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A class of analytic functions defined by Ruscheweyh derivative

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Abstract. The function $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k}$ ($p \in \mathbb{N} = \{1, 2, 3, \ldots\}$) analytic in the unit disk E is said to be in the class $K_{n,p}(h)$ if

$$\frac{D^{n+p} f}{D^{n+p-1} f} \prec h\,, \quad \text{where} \quad D^{n+p-1} f = \frac{z^p}{(1-z)^{p+n}} * f$$

and h is convex univalent in E with $h(0) = 1$. We study the class $K_{n,p}(h)$ and investigate whether the inclusion relation $K_{n+1,p}(h) \subseteq K_{n,p}(h)$ holds for $p > 1$. Some coefficient estimates for the class are also obtained. The class $A_{n,p}(a, h)$ of functions satisfying the condition

$$a \frac{D^{n+p} f}{D^{n+p-1} f} + (1-a) \frac{D^{n+p+1} f}{D^{n+p} f} \prec h$$

is also studied.

Introduction. Let $A(p)$ denote the class of functions of the form

$$f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k} z^{p+k} \qquad (p \in \mathbb{N} = \{1, 2, 3, \ldots\}) \tag{1}$$

which are analytic in the unit disk $E = \{z : |z| < 1\}$. We denote by $f * g(z)$ the Hadamard product of two functions $f(z)$ and $g(z)$ in $A(p)$.

Following Goel and Sohi [2] we put

$$D^{n+p-1} f(z) = \frac{z^p}{(1-z)^{n+p}} * f(z) \qquad (n > -p) \tag{2}$$

for the $(n + p - 1)$th order Ruscheweyh derivative of $f(z) \in A(p)$. Let h be convex univalent in E, with $h(0) = 1$.

DEFINITION 1. We say that a function $f(z) \in A(p)$ for which

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$D^{n+p-1}f(z) \neq 0$, $0 < |z| < 1$, is in $K_{n,p}(h)$ if and only if

$$\frac{D^{n+p}f}{D^{n+p-1}f} \prec h\,. \tag{3}$$

If we take $h(z) = 1/(1+z)$, then (3) reduces to $\operatorname{Re}(D^{n+p}f/D^{n+p-1}f) > \frac{1}{2}$ and the class $K_{n,p}(1/(1+z))$ reduces to the class K_{n+p-1} in the notation employed in [2] for $n+p \in \mathbb{N}$ and $p \in \mathbb{N}$. Further, for $p = 1$ this class $K_{n,1}$ reduces to the class K_n studied by Ruscheweyh [3] who proved that $K_n \subset K_{n-1}$, $n \in \mathbb{N}$.

In [3] it is proved that $K_{n+p} \subset K_{n+p-1}$. We are interested in investigating whether $K_{n+1,p}(h) \subseteq K_{n,p}(h)$ for an arbitrary h. We show that this is not true if $p > 1$, even for the choice of $h(z) = (1+Az)/(1+z)$, $0 \leq A < 1$.

Definition 2 [1]. Let β and γ be complex constants and let $h(z) = 1 + h_1(z) + \ldots$ be univalent in the unit disc E. The univalent function $q(z) = 1 + q_1(z) + \ldots$ analytic in E is said to be a *dominant* of the differential subordination

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \tag{4}$$

if and only if (4) implies that $p(z) \prec q(z)$ for all $p(z) = 1 + p_1 z + \ldots$ that are analytic in E. If $q(z) \prec \widetilde{q}(z)$ for all dominants $\widetilde{q}(z)$ of (4), then $q(z)$ is said to be the *best dominant* of (4).

We need the following theorems which provide a method for finding the best dominant for certain differential subordinations.

Theorem A [1]. *Let β and γ be complex constants, and let h be convex (univalent) in E, with $h(0) = 1$ and $\operatorname{Re}[\beta h(z) + \gamma] > 0$. If $p(z) = 1 + p_1 z + \ldots$ is analytic in E, then*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z) \Rightarrow p(z) \prec h(z)\,. \tag{5}$$

Theorem B [1]. *Let β and γ be complex constants, and let h be convex in E with $h(0) = 1$ and $\operatorname{Re}[\beta h(z) + \gamma] > 0$. Let $p(z) = 1 + p_1 z + \ldots$ be analytic in E, and let it satisfy the differential subordination*

$$p(z) + \frac{zp'(z)}{\beta p(z) + \gamma} \prec h(z)\,. \tag{6}$$

If the differential equation

$$q(z) + \frac{zp'(z)}{\beta q(z) + \gamma} = h(z)\,, \tag{7}$$

with $q(0) = 1$, has a univalent solution $q(z)$, then $p(z) \prec q(z) \prec h(z)$, and $q(z)$ is the best dominant of (6).

Remark 1 [1]. (i) The conclusion of Theorem B can be written in the form

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}\Rightarrow p(z)\prec q(z)\,.$$

(ii) The differential equation (7) has a formal solution given by

$$q(z)=\frac{zF'(z)}{F(z)}=\frac{\beta+\gamma}{\beta}\left[\frac{H(z)}{F(z)}\right]^{\beta}-\frac{\gamma}{\beta}\,, \tag{8}$$

where

$$F(z)=\left[\frac{\beta+\gamma}{z^{\gamma}}\int_0^z H^{\beta}(t)t^{\gamma-1}\,dt\right]^{1/\beta},$$

$$H(z)=z\exp\int_0^z\frac{h(t)-1}{t}\,dt\,.$$

Corollary 1 [1]. *Let $p(z)$ be analytic in E and let it satisfy the differential subordination*

$$p(z)+\frac{zp'(z)}{\beta p(z)+\gamma}\prec\frac{1-(1-2\delta)z}{1+z}\equiv h(z)\,,$$

with $\beta>0$ and $-\mathrm{Re}\,(\gamma/\beta)\le\delta<1$. Then the differential equation

$$q(z)+\frac{zq'(z)}{\beta q(z)+\gamma}=h(z)\,,\quad q(0)=1\,,$$

has a univalent solution $q(z)$. In addition, $p(z)\prec q(z)\prec h(z)$ and $q(z)$ is the best dominant of (8).

Finally, we study the class $A_{n,p}(a,h)$ of functions $f(z)\in A(p)$ satisfying the condition

$$a\frac{D^{n+p}f}{D^{n+p-1}f}+(1-a)\frac{D^{n+p+1}f}{D^{n+p}f}\prec h$$

for h univalent convex.

1. The classes $K_{n,p}(h)$

Theorem 1.1. *Let $f\in K_{n+1,p}(h)$, that is, $D^{n+p+1}f/D^{n+p}f\prec h$, $n+p>0$. Then*

$$\frac{D^{n+p}f}{D^{n+p-1}f}\prec K\quad \textit{where}\quad K=\frac{n+p+1}{n+p}h-\frac{1}{n+p}\,,$$

and for $h=(1+Az)/(1+z)$, $0\le A<1$, we have $D^{n+p}f/D^{n+p-1}f\prec q\prec K_1$

and q *is the best dominant given by*

$$q = \frac{z^{n+p}}{(n+p)(1+z)^{(1-A)(n+p+1)} \int\limits_0^z \frac{t^{n+p-1}\,dt}{(1+t)^{(1-A)(n+p+1)}}}\,, \tag{9}$$

where $K_1 = \dfrac{(n+p)(1+Az) - z(1-A)}{(n+p)(1+z)}$.

Proof. Set $g(z) = D^{n+p}f(z)/D^{n+p-1}f(z)$. Taking logarithmic derivatives and multiplying by z, we get

$$\frac{zg'(z)}{g(z)} = \frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} - \frac{z(D^{n+p-1}f(z))'}{D^{n+p-1}f(z)}\,.$$

Using the fact that

$$z(D^{n+p}f)' = (n+p+1)D^{n+p+1}f - (n+1)D^{n+p}f\,,$$

we obtain

$$\frac{zg'(z)}{(n+p)g(z)} + g(z) = \frac{n+p+1}{n+p} \cdot \frac{D^{n+p+1}f}{D^{n+p}f} - \frac{1}{n+p}\,.$$

This means that if $D^{n+p+1}f/D^{n+p}f \prec h$, then

$$\frac{zg'(z)}{(n+p)g(z)} + g(z) \prec \frac{n+p+1}{n+p}h(z) - \frac{1}{n+p} = K(z)\,.$$

Theorem A now implies that $g(z) \prec K(z)$ if $n+p > 0$ and $\operatorname{Re} K(z) > 0$, which will be true if $\operatorname{Re} h(z) > 1/(n+p+1)$. Next choose $h(z) = (1 + Az)/(1+z)$, $0 \le A < 1$. This choice of A is consistent with the condition on $\operatorname{Re} h$. Then the differential equation

$$\frac{zg'(z)}{(n+p)g(z)} + g(z) = K(z) \tag{10}$$

has a univalent solution $g(z) = q(z)$ by Corollary 1 and $g(z) \prec q(z) \prec K(z)$.

In the notation of Theorem B and Remark 1, we have

$$H(z) = z \exp \int\limits_0^z \{K(t) - 1\}t^{-1}\,dt\,,$$

which gives on substitution for $K(t)$ the following:

$$H(z) = z \exp \int\limits_0^z \left\{\frac{n+p+1}{n+p} \cdot \frac{1+At}{1+t} - \frac{1}{n+p} - 1\right\} t^{-1}\,dt\,.$$

On simplification we get

$$H(z) = \frac{z}{(1+z)^{(1-A)(n+p+1)/(n+p)}}\,, \tag{11}$$

$$F(z)=\left[\,(n+p)\int\limits_0^z \frac{t^{n+p}}{(1+t)^{(1-A)(n+p+1)}}\cdot\frac{1}{t}\,dt\right]^{1/(n+p)}. \tag{12}$$

From (11) and (12) we obtain $q(z)=[H(z)/F(z)]^{(n+p)}$. This leads to (9).

Corollary 1.1. *Let $f\in K_{n+1,p}(1/(1+z))$, that is $D^{n+p+1}f/D^{n+p}f\prec 1/(1+z)$. Then $D^{n+p}f/D^{n+p-1}f\prec 1/(1+z)$ or $f\in K_{n,p}(1/(1+z))$ so that*

$$K_{n+1,p}\left(\frac{1}{1+z}\right)\subset K_{n,p}\left(\frac{1}{1+z}\right),\qquad n+p\ge 0\,.$$

Proof. Now (11) becomes $H(z)=z/(1+z)^{(n+p+1)/(n+p)}$ and

$$F(z)=\left[(n+p)\int\limits_0^z \frac{t^{n+p}}{(1+t)^{(n+p+1)}}\cdot\frac{dt}{t}\right]^{1/(n+p)}=\frac{z}{1+z}\,,$$

$$q(z)=\left[\frac{H(z)}{F(z)}\right]^{(n+p)}=\frac{1}{1+z}\,.$$

Hence $D^{n+p}f/D^{n+p-1}f\prec 1/(1+z)$, that is, $f\in K_{n,p}(1/(1+z))$ or $\mathrm{Re}\,(D^{n+p}f/D^{n+p-1}f)>1/2$. This is the result obtained by Goel and Sohi [2].

In the above corollary put $p=1$; we then obtain the following:

Corollary 1.2. *Let $f\in K_{n+1}$ in Ruscheweyh's notation, that is, $D^{n+2}f(z)/D^{n+1}f(z)\prec 1/(1+z)$. Then $D^{n+1}f/D^nf\prec 1/(1+z)$ or $f\in K_n$ or equivalently* $\mathrm{Re}\,(D^{n+1}f/D^nf)>1/2$.

This is the same as Ruscheweyh's result [3], $K_{n+1}\subset K_n$.

Since

$$K_{n,p}\left(\frac{1}{1+z}\right)\subseteq K_{n-1,p}\left(\frac{1}{1+z}\right)\subseteq\ldots\subset K_{-(p-1),p}\left(\frac{1}{1+z}\right),\quad n+p\ge 0,$$

from Corollary 1.1 we obtain

Corollary 1.3. *Let $f\in K_{n,p}(1/(1+z))$, $n+p\ge 0$. Then $f\in K_{-(p-1),p}(1/(1+z))$, that is, $D^1f/D^0f=zf'/f\prec 1/(1+z)$, that is,* $\mathrm{Re}\,(zf'/f)>1/2$. *Such functions f of the form $f(z)=z^p+\sum_{k=1}^{\infty}a_{p+k}z^{p+k}$ are known to be p-valent* [4].

Now we proceed to investigate the case $A\ne 0$. In order that the best dominant q given by (9) may reduce to $(1+Az)/(1+z)$, we should have

$$\left[\frac{z}{(1+z)^{(1-A)(n+p+1)/(n+p)}}\right]^{n+p}=[F(z)]^{n+p}\frac{1+Az}{1+z}\,.$$

Taking derivative with respect to z we get

$$(13)\qquad [F(z)^{n+p}]' = \frac{(n+p)(1+Az)(1+z)^{n+p-1} - A(1+z)z^{n+p}}{(1+Az)^2(1+z)^{(1-A)(n+p+1)}} - \frac{[(n+p)(1-A)-A](1+Az)z^{n+p}}{(1+Az)^2(1+z)^{(1-A)(n+p+1)}}\,.$$

From (12) we get

$$(14)\qquad [F^{(n+p)}]' = \frac{(n+p)z^{n+p-1}}{(1+z)^{(1-A)(n+p+1)}}\,,$$

(13) and (14) must be identical; which on simplification gives the conditions $A = 0$ or $A = 1$. $A = 1$ forces h to be a constant. We rule out this possibility. Hence the best possible solution exists only when $A = 0$. Hence we conclude that $K_{n+1,p}(h)$ is not contained in $K_{n,p}(h)$ for $p > 1$, even for the choice of $h(z) = (1+Az)/(1+z)$.

Let $f \in K_{n,p}(h)$. Define

$$G(z) = z^p \left(\frac{D^{n+p-1}f(z)}{z^p}\right)^{p/(n+p)}\,.$$

Then $zG'/G = p(D^{n+p}f/D^{n+p-1}f)$. We observe that $f \in K_{n,p}(h)$ if and only if $(1/p)zG'/G \prec h$.

We now prove the following

Theorem 1.2. *Let $m, n \in \mathbb{N}_0$. Then $f \in K_{n,p}(h)$ if and only if*

$$g(z) = (m+p-1)!z^{1-m} \int\limits_0^z \int\limits_0^{x_{m+p-1}} \cdots \int\limits_0^{x_2} \left[\frac{1}{(n+p-1)!}(x_1^{n-1}f(x_1))^{(n+p-1)}\right]^{(m+p)/(n+p)} dx_1 \ldots dx_{m+p-1}$$

belongs to $K_{m,p}(h)$.

Proof. We have

$$\frac{g(z)z^{m-1}}{(m+p-1)!} = \int\limits_0^z \int\limits_0^{x_{m+p-1}} \cdots \int\limits_0^{x_2} \left[\frac{1}{(n+p-1)!}(x_1^{n-1}f(x_1))^{(n+p-1)}\right]^{(m+p)/(n+p)} dx_1 \ldots dx_{m+p-1}\,.$$

Differentiating $m+p-1$ times, we get

$$\left[\frac{g(z)z^{m-1}}{(m+p-1)!}\right]^{(m+p-1)} = \left[\frac{1}{(n+p-1)!}(z^{n-1}f(z))^{(n+p-1)}\right]^{(m+p)/(n+p)}\,.$$

Since $D^{n+p-1}f = z^p(z^{n-1}f)^{(n+p-1)}/(n+p-1)!$, we get

$$\frac{D^{m+p-1}g(z)}{z^p} = \left(\frac{D^{n+p-1}f}{z^p}\right)^{(m+p)/(n+p)} .$$

Set

$$G(z) = z^p \left(\frac{D^{m+p-1}g}{z^p}\right)^{p/(m+p)} = z^p \left(\frac{D^{n+p-1}f}{z^p}\right)^{p/(n+p)} .$$

As we have already observed we then have

$$\frac{zG'}{G} = p\left(\frac{D^{m+p}g}{D^{m+p-1}g}\right) = p\left(\frac{D^{n+p}f}{D^{n+p-1}f}\right) ,$$

which implies that

$$\frac{1}{p}\frac{zG'}{G} \prec h \Leftrightarrow g \in K_{m,p}(h) \Leftrightarrow f \in K_{n,p}(h) .$$

Coefficient estimates

THEOREM 1.3. *Let $f \in A(p)$ satisfy*

$$\operatorname{Re}\left\{\frac{zf'(z)}{pf(z)}\right\} > \frac{1}{2}, \qquad z \in E .$$

Then

$$|a_{p+k}| \le \frac{p(p+1)\dots(p+k-1)}{k!}, \qquad k = 1, 2, \dots \tag{15}$$

Proof. Let $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k}$ and

$$g(z) = 2\left(\frac{zf'(z)}{pf(z)} - \frac{1}{2}\right) . \tag{16}$$

Then $g(0) = 1$ and $\operatorname{Re} g(z) > 0$.

Writing $g(z) = 1 + \sum_{k=1}^{\infty} g_k z^k$, we note that $|g_k| \le 2$, $k = 1, 2, \dots$

From (16) we get

$$g(z) = \frac{2zf' - pf}{pf} .$$

Substituting for f, f' and g_k and simplifying we obtain

$$\Big(1 + \sum_{k=1}^{\infty} a_{p+k}z^k\Big)\Big(1 + \sum_{k=1}^{\infty} g_k z^k\Big) = \Big\{ 2 + \sum_{k=1}^{\infty} 2\frac{(p+k)}{p} a_{p+k}z^k\Big\} - \Big\{1 + \sum_{k=1}^{\infty} a_{p+k}z^k\Big\} .$$

Comparing the coefficients of z^n, we obtain

$$a_{p+n} + a_{p+n-1}g_1 + a_{p+n-2}g_2 + \dots + g_n = \left(1 + \frac{2n}{p}\right) a_{p+n} ,$$

$$a_{p+n} = \frac{p}{2n}[a_{p+n-1}g_1 + \ldots + g_n]\,.$$

The required coefficient estimate follows by induction, by using the fact $|g_k| \le 2$, $k = 1, 2, \ldots$

Theorem 1.4. *Let* $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k}$ *satisfy*

$$\operatorname{Re}\left\{\frac{D^{n+p}f}{D^{n+p-1}f}\right\} > \frac{1}{2}\,.$$

Then we have the sharp estimate

$$|a_{p+2} - a_{p+1}^2| \le (1 - |a_{p+1}|^2)/(n + p + 1)\,.$$

Proof. Since $\operatorname{Re}\{D^{n+p}f/D^{n+p-1}f\} > 1/2$, we can write $D^{n+p}f/D^{n+p-1}f = 1/(1+\omega(z))$, ω analytic in E, $|\omega(z)| \le 1$ for $z \in E$. Set $\omega(z) = \sum_{n=1}^{\infty} c_n z^n$. Using (2) we have

$$\frac{z^p + (n+p-1)a_{p+1}z^{p+1} + \frac{(n+p+1)(n+p+2)}{2!}a_{p+2}z^{p+2} + \ldots}{z^p + (n+p)a_{p+1}z^{p+1} + \frac{(n+p)(n+p+1)}{2!}a_{p+2}z^{p+2} + \ldots} = \frac{1}{1 + \sum_{n=1}^{\infty} c_n z^n}\,.$$

Simplifying and equating like powers of z we get

$$c_1 = -a_{p+1}\,, \tag{17}$$

$$c_2 + a_{p+1}c_1(n + p + 1) + a_{p+2}(n + p + 1) = 0\,. \tag{18}$$

From (17) and (18) we get

$$(n + p + 1)(a_{p+2} - a_{p+1}^2) = -c_2\,.$$

Using the well known fact $|c_2| \le 1 - |c_1|^2$, we obtain

$$|a_{p+2} - a_{p+1}^2| \le (1 - |a_{p+1}|^2)/(n + p + 1)\,.$$

For $p = 1$ this reduces to Theorem 3 of [3]. This fact increases the interest in estimates of the functional $|a_{n+p-1} - a_{p+1}^{k+p-2}|$ over the functions in the class $K_{n,p}(1/(1 + z))$. Such functions, as already observed, are p-valent.

Theorem 1.5. *Let* $f(z) = z^p + \sum_{k=1}^{\infty} a_{p+k}z^{p+k} \in K_{n,p}(1/(1 + z))$ *and*

$$\gamma(n, k, p) = \binom{(n+p)/p}{k-1}p^{k-1}\Big/\binom{n+p+k-2}{k-1}.$$

Then for $\mu \le \gamma(n, k, p)$, *we have the sharp estimate*

$$|a_{p+k-1} - \mu a_{p+1}^{k-1}| \le 1 - \mu\,, \quad k = 3, 4, \ldots \tag{19}$$

Proof. Let

$$f(z) = (n + p + 1)!z^{1-n} \int_0^z \int_0^{x_{n+p-1}} \cdots$$

$$\cdots \int_0^{x_2} \left[\frac{1}{(p-1)!} \left(\frac{g(x_1)}{x_1} \right)^{(p-1)} \right]^{(n+p)/p} dx_1 \ldots dx_{n+p-1} \,,$$

where $g(z) = z^p + \sum_{k=1}^{\infty} b_{p+k} z^{p+k}$. Using Theorem (1.2), from the above integral we find that $D^{n+p}f/D^{n+p-1}f = D^p g/D^{p-1}g$. Therefore, $\mathrm{Re}\,(D^{n+p}f/D^{n+p-1}f) > 1/2$ if and only if $\mathrm{Re}\,(D^p g/D^{p-1}g) > 1/2$. Since

$$\mathrm{Re}\,\left(\frac{D^p g}{D^{p-1} g} \right) = \mathrm{Re}\,\left(\frac{z(D^{p-1}g)'}{pD^{p-1}g} \right) ,$$

the hypothesis on f implies

$$\mathrm{Re}\left(\frac{z(D^{p-1}g)'}{pD^{p-1}g} \right) > \frac{1}{2} \,.$$

Applying Theorem 1.3 to the function $D^{p-1}g$, we conclude that $|b_{p+k}| \le 1$, $k = 1, 2, \ldots$ Further $a_{p+1} = b_{p+1}$. Put

$$\left[\left(\frac{g(z)}{z(p-1)!} \right)^{(p-1)} \right]^{(n+p)/p} = \sum_{j=0}^{\infty} c_{j+1} z^j \,,$$

so that

$$\left(1 + pb_{p+1} z + \frac{p(p+1)}{2!} b_{p+2} z^2 + \ldots \right)^{(n+p)/p} = \sum_{j=0}^{\infty} c_{j+1} z^j \,.$$

This yields

$$c_k = \binom{(n+p)/p}{k-1} p^{k-1} b_{p+1}^{k-1} + F(b_{p+1}, b_{p+2}, \ldots, b_{p+k-1}) \,. \tag{21}$$

Also from (20) we get

$$\begin{aligned} \frac{f(z) z^{n-1}}{(n+p-1)!} &= \frac{z^{n+p-1} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k+n-1}}{(n+p-1)!} \\ &= \int_0^z \int_0^{x_{n+p-1}} \cdots \int_0^{x_2} \sum_{j=0}^{\infty} c_{j+1} x_1^j \, dx_1 \ldots dx_{n+p-1} \,. \end{aligned}$$

This becomes on simplification

$$\frac{z^{p+n-1} + \sum_{k=1}^{\infty} a_{p+k} z^{p+k+n-1}}{(n+p-1)!} = \sum_{j=0}^{\infty} \frac{c_{j+1} z^{j+n+p-1}}{(j+1)(j+2)\ldots(j+n+p-1)} \,.$$

Equating coefficients of like powers we get

$$\frac{a_{p+k}}{(n+p-1)!} = \frac{c_{k+1}}{(k+1)(k+2)\ldots(k+n+p-1)} \,.$$

This yields

$$c_{k+1} = \binom{p+k+n-1}{n+p-1} a_{p+k} = \binom{p+k+n-1}{k} a_{p+k}\,. \tag{22}$$

Set $(1-z)^{-(n+p)} = \sum_{j=0}^{\infty} d_{j+1} z^j$ so that $d_k = \binom{n+p+k-2}{k-1}$. Set $\sigma = \mu\binom{n+p+k-2}{k-1}$. We now have from (21)

$$\begin{aligned} c_k - \sigma b_{p+1}^{k-1} &= F(b_{p+1}, b_{p+2}, \ldots, b_{p+k-1}) \\ &\quad + \left[\binom{(n+p)/p}{k-1} p^{k-1} - \sigma\right] b_{p+1}^{k-1}\,. \end{aligned} \tag{23}$$

Also it is easily seen that $d_k = c_k$ if $b_{p+1} = \ldots = b_{p+k-1} = 1$. So we write

$$\begin{aligned} \binom{n+p+k-2}{k-1} - \sigma &= d_k - \sigma \\ &= F(1,1,\ldots,1) + \left[\binom{(n+p)/p}{k-1} p^{k-1} - \sigma\right]. \end{aligned} \tag{24}$$

If $\sigma \le \binom{(n+p)/p}{k-1} p^{k-1}$, that is, if $\mu \le \binom{(n+p)/p}{k-1} p^{k-1} / \binom{n+p+k-2}{k-1}$, and $c_k = \binom{n+p+k-2}{k-1} a_{p+k-1}$, we have from (23) and (24)

$$\begin{aligned} \left| c_k - \binom{(n+p)/p}{k-1} p^{k-1} b_{p+1}^{k-1} \right| &= |F(b_{p+1}, b_{p+2}, \ldots, b_{p+k-1})| \\ &\le F(1,1,\ldots,1) = d_k - \binom{(n+p)/p}{k-1} p^{k-1}\,. \end{aligned}$$

(19) follows from this, since $b_{p+1} = a_{p+1}$. The coefficient bound in (19) is sharp for the function $f(z) = z^p/(1-z)$, which belongs to the class $K_{n,p}(1/(1+z))$, for all n. For $p = 1$, this reduces to Ruscheweyh's result ([3], Theorem 4).

Integral transform

For a function $f \in A(p)$ we consider the integral transform given by

$$g(z) = \frac{p+c}{z^c} \int_0^z t^{c-1} f(t)\, dt \qquad (n > -p,\ p \in \mathbb{N})\,.$$

We prove the following

Theorem 1.6. *Let $f \in A(p)$ be in the class $K_{n+1,p}(h)$ for $n > -p$ and $p \in \mathbb{N}$. Then $g(z) \in K_{n+1,p}(h)$, provided* $\operatorname{Re}\{(n+p+1)h - (n-c+1)\} > 0$.

Proof. By definition of $g(z)$,

$$zg'(z) + cg(z) = (p+c) f(z)\,,$$

and therefore

$$D^{n+p}(zg'(z)) + D^{n+p}(cg(z)) = D^{n+p}((p+c)f(z))\,. \tag{25}$$

By using $D^{n+p}(zg'(z)) = z(D^{n+p}g(z))'$ and

$$z(D^{n+p}g(z))' = (n+p+1)D^{n+p+1}g(z) - (n+1)D^{n+p}g(z) \tag{26}$$

equation (25) reduces to

$$(n+p+1)\frac{D^{n+p+1}g(z)}{D^{n+p}g(z)} - (n-c+1) = (p+c)\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\,.$$

Setting $D^{n+p+1}g(z)/D^{n+p}g(z) = R(z)$, this reduces to

$$R(z) - \frac{(n-c+1)}{(n+p+1)} = \frac{p+c}{n+p+1}\frac{D^{n+p}f(z)}{D^{n+p}g(z)}\,.$$

Taking logarithmic derivative and multiplying by z we get

$$\frac{zR'(z)}{R(z)-(n-c+1)/(n+p+1)} = \frac{z(D^{n+p}f(z))'}{D^{n+p}f(z)} - \frac{z(D^{n+p}g(z))'}{D^{n+p}g(z)}\,.$$

Using (26) and simplifying we get

$$\frac{zR'(z)}{(n+p+1)R(z)-(n-c+1)} + R(z) = \frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} \prec h(z)\,,$$

since $f \in K_{n+1,p}(h)$. Hence we conclude that $R(z) \prec h(z)$, that is, $D^{n+p+1}g(z)/D^{n+p}g(z) \prec h(z)$ if $\operatorname{Re}\{(n+p+1)h - (n-c+1)\} > 0$. This completes the proof.

Remark. For $p = 1$, Theorem 1.6 reduces to Theorem 5 in [3].

2. The classes $A_{n,p}(a,h)$

DEFINITION 2.1. Let h be convex univalent in E with $h(0) = 1$. The function $f(z) \in A(p)$ such that $D^{n+p-1}f(z) \neq 0$ and $D^{n+p}f(z) \neq 0$ for $0 < |z| < 1$ is said to be in $A_{n,p}(a,h)$ if

$$a\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + (1-a)\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} \prec h(z) \quad (a \text{ real})\,.$$

THEOREM 2.1. *Let $n \in \mathbb{N}_0$, $p \in \mathbb{N}$, $0 \le t \le 1$. Then*

$$A_{n,p}(a,h) \cap A_{n,p}(1,h) \subset A_{n,p}((a-1)t+1,h)\,.$$

Proof. If $f \in A_{n,p}(a,h)$ then

$$a\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + (1-a)\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} \prec h(z)\,.$$

Again, $f \in A_{n,p}(1,h)$ implies $D^{n+p}f(z)/D^{n+p-1}f(z) \prec h(z)$. Let

$$a\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} + (1-a)\frac{D^{n+p+1}f(z)}{D^{n+p}f(z)} = h_1(z)\,,$$

$$\frac{D^{n+p}f(z)}{D^{n+p-1}f(z)} = h_2(z)\,.$$

Then $h_1 \prec h$ and $h_2 \prec h$ so that $th_1 + (1-t)h_2 \prec h$. But

$$[1 + t(a-1)]\frac{D^{n+p}f}{D^{n+p-1}f} + (1-a)t\frac{D^{n+p+1}f}{D^{n+p}f} = th_1 + (1-t)h_2 \prec h\,.$$

Thus $f \in A_{n,p}((a-1)t+1, h)$.

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