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L^p-L^q-Time decay estimate for solution of the Cauchy problem for hyperbolic partial differential equations of linear thermoelasticity

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Abstract. We prove the L^p-L^q-time decay estimates for the solution of the Cauchy problem for the hyperbolic system of partial differential equations of linear thermoelasticity. In our proof based on the matrix of fundamental solutions to the system we use Strauss–Klainerman's approach [12], [5] to the L^p-L^q-time decay estimates.

0. Introduction. We consider the Cauchy problem for the hyperbolic system of partial differential equations of linear thermoelasticity (cf. [13]):

$$\rho\partial_t^2 u - \mu\Delta u - (\lambda+\mu)\,\mathrm{grad}\,\mathrm{div}\,u + \beta\,\mathrm{grad}\,\partial_t T = 0\,, \tag{0.1}$$

$$\beta\,\mathrm{div}\,\partial_t u + \rho\tau\partial_t^2 T - k\Delta T = 0\,, \tag{0.2}$$

with initial conditions

$$\begin{aligned} u(+0,x) &= u^0(x)\,, \qquad (\partial_t u)(+0,x) = u^1(x)\\ T(+0,x) &= T^0(x)\,, \qquad (\partial_t T)(+0,x) = T^1(x) \end{aligned} \tag{0.3}$$

where $u = (u_1,u_2,u_3)$ is the displacement vector field of the medium, T the temperature of the medium, $t \ge 0$, $x \in \mathbb{R}^3$, $\partial_t = \partial/\partial t$, $\Delta = \sum_{j=1}^3 \partial_j^2$; ρ, μ, λ, β, τ, k are positive physical constants; u^0, u^1, T^0, T^1 are given functions.

Remark 0.1. The system (0.1)–(0.2) is the principal part of a hyperbolic system of partial differential equations describing the evolution of a thermoelastic medium (cf. E. S. Suhubi [13], p. 199, formulae (2.7.36)). For the sake of simplicity we assume in system (2.7.36) that $\gamma = 1$ and $T_0 = 1$.

Under the assumption that the Cauchy data u^0, u^1, T^0, T^1 are smooth enough (cf. [4], formulae (5.1)–(5.3)) the solution of the problem (0.1)–(0.3)

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is given by

$$U(t,x)=(u(t,x),T(t,x))=H(t,\cdot)*\overline{g}(x)+\partial_t H(t,\cdot)*\widetilde{h}(x) \tag{0.4}$$

where $\overline{g}(\cdot)=\widetilde{g}(\cdot)+D(\partial)\widetilde{h}(\cdot)$, $\widetilde{g}(x)=(u^1(x),T^1(x))$, $\widetilde{h}(x)=(u^0(x),T^0(x))$,

$$D(\partial)=\begin{pmatrix} 0 & 0 & 0 & \beta\partial_1 \\ 0 & 0 & 0 & \beta\partial_2 \\ 0 & 0 & 0 & \beta\partial_3 \\ \beta\partial_1 & \beta\partial_2 & \beta\partial_3 & 0 \end{pmatrix},$$

$*$ denotes the three-dimensional convolution in $\mathbb{R}^3$ and $H(t,x)$ is the matrix of fundamental solutions of the system (0.1)–(0.2) constructed in [4] of the form

$$\begin{aligned}
(0.5)\qquad & H_{jk}(t,x) \\
&= (32\pi 7!\rho^4\tau)^{-1}\Big\{\delta_{jk}(1-\delta_{k4})\Big[\frac{A_1}{a_1^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_1}\Big) \\
&\quad -\frac{A_2}{a_2^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_2}\Big)-\frac{B}{b^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{b}\Big)\Big] \\
&\quad +\delta_{jk}\delta_{k4}\Big[\frac{A_3}{a_1^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_1}\Big)-\frac{A_4}{a_2^2}\frac{1}{|x|}\delta\Big(t-\frac{|x|}{a_2}\Big)\Big] \\
&\quad +\delta_{jj}(1-\delta_{k4})\Big[\frac{A_5}{a_1^2}\frac{x_jx_k}{|x|^3}\delta\Big(t-\frac{|x|}{a_1}\Big) \\
&\quad -\frac{A_6}{a_2^2}\frac{x_jx_k}{|x|^3}\delta\Big(t-\frac{|x|}{a_2}\Big)+\frac{B}{b^2}\frac{x_jx_k}{|x|^3}\delta\Big(t-\frac{|x|}{b}\Big)\Big] \\
&\quad -\delta_{jj}(1-\delta_{k4})\Big[A_7t\Big(\frac{\delta_{jk}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\Big[\varepsilon\Big(t-\frac{|x|}{a_1}\Big)-\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\Big] \\
&\quad +A_8t\Big(\frac{\delta_{jk}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\Big[\varepsilon\Big(t-\frac{|x|}{b}\Big)-\varepsilon\Big(t-\frac{|x|}{a_1}\Big)\Big] \\
&\quad +A_9t\Big(\frac{\delta_{jk}}{|x|^3}-\frac{3x_jx_k}{|x|^5}\Big)\Big[\varepsilon\Big(t-\frac{|x|}{b}\Big)-\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\Big]\Big] \\
&\quad +A_{10}\frac{x_j}{|x|^3}\Big[\varepsilon\Big(t-\frac{|x|}{a_1}\Big)-\varepsilon\Big(t-\frac{|x|}{a_2}\Big)\Big]\delta_{4j}(1-\delta_{kj}) \\
&\quad +A_{10}\,\frac{x_j}{|x|^2}\Big[\delta\Big(t-\frac{|x|}{a_1}\Big)-\delta\Big(t-\frac{|x|}{a_2}\Big)\Big]\delta_{4j}(1-\delta_{kj})\Big\}, \\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad j,k=1,2,3,4,
\end{aligned}$$

where δ_{jk} denotes the Kronecker symbol, $\delta(\cdot)$ is Dirac's distribution, $\varepsilon(\cdot)$ is Heaviside's function

$$\varepsilon(t)=\begin{cases} 1 & \text{for } t>0, \\ 0 & \text{for } t<0, \end{cases}$$

and a_1, a_2, b, $A_1, \ldots, A_{10}$, B are constants.

The aim of this paper is to prove the L^∞-L^1-time decay estimates for the solution of the problem (0.1)–(0.3) using the formula (0.4), then to prove the L^2-L^2-time decay estimates for this solution applying some theorems of the theory of symmetric first order hyperbolic systems (cf. Yu. V. Egorov [3], p. 320–333) and finally the L^p-L^q-time decay estimates using interpolation inequalities in Sobolev spaces (cf. [10], [7], [14]).

Such L^p-L^q-time decay estimates play an important role in the proof of global (in time) existence of solution to the Cauchy problem for nonlinear wave equations (cf. [5], [7]).

Notation: Besides other standard notation we use the symbol $L^p_{,m}(\mathbb{R}^3) = W^{m,p}(\mathbb{R}^3)$ $(1 \le p \le \infty,\ m \in \mathbb{N} \cup \{0\})$ for the well-known Sobolev spaces with norm $\|\cdot\|_{L^p_{,m}(\mathbb{R}^3)} = \|\cdot\|_{W^{m,p}(\mathbb{R}^3)}$ (cf. [1], [11]); $W^{0,p}(\mathbb{R}^3) = L^p(\mathbb{R}^3)$ with norm $\|\cdot\|_{L^p(\mathbb{R}^3)}$.

We also write

$$\nabla f = (\partial_t f, \partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$$

for the space-time gradient of a function f, and

$$Df = (\partial_{x_1} f, \partial_{x_2} f, \partial_{x_3} f)$$

for the space gradient of f.

$[s]$ denotes the smallest integer larger than or equal to s for $s \in \mathbb{R}$.

1. The L^∞-L^1-time decay estimates. We shall prove the following theorem:

THEOREM 1.1 (L^∞-L^1-time decay estimates). *Let the Cauchy data u^0, u^1, T^0, T^1 be functions vanishing at infinity. Moreover, let*

$$(u^1, Du^0, T^1, DT^0) \in L^1_{,3}(\mathbb{R}^3)\,.$$

Then the solution (u, T) of the problem (0.1)–(0.3) *given by the formula* (0.4) *satisfies the following estimates*:

$$\|(u(t,\cdot), T(t,\cdot))\|_{L^\infty(\mathbb{R}^3)} \le C(1+t)^{-1} \|(u^1, Du^0, T^1, DT^0)\|_{L^1_{,3}(\mathbb{R}^3)}, \tag{1.1}$$

$$\|(\nabla u(t,\cdot), \nabla T(t,\cdot))\|_{L^\infty(\mathbb{R}^3)} \le C(1+t)^{-1} \|(u^1, Du^0, T^1, DT^0)\|_{L^1_{,3}(\mathbb{R}^3)} \tag{1.2}$$

for $t \ge 0$, where C is a constant independent of u^0, u^1, T^0, T^1 and t.

Proof. We prove (1.2). The proof of (1.1) runs in the same way. Writing the solution $U(t,x) = (u(t,x), T(t,x))$ given by the formula (0.4) in the form

$$U_j(t,x) = \sum_{k=1}^{4} H_{jk}(t,\cdot) * \overline{g}^k(x) + \sum_{k=1}^{4} \partial_t H_{jk}(t,\cdot) * \widetilde{h}^k(x)\,, \tag{1.3}$$

$j = 1,2,3,4$, where $U_j(t,x) = u_j(t,x)$, $j = 1,2,3$, $U_4(t,x) = T(t,x)$, and differentiating (1.3) with respect to t and x_l (for $l = 1,2,3$) we get

$$\begin{aligned}(1.4)\qquad \partial_t U_j(t,x) = {} & \sum_{k=1}^{4} \partial_t H_{jk}(t,\cdot) * \overline{g}^k(x) \\ & + \sum_{k=1}^{4}\sum_{l=1}^{3} \left[\frac{\mu}{\rho}\partial_l H_{jk}(t,\cdot) + \frac{\lambda+\mu}{\rho}\partial_j H_{lk}(t,\cdot)\right] * \widetilde{h}_l^k(x) \\ & - \sum_{k=1}^{4} \frac{\beta}{\rho}\partial_t H_{jk}(t,\cdot) * \widetilde{h}_j^k(x), \qquad j = 1,2,3\,,\end{aligned}$$

$$\begin{aligned}(1.5)\qquad \partial_t U_4(t,x) = {} & \sum_{k=1}^{4} \partial_t H_{4k}(t,\cdot) * \overline{g}^k(x) \\ & + \sum_{k=1}^{4}\sum_{l=1}^{3} \frac{k}{\rho}\partial_l H_{4k}(t,\cdot) * \widetilde{h}_l^k(x) \\ & - \sum_{k=1}^{4}\sum_{l=1}^{3} \frac{\beta}{\rho}\partial_t H_{lk}(t,\cdot) * \widetilde{h}_l^k(x)\,,\end{aligned}$$

$$(1.6)\qquad \partial_l U_j(t,x) = \sum_{k=1}^{4} \partial_l H_{jk}(t,\cdot) * \overline{g}^k(x) + \sum_{k=1}^{4} \partial_t H_{jk}(t,\cdot) * \widetilde{h}_l^k(x)\,,$$

$l = 1,2,3$, $j = 1,2,3,4$, $\widetilde{h}_m^k = \partial_{x_m}\widetilde{h}^k$, $m = 1,2,3$. We can write (1.4)–(1.6) in vector form as follows:

$$(1.7)\qquad V(t,x) = R(t,\cdot) * V^0(x)$$

where

$$(1.8)\qquad V(t,x) = (\nabla u, \nabla T), \qquad V^0(x) = (u^1, Du^0, T^1, DT^0)$$

and $R(t,x)$ is a 16×16 matrix with elements which are linear combinations of the terms $\partial_l H_{jk}(t,x)$ and $\partial_t H_{jk}(t,x)$ (cf. (1.3)–(1.6)). From (1.3)–(1.6) and (1.7) it follows that in order to prove the estimate (1.2) it is sufficient to prove the following estimates:

$$(1.9)\qquad \|\partial_t H_{jk}(t,\cdot) * f\|_{L^\infty(\mathbb{R}^3)} \le C(1+t)^{-1}\|f\|_{L^1_{,3}(\mathbb{R}^3)}\,,$$

$$(1.10)\qquad \|\partial_l H_{jk}(t,\cdot) * f\|_{L^\infty(\mathbb{R}^3)} \le C(1+t)^{-1}\|f\|_{L^1_{,3}(\mathbb{R}^3)}$$

for $j,k = 1,2,3,4$ and any scalar function $f(x)$ satisfying the assumptions of Theorem 1.1. Taking into account the form of the matrix $H(t,x)$ (cf. (0.5)) we get

$$(1.11)\qquad H_{jk}(t,\cdot) * f(x)$$

$$
\begin{aligned}
&= (32\pi 7!\,\rho^4\tau)^{-1}\Big\{\delta_{jk}(1-\delta_{k4})\Big[\frac{A_1}{a_1}\int\limits_{|z|=a_1}\frac{t}{a_1}f(x+tz)\,dS_z \\
&\quad-\frac{A_2}{a_2}\int\limits_{|z|=a_2} t\frac{1}{a_2}f(x+tz)\,dS_z-\frac{B}{b}\int\limits_{|z|=b} t\frac{1}{b}f(x+tz)\,dS_z\Big] \\
&\quad+\delta_{jk}\delta_{k4}\Big[\frac{A_3}{a_1}\int\limits_{|z|=a_1} t\frac{1}{a_1}f(x+tz)\,dS_z-\frac{A_4}{a_2}\int\limits_{|z|=a_2} t\frac{1}{a_2}f(x+tz)\,dS_z\Big] \\
&\quad+\delta_{jj}(1-\delta_{k4})\Big[\frac{A_5}{a_1}\int\limits_{|z|=a_1} t\frac{z_jz_k}{a_1^3}f(x+tz)\,dS_z \\
&\quad-\frac{A_6}{a_2}\int\limits_{|z|=a_2} t\frac{z_jz_k}{a_2^3}f(x+tz)\,dS_z+\frac{B}{b}\int\limits_{|z|=b} t\frac{z_jz_k}{b^3}f(x+tz)\,dS_z\Big] \\
&\quad-\delta_{jj}(1-\delta_{k4})\Big[A_7\int\limits_{a_1\le|z|\le a_2} t\Big(\frac{\delta_{jk}}{|z|^3}-\frac{3z_jz_k}{|z|^5}\Big)f(x+tz)\,dz \\
&\quad+A_8\int\limits_{b\le|z|\le a_1} t\Big(\frac{\delta_{jk}}{|z|^3}-\frac{3z_jz_k}{|z|^5}\Big)f(x+tz)\,dz \\
&\quad+A_9\int\limits_{b\le|z|\le a_2} t\Big(\frac{\delta_{jk}}{|z|^3}-\frac{3z_jz_k}{|z|^5}\Big)f(x+tz)\,dz\Big] \\
&\quad+\delta_{4j}(1-\delta_{kj})A_{10}\int\limits_{a_1\le|z|\le a_2}\frac{z_j}{|z|^3}f(x+tz)\,dz \\
&\quad+\delta_{4j}(1-\delta_{kj})A_{10}\Big[\int\limits_{|z|=a_1} t\frac{z_j}{a_1^2}f(x+tz)\,dS_z \\
&\quad-\int\limits_{|z|=a_2} t\frac{z_j}{a_2^2}f(x+tz)\,dS_z\Big]\Big\}
\end{aligned}
$$

where dS_z is the area element of the sphere $|z| = a_j$, $j = 1,2$, or of $|z| = b$.

For simplicity we consider two typical integrals occurring on the right hand side of (1.11) (other integrals in (1.11) are estimated similarly):

$$I^1 = \int\limits_{|y|=b} tf(x+ty)\,dS_y\,, \tag{1.12}$$

$$I^2 = \int\limits_{b\le|y|\le a} t\frac{f(x+ty)}{|y|^3}\,dy\,. \tag{1.13}$$

Differentiating the integrals I^1 and I^2 with respect to t and x_l $(l = 1,2,3)$

we obtain [1]

$$I_t^1 = \int\limits_{|y|=b} f(x+ty)\,dS_y + \int\limits_{|y|=b} t\partial_t f(x+ty)\,dS_y\,, \tag{1.14}$$

$$I_t^2 = \int\limits_{b\le|y|\le a} \frac{f(x+ty)}{|y|^3}\,dy + \int\limits_{b\le|y|\le a} \frac{t}{|y|^3}\partial_t f(x+ty)\,dy\,, \tag{1.15}$$

$$I_{x_l}^1 = \int\limits_{|y|=b} t\partial_{x_l} f(x+ty)\,dS_y\,, \tag{1.16}$$

$$I_{x_l}^2 = \int\limits_{b\le|y|\le a} \frac{t}{|y|^3}\partial_{x_l} f(x+ty)\,dy\,. \tag{1.17}$$

Following S. Klainerman (cf. [5], pp. 53–59) we get

$$\begin{aligned} f(x+ty) &= -\int\limits_t^\infty \partial_s f(x+sy)\,ds = \int\limits_t^\infty (s-t)\partial_s^2 f(x+sy)\,ds \\ &= -\tfrac{1}{2}\int\limits_t^\infty (s-t)^2\partial_s^3 f(x+sy)\,ds\,, \end{aligned} \tag{1.18}$$

$$\partial_t f(x+ty) = -\int\limits_t^\infty \partial_s^2 f(x+sy)\,ds = \int\limits_t^\infty (s-t)\partial_s^3 f(x+sy)\,ds\,. \tag{1.19}$$

In view of (1.18), (1.19) we have

$$\begin{aligned} I_t^1 &= \int\limits_{|y|=b}\int\limits_t^\infty (s-t)\partial_s^2 f(x+sy)\,ds\,dS_y \\ &\quad - \int\limits_{|y|=b} t\int\limits_t^\infty \partial_s^2 f(x+sy)\,ds\,dS_y \\ &= t^{-1}\Big[\int\limits_{|y|=b}\int\limits_t^\infty t(s-t)\partial_s^2 f(x+sy)\,ds\,dS_y \\ &\quad - \int\limits_{|y|=b}\int\limits_t^\infty t^2\partial_s^2 f(x+sy)\,ds\,dS_y\Big] \end{aligned} \tag{1.20}$$

for $t>0$.

Taking into account that

$$|\partial_s^2 f(x+sy)| = \Big|\sum_{j,k=1}^3 \partial^2_{x_jx_k} f(x+sy)y_jy_k\Big| \le \tfrac{1}{2}b^2|D_x^2 f(x+sy)| \tag{1.21}$$

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[1] We use the notation $\partial_t I^j = I_t^j$, $\partial_l I^j = I_{x_l}^j$, $j=1,2$.

for $|y| = b$ and $t(s-t) \le s^2$, $t^2 \le s^2$ for $0 \le t \le s \le \infty$ we get

$$|I_t^1| \le t^{-1} b^2 \int\limits_{|y|=b} \int\limits_t^\infty s^2 |D_x^2 f(x+sy)|\, ds\, dS_y\,. \tag{1.22}$$

Using the spherical coordinates we have

$$|I_t^1| \le bt^{-1} \|D_x^2 f\|_{L^1(\mathbb{R}^3)} \quad \text{for } t > 0\,. \tag{1.23}$$

Acting in the same way we get

$$I_{x_l}^1 = -\int\limits_{|y|=b} t \int\limits_t^\infty \partial_s[\partial_{x_l} f(x+sy)]\, ds\, dS_y\,. \tag{1.24}$$

Since

$$|\partial_s[\partial_{x_l} f(x+sy)]| = \Big|\sum_{j=1}^3 \partial^2_{x_l x_j} f(x+sy) y_j\Big| \le b|D_x^2 f(x+sy)| \quad \text{for } |y| = b$$

we have

$$\begin{aligned} |I_{x_l}^1| &\le t^{-1} b \int\limits_{|y|=b} \int\limits_t^\infty s^2 |D_x^2 f(x+sy)\, ds|\, dS_y \\ &\le t^{-1} \|D_x^2 f\|_{L^1(\mathbb{R}^3)} \quad \text{for } t > 0\,. \end{aligned} \tag{1.25}$$

Similarly

$$\begin{aligned} |I_t^2| &\le b^{-3}\Big[\int\limits_{b\le|y|\le a} |f(x+ty)|\, dy + \int\limits_{b\le|y|\le a} t|\partial_t f(x+ty)|\, dy\Big] \\ &\le b^{-3}\Big[\int\limits_{b\le|y|\le a} |f(x+ty)|\, dy + a \int\limits_{b\le|y|\le a} t|D_x^1 f(x+ty)|\, dy\Big]\,, \end{aligned} \tag{1.26}$$

$$|I_{x_l}^2| \le b^{-3} \int\limits_{b\le|y|\le a} t|D_x^1 f(x+ty)|\, dy\,. \tag{1.27}$$

Changing the variable ty to z in the above integrals we derive

$$\begin{aligned} |I_t^2| &\le b^{-3}\Bigg[\frac{1}{t^3} \int\limits_{bt\le|z|\le at} f(x+z)\, dz \\ &\qquad + \frac{a}{t^2} \int\limits_{bt\le|z|\le at} |D_x^1 f(x+z)|\, dz\Bigg] \\ &\le b^{-3}\Bigg[\frac{1}{t^3}\|f\|_{L^1(\mathbb{R}^3)} + \frac{a}{t^2}\|D_x^1 f\|_{L^1(\mathbb{R}^3)}\Bigg]\,, \end{aligned} \tag{1.28}$$

$$|I_{x_l}^2| \le \frac{b^{-3}}{t^2} \int\limits_{bt\le|z|\le at} |D_x^1 f(x+z)|\, dz \le \frac{b^{-3}}{t^2}\|D_x^1 f\|_{L^1(\mathbb{R}^3)}\,. \tag{1.29}$$

Noting that $1/t^3 \le 1/t^2 \le 1/t$ for $t \ge 1$ we get

$$(1.30) \qquad |I_t^2| + |I_{x_l}^2| \le Ct^{-1} \|D_x^1 f\|_{L^1(\mathbb{R}^3)} \quad \text{for } t \ge 1\,.$$

From (1.23), (1.25), (1.30) we obtain

$$(1.31) \qquad \|\nabla H_{jk}(t,\cdot) * f(\cdot)\|_{L^\infty(\mathbb{R}^3)} \le Ct^{-1} \|f\|_{L^1_{,2}(\mathbb{R}^3)}$$

for $t \ge 0$ and $j, k = 1, 2, 3, 4$.

In order to obtain an estimate analogous to (1.30) for $0 \le t \le 1$ we proceed as above expressing the integrals I_t^1 and I_t^2 (cf. (1.18), (1.19)) in the following form:

$$(1.32) \qquad I_t^1 = -\frac{1}{2} \int\limits_{|y|=b} \int\limits_t^\infty (s-t)^2 \partial_s^3 f(x+sy)\, ds\, dS_y + \int\limits_{|y|=b} t \int\limits_t^\infty (s-t) \partial_s^3 f(x+sy)\, ds\, dS_y\,,$$

$$(1.33) \qquad I_{x_l}^1 = \int\limits_{|y|=b} t \int\limits_t^\infty (s-t) \partial_s^2 [\partial_{x_l} f(x+sy)]\, ds\, dS_y\,.$$

After some calculations we get

$$(1.34) \qquad |I_t^1| + |I_{x_l}^1| \le C\|f\|_{L^1_{,3}(\mathbb{R}^3)} \quad \text{for } t \ge 0\,.$$

It is easy to see that for $0 \le t \le 1$

$$(1.35) \qquad \Big| \int\limits_{b\le|y|\le a} \frac{f(x+ty)}{|y|^3}\, dy \Big| \le b^{-3} \int\limits_{b\le|y|\le a} |f(x+ty)|\, dy \le b^{-3} \|f\|_{L^1(\mathbb{R}^3)}\,,$$

$$(1.36) \qquad \Big| \int\limits_{b\le|y|\le a} t \frac{\partial_t f(x+ty)}{|y|^3}\, dy \Big| \le b^{-3} \int\limits_{b\le|y|\le a} \Big| \sum_{j=1}^3 \partial_{x_j} f(x+ty) y_j \Big|\, dy \le b^{-3} a \|D_x^1 f\|_{L^1(\mathbb{R}^3)}\,,$$

$$(1.37) \qquad \Big| \int\limits_{b\le|y|\le a} \frac{t}{|y|^3} \partial_{x_l} f(x+ty)\, dy \Big| \le b^{-3} \int\limits_{b\le|y|\le a} |\partial_{x_l} f(x+ty)|\, dy \le b^{-3} \|D_x^1 f\|_{L^1(\mathbb{R}^3)}\,.$$

Hence

$$|I_t^2| + |I_{x_l}^2| \le C\|f\|_{L^1_{,1}(\mathbb{R}^3)} \quad \text{for } 0 \le t \le 1\,. \tag{1.38}$$

Thus from (1.38) and (1.34) we obtain

$$\|\nabla H_{jk}(t,\cdot) * f(\cdot)\|_{L^\infty(\mathbb{R}^3)} \le C\|f\|_{L^1_{,3}(\mathbb{R}^3)} \quad \text{for } 0 \le t \le 1\,. \tag{1.39}$$

Now, in view of $1 \le 2(1+t)^{-1}$ for $0 \le t \le 1$ and $t^{-1} \le 2(1+t)^{-1}$ for $t \ge 1$ and taking into account (1.31), (1.39) we conclude that

$$\|\nabla H_{jk}(t,\cdot) * f(\cdot)\|_{L^\infty(\mathbb{R}^3)} \le C(1+t)^{-1}\|f\|_{L^1_{,3}(\mathbb{R}^3)} \quad \text{for } t \ge 0\,. \tag{1.40}$$ ■

2. The L^2-L^2-time decay estimates. We derive the L^2-L^2-time decay estimates for solution of the Cauchy problem (0.1)–(0.3). More precisely, we formulate the following theorem:

THEOREM 2.1 (L^2-L^2-time decay estimates). *Let the Cauchy data u^0, u^1, T^0, T^1 be functions vanishing at infinity. Moreover, let*

$$(u^1, Du^0, T^1, DT^0) \in L^2(\mathbb{R}^3)\,.$$

Then the solution (u,T) of the problem (0.1)–(0.3) *given by the formula* (0.4) *satisfies the following estimates*:

$$\|(u(t,\cdot), T(t,\cdot))\|_{L^2(\mathbb{R}^3)} \le C\|(u^1, Du^0, T^1, DT^0)\|_{L^2(\mathbb{R}^3)} \quad \textit{for } t \ge 0\,, \tag{2.1}$$

$$\|(\nabla u(t,\cdot), \nabla T(t,\cdot))\|_{L^2(\mathbb{R}^3)} \le C\|(u^1, Du^0, T^1, DT^0)\|_{L^2(\mathbb{R}^3)} \quad \textit{for } t \ge 0\,, \tag{2.2}$$

where C is constant independent of u^0, u^1, T^0, T^1 and t.

Sketch of proof. Following Yu. V. Egorov (cf. [3], pp. 320–322, 326–333) we reduce the Cauchy problem (0.1)–(0.3) to an equivalent Cauchy problem for a linear symmetric hyperbolic system of first order. Next, applying the existence and uniqueness theorems (cf. [3], Theorem 3.2, p. 329) we obtain the estimates (2.1), (2.2).

3. The L^p-L^q-time decay estimates. In this section we express the L^p-L^q-time decay estimates for solutions of the Cauchy problem (0.1)–(0.3) in terms of their gradient.

We consider the operator Π_* defined as follows:

$$\Pi_* f(x) = R(t,\cdot) * f(x) \quad \text{for any function } f(x) \tag{3.1}$$

satisfying the assumptions of Theorems 1.1 and 2.1, where $R(t,x)$ is defined by (1.7).

From Theorems 1.1 and 2.1 it follows that

$$\varPi^0_* : L^1_{,3}(\mathbb{R}^3) \longrightarrow L^\infty(\mathbb{R}^3)\,, \quad \|\varPi^0_*\| \le C(1+t)^{-1}\,, \tag{3.2}$$

$$\varPi^1_* : L^2(\mathbb{R}^3) \longrightarrow L^2(\mathbb{R}^3)\,, \quad \|\varPi^1_*\| \le C\,. \tag{3.3}$$

By interpolation (cf. J. Shatah [10], S. Klainerman and G. Ponce [7]) we have

$$\begin{gathered} \varPi^\theta_* : \; [L^1_{,3}, L^2]_\theta \longrightarrow [L^\infty, L^2]_\theta\,, \\ \|\varPi^\theta_*\| = \|\varPi^0_*\|^{1-\theta}\|\varPi^1_*\|^\theta \quad \text{with } 0 \le \theta \le 1\,, \end{gathered} \tag{3.4}$$

where $[X,Y]_\theta$ $(0 \le \theta \le 1)$ denotes the complex interpolation space (cf. [8], [14]) with respect to X and Y.

In order to obtain the L^p-L^q-time decay estimates (where $q = 2\alpha+2$, $p = (2\alpha+2)/(2\alpha+1)$, $1/p + 1/q = 1$, α is a nonnegative integer) we notice that for $\theta = 1/(\alpha+1)$

$$[L^1_{,3}, L^2]_{1/(\alpha+1)} = L^p_{,s_0} \quad \text{where } s_0 = \left[\frac{3\alpha}{\alpha+1}\right],\ p = \frac{2\alpha+2}{2\alpha+1}\,, \tag{3.5}$$

$$[L^\infty, L^2]_{1/(\alpha+1)} = L^{2\alpha+2}\,. \tag{3.6}$$

Hence, we have

$$\varPi^{(\alpha)}_* : \; L^p_{,s_0}(\mathbb{R}^3) \longrightarrow L^{2\alpha+2}(\mathbb{R}^3)\,, \tag{3.7}$$

$$\|\varPi^{(\alpha)}_*\| = \|\varPi^0_*\|^{1-1/(\alpha+1)}\|\varPi^1_*\|^{1/(\alpha+1)} \le C(1+t)^{-\alpha/(\alpha+1)}\,. \tag{3.8}$$

So, we have proved the following theorem:

Theorem 3.1 (L^p-L^q-time decay estimates). *Let the Cauchy data* u^0, u^1, T^0, T^1 *be functions vanishing at infinity. Moreover*, *let*

$$(u^1, Du^0, T^1, DT^0) \in L^p_{,s_0}(\mathbb{R}^3) \quad \text{for } p = \frac{2\alpha+2}{2\alpha+1}\,,$$

$s_0 = [3\alpha/(\alpha+1)]$ *and* α *a nonnegative integer. Then the solution of the problem* (0.1)–(0.3) *given by the formula* (0.4) *satisfies the following estimates*:

$$\begin{aligned} \|(\nabla u(t,\cdot), \nabla T(t,\cdot))\|_{L^{2\alpha+2}(\mathbb{R}^3)} &\le C(1+t)^{-\alpha/(\alpha+1)} \\ &\quad \times \|(u^1, Du^0, T^1, DT^0)\|_{L^p_{,s_0}(\mathbb{R}^3)} \quad \text{for } t \ge 0 \end{aligned} \tag{3.9}$$

where C *is a constant independent of* u^0, u^1, T^0, T^1 *and* t.

Remark 3.1. In a subsequent paper, we shall apply Theorem 3.1 in the proof of global (in time) existence of solution of the Cauchy problem for the nonlinear hyperbolic system of partial differential equations describing a thermoelastic medium.

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