \nmid j \\ 0 & \text{if } M_n \nmid j. \end{cases} \quad (a_n(0) := 0, n \in \mathbb{N}_0),$$

Then

$$a_n(j) = D_{M_{n+1}}(j,0) - D_{M_n}(j,0) = \sum_{k=M_n}^{M_{n+1}-1} \exp(2\pi i \check{k} j)$$

is even. Let $M_n \le b < 2M_n$. Thus $b = M_n + b_{n-1}M_{n-1} + \ldots + b_0 M_0$ and

$$D_b(r,j) = D_{M_n}(r,j) + \sum_{k=M_n}^{b-1} \exp\left(2\pi i \check{k}(r-j)\right).$$

Since $k = M_n + k_{n-1}M_{n-1} + \ldots + k_0 M_0 =: M_n + k^-$ therefore $\check{k} = 1/M_{n+1} + (k^-)^\vee$. This implies that

$$D_b(r,j) = D_{M_n}(r,j) + \exp\left(2\pi i \frac{1}{M_n}(r-j)\right) \sum_{k^-=0}^{b-M_n-1} \exp\left(2\pi i (k^-)^\vee (r-j)\right).$$

Now

$$\begin{aligned} S_b a_n(r) &= M\Big\{ \sum_{k=M_n}^{M_{n+1}-1} \exp\left(2\pi i \check{k} j\right) \Big\{ \sum_{k=0}^{M_n-1} \exp\left(2\pi i \check{k}(r-j)\right) \\ &\quad + \sum_{k=M_n}^{b-1} \exp(2\pi i \check{k}(r-j)) \Big\} \Big\} \\ &= \sum_{k=M_n}^{b-1} \exp(2\pi i \check{k} r) = \exp\left(2\pi i \frac{1}{M_{n+1}} r\right) D_{b-M_n}(r,0). \end{aligned}$$

Hence

$$M_n^{-1} \sum_{b=M_n+1}^{2M_n} \|S_b a_n\|_1 = M_n^{-1} \sum_{k=1}^{M_n} \|D_k(\cdot,0)\|_1 =: D_n^0.$$

Next we give a lower bound for D_n^0:

$$D_n^0 \ge M_n^{-1} \sum_{s=0}^{n-1} M\Big\{ \sum_{k=1}^{M_n} |D_k(\cdot,0)| \beta_s(\cdot) \Big\}. \tag{11}$$

(The arithmetical functions β_s are defined in the proof of Theorem 23.) The product $|D_k(j,0)|\beta_s(j)$ can be different from zero only in the case when $M_s \mid j$ and $M_{s+1} \nmid j$. In this case it is equal to $|D_k(j,0)|$. Suppose that

$k = \sum_{i=0}^{t} k_i M_i$ $(t \le n)$. We have

$$\begin{aligned}
D_k(j,0) = D_{M_t}(j,0) &+ \exp\left(2\pi i \frac{1}{M_{t+1}} j\right) D_{M_t}(j,0) + \dots \\
&+ \exp\left(2\pi i j \frac{k_t - 1}{M_{t+1}}\right) D_{M_t}(j,0) + \exp\left(2\pi i j \frac{k_t}{M_{t+1}}\right) D_{M_{t-1}}(j,0) \\
&+ \exp\left(2\pi i j \left(\frac{k_t}{M_{t+1}} + \frac{1}{M_t}\right)\right) D_{M_{t-1}}(j,0) + \dots \\
&+ \exp\left(2\pi i j \left(\frac{k_t}{M_{t+1}} + \frac{k_{t-1} - 1}{M_t}\right)\right) D_{M_{t-1}}(j,0) \\
&+ \exp\left(2\pi i j \left(\frac{k_t}{M_{t+1}} + \frac{k_{t-1}}{M_t}\right)\right) D_{M_{t-2}}(j,0) + \dots \\
&+ \exp\left(2\pi i j \left(\frac{k_t}{M_{t+1}} + \dots + \frac{k_{s+1}}{M_{s+2}}\right)\right) D_{M_s}(j,0) + \dots \\
&+ \exp\left(2\pi i j \left(\frac{k_t}{M_{t+1}} + \dots + \frac{k_1}{M_2} + \frac{k_0 - 1}{M_1}\right)\right) D_{M_0}(j,0).
\end{aligned}$$

Since $M_s \mid j$ and $M_{s+1} \nmid j$, assuming $k_s \ge 1$ we get

$$\begin{aligned}
D_k(j,0) &= \sum_{l=0}^{s} M_l \sum_{n=0}^{k_l - 1} \exp\left(2\pi i j \left(\frac{k_t}{M_{t+1}} + \dots + \frac{k_{l+1}}{M_{l+2}} + \frac{n}{M_{l+1}}\right)\right) \\
&= M_s \sum_{n=0}^{k_s - 1} \exp\left(2\pi i j \left(\frac{k_t}{M_{t+1}} + \dots + \frac{k_{s+1}}{M_{s+2}}\right)\right).
\end{aligned}$$

As a consequence,

$$|D_k(j,0)| = M_s \left| \sum_{n=0}^{k_s - 1} \exp\left(2\pi i j \frac{n}{M_{s+1}}\right) \right| = M_s \left| \sum_{n=0}^{k_s - 1} \exp\left(2\pi i \frac{j_s n}{m_s}\right) \right|.$$

This gives

$$\begin{aligned}
M\Big\{\sum_{k=1}^{M_n} |D_k(j,0)| \beta_s(\cdot)\Big\} &\ge \sum_{k=1}^{M_n} M_s \frac{1}{M_{s+1}} \sum_{j_s=1}^{m_s - 1} \left| \frac{\exp(2\pi i j_s k_s / m_s) - 1}{\exp(2\pi i j_s / m_s) - 1} \right| \\
&= \frac{1}{m_s} \sum_{\substack{k=M_s \\ k_s \ge 1}}^{M_n} \sum_{j=1}^{m_s - 1} \frac{|\sin(\pi j k_s / m_s)|}{\sin(\pi j / m_s)} \\
&\ge \frac{1}{m_s} \frac{M_n - M_s}{m_s} \sum_{k_s=1}^{m_s - 1} \sum_{j=1}^{m_s - 1} \frac{|\sin(\pi j k_s / m_s)|}{\sin(\pi j / m_s)}
\end{aligned}$$

$$\geq cM_n m_s^{-2} \sum_{j=1}^{[m_s/2]} \frac{m_s}{j} \sum_{j=1}^{m_s-1} |\sin(\pi j k_s/m_s)|$$

$$\geq cM_n m_s^{-2} \sum_{j=1}^{[m_s/2]} \frac{m_s}{j} j \sum_{k_s=1}^{[m_s/(2j)]} \frac{jk_s}{m_s} \geq cM_n \sum_{j=1}^{[m_s/2]} \frac{1}{j} \geq cM_n \log m_s$$

for some absolute constant $c > 0$ (which may vary from line to line). Substituting this last inequality into (11) we get

$$D_n^0 \geq c \sum_{s=0}^{n-1} \log m_s = c \log M_n \geq cn$$

for some absolute constant $c > 0$. This implies that there exists a $b = b(a_n)$ with $M_n < b \leq 2M_n$ such that $\|S_b a_n\|_1 \geq cn$.

Define

$$f := \sum_{n=0}^{\infty} \lambda_n a_{\nu_n}, \qquad \text{where} \qquad \lambda_n \in \mathbb{C},\ \sum_{n=0}^{\infty} |\lambda_n| < \infty.$$

Then $f^K := \sum_{n=0}^{K} \lambda_n a_{\nu_n}$ is an even arithmetical function. We have

$$\|f - f^K\|_1 \leq c \sum_{n=K+1}^{\infty} |\lambda_n| (M_{\nu_n+1} - M_{\nu_n}) \frac{1}{M_{\nu_n+1}}$$
$$\leq c \sum_{n=K+1}^{\infty} |\lambda_n| \to 0 \qquad (K \to +\infty).$$

Thus $f \in \mathcal{B}^1$ is almost even.

If $j < n$, then $S_{b(a_j)} a_n = 0$, hence in this case $\|S_{b(a_j)} a_n\|_1 \leq c$. Finally, we get

$$\|S_{b(a_{\nu_n})} f\|_1 = \Big\| S_{b(a_{\nu_n})} \Big(\sum_{j=0}^{n} \lambda_j a_{\nu_j} \Big) \Big\|_1 \geq \|S_{b(a_{\nu_n})} a_{\nu_n}\|_1 |\lambda_n| - c \sum_{k=0}^{\infty} |\lambda_k|$$
$$\geq c\nu_n |\lambda_n| - c \sum_{k=0}^{\infty} |\lambda_k|.$$

Now, take $\nu_n > n/|\lambda_n|$ ($n \in \mathbb{N}_0$) and $\lambda_n \neq 0$ ($n \in \mathbb{N}_0$). We get $\sup \|S_n f\|_1 = \infty$, that is, the proof of Theorem 25 is complete.

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