ANNALES
POLONICI MATHEMATICI
55 (1991)

Anisotropic complex structure on the pseudo-Euclidean Hurwitz pairs

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Abstract. The concept of supercomplex structure is introduced in the pseudo-Euclidean Hurwitz pairs and its basic algebraic and geometric properties are described, e.g. a necessary and sufficient condition for the existence of such a structure is found.

1. Introduction. In 1923 A. Hurwitz [2] proved that any normed division algebra over $\mathbb{R}$ with unity is isomorphic to either $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ or $\mathbb{O}$, the real, complex, quaternion or octonion number algebras. In particular, Hurwitz showed that all the positive integers n and all the systems $c^k_{j\alpha} \in \mathbb{R}$, j, k, $\alpha = 1, \ldots, n$, such that the collection of bilinear forms $\eta_j := x_\alpha c^k_{j\alpha} y_k$ satisfies the condition

$$\sum_j \eta_j^2 = \Big(\sum_\alpha x_\alpha^2\Big)\Big(\sum_k y_k^2\Big)$$

are restricted to the cases $n = 1, 2, 4$ or 8.

The results of Hurwitz were the starting point for Ławrynowicz and Rembieliński to introduce the concept of the so-called Hurwitz pairs. They developed the theory obtaining many interesting results. Using the geometric concept of pseudo-Euclidean Hurwitz pairs, they gave their systematic classification in connection with real Clifford algebras. Moreover, they showed that the theory of Hurwitz pairs provided a convenient framework for some problems in mathematical physics (e.g. Dirac equation, Kałuża–Klein theories, spontaneous symmetry breaking and others).

We generalize the concept of supercomplex structure introduced by Ławrynowicz and Rembieliński [3] to pseudo-Euclidean Hurwitz pairs. We describe the basic algebraic and geometric properties of supercomplex structures and find a necessary and sufficient condition for their existence. This is the main result of our paper. We prove that if $O(n,k)$ denotes the orthogonal group preserving the norm $x_1^2 + \ldots + x_n^2 - x_{n+1}^2 - \ldots - x_{n+k}^2$ then

1991 *Mathematics Subject Classification*: Primary 15A63, 15A66.

a complex structure J ($J \in O(n,k)$, $J^2 = -I_{n+k}$, where I_{n+k} stands for the identity $(n+k)\times(n+k)$-matrix) exists if and only if n and k are even.

The concept of a supercomplex structure for Hurwitz pairs is strongly motivated by possible quantum-mechanical applications of anisotropic Hilbert spaces (see e.g. [5]).

2. Pseudo-Euclidean Hurwitz pairs and Clifford algebras. Let us recall fundamental notions and basic results from the theory of pseudo-Euclidean Hurwitz pairs. More details can be found in [3–5].

Consider two real vector spaces S and V, equipped with non-degenerate pseudo-Euclidean real scalar products $(\,,)_S$ and $(\,,)_V$ with standard properties (see e.g. [3]). For $f,g,h \in V$, $a,b,c \in S$ and $\alpha,\beta \in \mathbb{R}$ we assume that

$$
\begin{aligned}
&(a,b)_S \in \mathbb{R}\,, && (f,g)_V \in \mathbb{R}\,,\\
&(b,a)_S = (a,b)_S\,, && (g,f)_V = \delta(f,g)_V, \quad \delta = 1 \text{ or } -1\,,\\
&(\alpha a,b)_S = \alpha(a,b)_S\,, && (\alpha f,g)_V = \alpha(f,g)_V\,,\\
&(a,b+c)_S = (a,b)_S + (a,c)_S\,, && (f,g+h)_V = (f,g)_V + (f,h)_V\,.
\end{aligned}
\tag{1}
$$

In S and V we choose some bases (ε_α) and (e_j), respectively, with $\alpha = 1,\ldots,\dim S = p$; $j = 1,\ldots,\dim V = n$. We assume that $p \le n$. Set

$$
\eta \equiv [\eta_{\alpha\beta}] := [(\varepsilon_\alpha,\varepsilon_\beta)_S], \qquad \kappa \equiv [\kappa_{jk}] := [(e_j,e_k)_V]\,. \tag{2}
$$

By (1), we immediately get

$$
\begin{aligned}
&\det\eta \ne 0\,, \quad && \eta^{-1} \equiv [\eta^{\alpha\beta}]\,, \quad && \eta^T = \eta\,,\\
&\det\kappa \ne 0\,, \quad && \kappa^{-1} \equiv [\kappa^{jk}]\,, \quad && \kappa^T = \delta\kappa\,.
\end{aligned}
$$

Now, without any loss of generality, we can choose the bases (ε_α) in S and (e_j) in V so that

$$
\begin{aligned}
\eta &= \operatorname{diag}(\underbrace{1,\ldots,1}_{r},\underbrace{-1,\ldots,-1}_{s}), \qquad r+s = p\,,\\
\kappa &= \operatorname{diag}(\underbrace{1,\ldots,1}_{k},\underbrace{-1,\ldots,-1}_{l})\,, \qquad k+l = n\,,
\end{aligned}
\tag{3}
$$

and hence $\eta^{-1} = \eta$, $\kappa^{-1} = \kappa$.

Next, multiplication of elements of S by elements of V is defined as a mapping $F: S\times V \to V$ with the properties

(i) $F(a+b,f) = F(a,f) + F(b,f)$ and $F(a,f+g) = F(a,f) + F(a,g)$ for $f,g \in V$ and $a,b \in S$,

(ii) $(a,a)_S(f,g)_V = (F(a,f),F(a,g))_V$, the *generalized Hurwitz condition*,

(iii) there exists a unit element ε_0 in S for multiplication; $F(\varepsilon_0, f) = f$ for $f \in V$.

The product $a \cdot f := F(a, f)$ is uniquely determined by the multiplication scheme for base vectors:

$$F(\varepsilon_\alpha, e_j) = C^k_{j\alpha} e_k\,, \qquad \alpha = 1, \ldots, p;\ j, k = 1, \ldots, n\,. \tag{4}$$

Hereafter we shall require the irreducibility of the multiplication $F : S \times V \to V$, which means that it does not leave invariant proper subspaces of V. In such a case we shall call (V, S) a *pseudo-Euclidean Hurwitz pair*.

It turns out that the generalized Hurwitz condition is equivalent to the relations

$$C_\alpha C^+_\beta + C_\beta C^+_\alpha = 2\eta_{\alpha\beta} I_n\,, \qquad \alpha, \beta = 1, \ldots, p\,, \tag{5}$$

where we use matrix notation

$$C_\alpha := [C^k_{j\alpha}]\,, \qquad C^+_\alpha := \kappa C^T_\alpha \kappa^{-1}\,, \tag{6}$$

and I_n stands for the identity $n \times n$-matrix. On setting

$$C_\alpha = i\gamma_\alpha C_t\,, \qquad t \text{ fixed}, \ \alpha = 1, \ldots, p\,,\ \alpha \neq t\,, \tag{7}$$

where i denotes the imaginary unit, we arrive at the following system equivalent to (5):

$$\begin{cases} C_t C^+_t = \eta_{tt} I_n\,, \quad t \text{ fixed}, \\ \gamma^+_\alpha = -\gamma_\alpha\,, \quad \operatorname{Re}\gamma_\alpha = 0\,,\ \alpha = 1, \ldots, p,\ \alpha \neq t\,, \\ \gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha = 2\widehat{\eta}_{\alpha\beta} I_n\,, \quad \alpha, \beta = 1, \ldots, p,\ \alpha, \beta \neq t\,, \end{cases} \tag{8}$$

where

$$\widehat{\eta}_{\alpha\beta} := \eta_{\alpha\beta}/\eta_{tt}\,, \tag{9}$$

$[\eta_{\alpha\beta}]$ is the matrix (3). Clearly $\eta_{tt} = 1$ or -1.

From (8) it follows that $\{\gamma_\alpha\}$ are generators of a real Clifford algebra $C^{(r,s-1)}$ or $C^{(r-1,s)}$ with $(r, s-1)$ and $(r-1, s)$ determined by the signature of $\widehat{\eta} := [\widehat{\eta}_{\alpha\beta}]$ and by $r + s = p$. Thus, following Ławrynowicz and Rembieliński [3] we have

THEOREM 1. *The problem of classifying pseudo-Euclidean Hurwitz pairs (V, S) is equivalent to the classification problem for real Clifford algebras $C^{(r,s)}$ with generators $\{\gamma_\alpha\}$ imaginary and antisymmetric or symmetric according as $\alpha \le r$ or $\alpha > r$, given by the formulae*

$$i\gamma_\alpha C_t = C_\alpha\,, \qquad \alpha = 1, \ldots, r+s, \quad \alpha \neq t\,,$$
$$C_t C^+_t = \eta_{tt} I_n\,, \qquad t \text{ fixed}\,,$$

the matrices C_α being determined by (2), (5) *and* (6). *The relationship is given by the formulae* (8).

Corollary 1. *Without any loss of generality, in Theorem* 1 *we may set* $C_t = I_n$ *and* $t = r$, *so that* $\eta_{tt} = 1$ *and* $\widehat{\eta}_{\alpha\beta} = \eta_{\alpha\beta}$ *for* $\alpha, \beta \neq t$.

Lemma 1. *Pseudo-Euclidean Hurwitz pairs are of bidimension* (n, p), $n = \dim V$, $p = \dim S = r' + s' + 1$,

$$n = \begin{cases} 2^{[p/2-1/2]} & \text{for } r' - s' \equiv 6, 7, 0 \pmod 8, \\ 2^{[p/2+1/2]} & \text{for } r' - s' \equiv 1, 2, 3, 4, 5 \pmod 8, \end{cases}$$

where [ ] *stands for the function "entier".*

3. Supercomplex structure: an anisotropic complex structure involving a real Clifford algebra connected with the pseudo-Euclidean Hurwitz pairs

Definition. A *Hurwitz type vector space* $\boldsymbol{E}$ on (V, κ) is the p-dimensional subspace of the space $\mathrm{End}(V, \kappa)$ $(\dim \mathrm{End}\, V = \dim V)$ of endomorphisms of (V, κ), which consists of all endomorphisms E not leaving invariant proper subspaces of V, with the property

$$(Ef, Ef)_V = \|E\|^2 (f, f)_V \qquad \text{for } f \in V, E \in \boldsymbol{E}, \tag{10}$$

where $\|E\| := (\mathrm{Tr}\, E^T E)^{1/2}$, $E^T E$ being considered in an arbitrary matrix representation of E in an orthonormal basis (e_j) of V. We assume that $\boldsymbol{E}$ contains the identity endomorphism E_0.

Consider next a system (γ_α) of $p-1$ imaginary $n \times n$ matrices determined by the formulae

$$\gamma_\alpha \gamma_\beta + \gamma_\beta \gamma_\alpha = 2\widehat{\eta}_{\alpha\beta} I_n, \qquad \alpha, \beta = 1, \ldots, p, \ \alpha, \beta \neq t,$$
$$\gamma_\alpha^+ = -\gamma_\alpha, \qquad \mathrm{Re}\, \gamma_\alpha = 0, \ \alpha = 1, \ldots, p, \ \alpha \neq t,$$
$$\gamma_\alpha^+ := \kappa \gamma_\alpha^T \kappa^{-1},$$

where I_n is the identity $n \times n$-matrix and $\widehat{\eta}_{\alpha\beta}$ is determined by (9). Then the matrices γ_α generate a real Clifford algebra. Choose the basic endomorphism (E_0, E_α), $\alpha = 1, \ldots, p$, $\alpha \neq t$ in $\boldsymbol{E}$ so that

$$E_0 e_j = e_j, \qquad E_\alpha e_j = i\gamma^k_{j\alpha} e_k, \qquad \alpha = 1, \ldots, p, \ \alpha \neq t, \ j, k = 1, \ldots, n, \tag{11}$$

where i denotes the imaginary unit. The choice (11) is motivated by

Lemma 2. *The endomorphisms* E_0, E_α *satisfy the relations*

$$E_0 = E_I, \qquad E_\alpha e_j = C^k_{j\alpha} e_k, \qquad E_I \text{ the identity endomorphism in } \boldsymbol{E}, \tag{12}$$

for $\alpha = 1, \ldots, p, \alpha \neq t, j, k = 1, \ldots, n$, *where* $C^k_{j\alpha}$ *can be chosen as*

$$C_\alpha = i\gamma_\alpha, \qquad \alpha = 1, \ldots, p, \ \alpha \neq t, \ C_t = I_n.$$

Proof. The lemma follows directly from (8) and Corollary 1.

Consider a fixed direction in $\boldsymbol{E}$ determined by the endomorphisms E_α, $\alpha = 1, \ldots, p$, $\alpha \neq t$. Define

$$\widetilde{n} := \sum_{\substack{\alpha=1 \\ \alpha\neq t}}^{p} E_\alpha n^\alpha \,, \qquad \sum_{\substack{\alpha,\beta=1 \\ \alpha,\beta\neq t}}^{p} \widehat{\eta}_{\alpha\beta} n^\alpha n^\beta = 1 \,, \tag{13}$$

where (n^α) is a system of $p-1$ real numbers. Then we have

LEMMA 3. *The endomorphisms E_0 and $\widetilde{n}$ replace* 1 *and i of $\mathbb{C}$ in the field of "numbers" $qE_0 + s\widetilde{n}$, where $q, s \in \mathbb{R}$:*

$$E^2 = E_0, \quad E_0\widetilde{n} = \widetilde{n}E_0 = \widetilde{n}, \quad \widetilde{n}^2 = -E_0 \,. \tag{14}$$

Proof. We only prove the third equality. Notice that

$$\begin{aligned} \widetilde{n}^2(e_j) &= \widetilde{n}(\widetilde{n}e_j) = E_\beta n^\beta (E_\alpha n^\alpha) e_j \\ &= -n^\alpha n^\beta \gamma^k_{j\alpha} \gamma^m_{k\beta} e_m = -n^\alpha n^\beta [\gamma_\alpha \gamma_\beta]^m_j e_m \,. \end{aligned}$$

On the other hand, we have

$$\widetilde{n}^2(e_j) = -n^\beta n^\alpha [\gamma_\beta \gamma_\alpha]^m_j e_m \,.$$

Using the above equalities we obtain

$$\begin{aligned} 2\widetilde{n}^2(e_j) &= -n^\alpha n^\beta [\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha]^m_j e_m = -2n^\alpha n^\beta \widehat{\eta}_{\alpha\beta} [I_n]^m_j e_m \\ &= -2(n^\alpha n^\beta \widehat{\eta}_{\alpha\beta}) \delta^m_j e_m = -2e_j = -2E_0(e_j) \,. \end{aligned}$$

Hence $\widetilde{n}^2 = -E_0$, as required. ∎

The endomorphism $\widetilde{n}$ is represented in the basis (e_j) by the matrix

$$J = i n^\alpha \gamma_\alpha \,.$$

Now, we shall show some important properties of this matrix.

REMARK 1. $J^2 = -I_n$.

Proof. On the one hand, by the definition we have

$$J^2 = (i n^\alpha \gamma_\alpha)(i n^\beta \gamma_\beta) = -n^\alpha n^\beta \gamma_\alpha \gamma_\beta \,.$$

On the other hand, changing the indices we get $J^2 = -n^\beta n^\alpha \gamma_\beta \gamma_\alpha$. Thus,

$$2J^2 = -n^\alpha n^\beta [\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha] = -2n^\alpha n^\beta \widetilde{\eta}_{\alpha\beta} I_n = -2I_n \,. \ \blacksquare$$

Denote by $O(k,l)$ the group of orthogonal transformations of the space (V, κ) ($\kappa = \mathrm{diag}(\underbrace{1,\ldots,1}_{k}, \underbrace{-1,\ldots,-1}_{l})$). It is well-known that a matrix B belongs to $O(k,l)$ if and only if

$$B^T \kappa B = \kappa \quad \text{or} \quad B\kappa B^T = \kappa \,. \tag{15}$$

By the definition of the conjugation "+", given in (6), the above condition is equivalent to

$$B^+B = I_n \quad \text{or} \quad BB^+ = I_n\,.$$

Remark 2. $J \in O(k,l)$.

P r o o f. Directly by the definition of J we have

$$J\kappa J^T = -n^\alpha n^\beta \gamma_\alpha \kappa \gamma_\beta^T\,.$$

By (8) ($\gamma_\alpha^+ = -\gamma_\alpha$) we get $\kappa\gamma_\beta^T\kappa^{-1} = -\gamma_\beta$. Thus

$$J\kappa J^T = n^\alpha n^\beta \gamma_\alpha \gamma_\beta \kappa\,.$$

On the other hand, changing the indices we obtain

$$J\kappa J^T = n^\beta n^\alpha \gamma_\beta \gamma_\alpha \kappa\,.$$

Thus

$$2J\kappa J^T = n^\alpha n^\beta [\gamma_\alpha\gamma_\beta + \gamma_\beta\gamma_\alpha]\kappa = 2n^\alpha n^\beta \widehat{\eta}_{\alpha\beta} I_n \kappa = 2\kappa\,. \ \blacksquare$$

The standard complex structure in the Euclidean space E_n is the endomorphism represented by the matrix

$$J_0 = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}.$$

It is clear that $J_0 \in O(n)$.

Remark 3. *For each pair (k,l) of positive integers such that $k+l=n$, we have $J_0 \notin O(k,l)$.*

P r o o f. It suffices to show that $J_0\kappa \neq \kappa J_0$. Otherwise, we would have $J_0\kappa J_0^T = \kappa J_0 J_0^T = \kappa$ and J_0 would belong to $O(k,l)$.

We divide our proof into 3 parts.

I. $k = l = n/2$. In this case we have

$$J_0\kappa = \begin{pmatrix} 0 & -I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix}, \qquad \kappa J_0 = \begin{pmatrix} 0 & I_{n/2} \\ I_{n/2} & 0 \end{pmatrix},$$

so $J_0\kappa \neq \kappa J_0$.

II. $k < n/2$. Then

$$J_0\kappa = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \left(\begin{array}{cc|c} I_k & & 0 \\ & -I & \\ \hline & 0 & -I_{n/2} \end{array}\right) = \left(\begin{array}{cc|c} & 0 & -I_{n/2} \\ \hline -I_k & & \\ & I & 0 \end{array}\right),$$

$$\kappa J_0 = \left(\begin{array}{c|cc} & I_k & \\ 0 & & -I \\ \hline I_{n/2} & & 0 \end{array}\right),$$

where I denotes $I_{n/2-k}$, so in this case $J_0\kappa \neq \kappa J_0$ as well.

III. $k > n/2$. Then

$$J_0\kappa = \begin{pmatrix} 0 & I_{n/2} \\ -I_{n/2} & 0 \end{pmatrix} \left(\begin{array}{c|cc} I_{n/2} & \multicolumn{2}{c}{0} \\ \hline 0 & I & \\ & & -I_l \end{array}\right) = \left(\begin{array}{c|cc} 0 & I & \\ & & -I_l \\ \hline -I_{n/2} & \multicolumn{2}{c}{0} \end{array}\right),$$

$$\kappa J_0 = \left(\begin{array}{cc|c} \multicolumn{2}{c|}{0} & I_{n/2} \\ \hline -I & & \\ & I_l & 0 \end{array}\right),$$

where I denotes $I_{n/2-l}$. Again $J_0\kappa \neq \kappa J_0$. This completes the proof. ∎

The following problem arises:

Problem 1. For which pairs (k, l) of positive integers does there exist a matrix $J \in O(k, l)$ satisfying $J^2 = -I_n$, $n = k + l$?

We are looking for a matrix $J \in M(n)$ which satisfies

$$\text{(a) } J^T \kappa J = \kappa\,, \qquad \text{(b) } J^2 = -I_n\,. \tag{16}$$

Notice that the above conditions are equivalent to

$$\text{(a) } (\kappa J)^T = -\kappa J\,, \qquad \text{(b) } J^2 = -I_n\,. \tag{17}$$

Lemma 4. *Let*

$$\kappa = \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix}, \qquad k, l \neq 0\,.$$

If $B \in O(k, l)$, *then*

1) B *is of the form*

$$B = \begin{pmatrix} A & C_1 \\ C_2 & B \end{pmatrix}, \tag{18}$$

where $A \in M(k)$, $A \neq 0$; $B \in M(l)$, $B \neq 0$; $C_1 \in M(l \times k)$, $C_2 \in M(k \times l)$ *and the following conditions are satisfied*:

$$\begin{array}{ll} \text{(a) } A^T A - C_2^T C_2 = I_k\,, & \text{(b) } A^T C_1 - C_2^T B = 0\,, \\ \text{(c) } C_1^T A - B^T C_2 = 0\,, & \text{(d) } B^T B - C_1^T C_1 = I_l\,. \end{array} \tag{19}$$

2) $\det B = \pm 1$.

Proof. The condition 2) is a straightforward consequence of (15). To prove 1) assume that B is of the form (18). Then

$$B^T = \begin{pmatrix} A^T & C_2^T \\ C_1^T & B^T \end{pmatrix}. \tag{20}$$

By (15), we have, say,

$$\begin{pmatrix} A^T & C_2^T \\ C_1^T & B^T \end{pmatrix} \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix} \begin{pmatrix} A & C_1 \\ C_2 & B \end{pmatrix}$$

$$= \begin{pmatrix} A^T A - C_2^T C_2 & A^T C_1 - C_2^T B \\ C_1^T A - B^T C_2 & C_1^T C_1 - B^T B \end{pmatrix} = \begin{pmatrix} I_k & 0 \\ 0 & -I_l \end{pmatrix}.$$

This is nothing but (19).

Assume that $A = 0$. Then by (19a) we would have $C_2^T C_2 = -I_k$. If $(a_1, \ldots, a_l)$ is the first column of C_2, then we would get $a_1^2 + \ldots + a_l^2 = -1$, which is impossible. Thus $A \neq 0$. Analogously, we show that $B \neq 0$. ■

THEOREM 2. *Let κ be as in Lemma 4. If $J \in O(k,l)$ and J satisfies $J^2 = -I_n$, $n = k + l$, then*

1) *J has the form*

$$J = \begin{pmatrix} A & C \\ C^T & B \end{pmatrix}, \tag{21}$$

where $A \in M(k)$, $A \neq 0$, $A^T = -A$; $B \in M(l)$, $B \neq 0$, $B^T = -B$; $C \in M(l \times k)$, and the matrices A, B, C satisfy (19) *with $C_1 = C_2 = C$.*

2) *The integers k and l are even.*

Proof. By the assumptions, J satisfies (17a) so we have

$$(\kappa J)^r_s = -(\kappa J)^s_r, \quad \sum_{m=1}^{n} \kappa^r_m J^m_s = -\sum_{w=1}^{n} \kappa^s_w J^w_r \quad \text{for } r, s = 1, \ldots, n.$$

Since κ is a diagonal matrix, the above equality is equivalent to

$$\kappa^r_r J^r_s = -\kappa^s_s J^s_r \quad \text{for } r, s = 1, \ldots, n. \tag{22}$$

By the assumption $\kappa = \mathrm{diag}(\underbrace{1, \ldots, 1}_{k}, \underbrace{-1, \ldots, -1}_{l})$, so by (22) we get the following:

I. If $r \le k$, $s \le k$, then $J^r_s = -J^s_r$.
II. If $r > k$, $s > k$, then $J^r_s = -J^s_r$.
III. If $r \le k$, $s > k$, then $J^r_s = J^s_r$.
IV. If $r > k$, $s \le k$, then $J^r_s = J^s_r$.

We conclude that J has the form (21). Thus

$$J^T = \begin{pmatrix} -A & C \\ C^T & -B \end{pmatrix}.$$

Using (17) we get

$$J^T \kappa J = \begin{pmatrix} -A^2 - CC^T & -AC - CB \\ C^T A + BC^T & C^T C + B^2 \end{pmatrix}$$

and

$$J^2 = \begin{pmatrix} A^2 + CC^T & AC + CB \\ C^T A + BC^T & C^T C + B^2 \end{pmatrix}.$$

Thus A, B, C satisfy (19) with $C_1 = C_2 = C$. Analogously to Lemma 4, we prove that $A, B \neq 0$.

In order to prove the second assertion of our theorem we assume that k and l are odd ($k + l = n$, and by Lemma 1, n is always even). Since A and B are antisymmetric, we then have

$$\det A = \det B = 0\,. \tag{23}$$

We now show that (23) contradicts (19). Indeed, to the matrix A^2 we can associate a quadratic form F_{A^2} defined by $F_{A^2}(x, x) := \langle x, A^2 x\rangle$, where $\langle\ ,\ \rangle$ denotes the usual scalar product. By (19a) we have

$$\begin{aligned} F_{A^2}(x, x) &= \langle x, (-I_k - CC^T)x\rangle = \langle x, -x - CC^T x\rangle \\ &= \langle x, -x\rangle - \langle x, CC^T x\rangle = -\|x\|^2 - \langle C^T x, C^T x\rangle \\ &= -\|x\|^2 - \|C^T x\|^2 < 0 \end{aligned}$$

for $x \neq 0$. The form F_{A^2} is thus negative definite, so $\det A^2 < 0$, which contradicts (23). ∎

Remark 4. If k and l are even integers ($k + l = n$, $k, l \neq 0$), then the matrix $J \in O(k, l)$ satisfying $J^2 = -I_n$ can be chosen as follows:

$$J = J^0 := \left(\begin{array}{c|c|c|c|c} \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & 0 & \ldots & 0 \\ \hline 0 & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} & 0 & \ldots & 0 \\ \hline \vdots & \vdots & & \ldots & \\ \hline 0 & 0 & 0 & & \begin{matrix} 0 & 1 \\ -1 & 0 \end{matrix} \end{array}\right). \tag{24}$$

Of course, $(J^0)^T = -J^0$.

Denote by F the family of all matrices $A \in M(n)$ satisfying one of the equivalent conditions

$$A^+ = -A, \quad \kappa A^T \kappa^{-1} = -A, \quad (A\kappa)^T = -(A\kappa)\,,$$

where $\kappa^T = \kappa = \kappa^{-1}$.

Remark 5. *Any $A \in F$ satisfies*

$$\operatorname{Tr} A = 0\,. \tag{25}$$

Proof. Indeed,

$$(A\kappa)^i_j = \sum_{m=1}^{n} A^i_m \kappa^m_j = A^i_j \kappa^j_j$$

because κ is diagonal. Now, since $A\kappa$ is antisymmetric, we get

$$0 = (A\kappa)^j_j = A^j_j \kappa^j_j \Rightarrow A^j_j = 0\,. \ ∎$$

Corollary 2. *The matrices* γ_α, $\alpha = 1, \ldots, p$, $\alpha \neq t$, *determined by* (7)–(9), *belong to* F.

Corollary 3. *If* γ_α, $\alpha = 1, \ldots, p$, $\alpha \neq t$, *are the matrices described by* (7)–(9) *and* (n^α) *is an arbitrary system of* $p-1$ *real numbers satisfying* $\sum_{\alpha,\beta=1,\alpha,\beta\neq t}^{p} \widehat{\eta}_{\alpha\beta} n^\alpha n^\beta = 1$, *then*

$$\mathrm{Tr}(in^\alpha \gamma_\alpha) = 0\,. \tag{26}$$

Here the following problem arises:

Problem 2. Determine all matrices C_α, $\alpha = 1, \ldots, p$, satisfying (5).

Lemma 5. *The general formula describing the admissible matrices* C'_α *satisfying* (5) *is*

$$C'_\alpha = \sum_\beta O^\beta_\alpha R C_\beta R^{-1}\,, \tag{27}$$

where $O \in O(\widehat{\eta})$, $R \in O(\kappa)$.

Proof. The matrices C_α only depend on the choice of the bases in S and V. We shall show how the matrices C_α transform with the change of the bases. Let

$$\varepsilon'_\alpha = O^\beta_\alpha \varepsilon_\beta\,, \quad e'_j = R^k_j e_k\,, \quad R \in O(\kappa)\,,\ O \in O(\widehat{\eta})\,,$$

and

$$F(\varepsilon'_\alpha, e'_j) = C'^{\,k}_{\alpha_j}\, e'_k\,.$$

Then

$$F(O^\beta_\alpha \varepsilon_\beta, R^k_j e_k) = C'^{\,k}_{\alpha j}\, R^m_k e_m\,,$$
$$O^\beta_\alpha R^k_j F(\varepsilon_\beta, e_k) = C'^{\,k}_{\alpha j}\, R^m_k e_m\,,$$
$$O^\beta_\alpha R^k_j C^l_{\beta_k} e_l = C'^{\,k}_{\alpha j}\, R^m_k e_m\,.$$

Since $R \in O(\kappa)$, it follows that $\kappa R^T \kappa^{-1} = R^{-1}$, $\kappa^{-1} = \kappa$, and

$$R^m_k (\kappa R^T \kappa)^w_m = \delta^w_k\,.$$

Thus,

$$O^\beta_\alpha R^k_j C^l_{\beta k} e_l = C'^{\,k}_{\alpha j}\, R^m_k \delta^l_m e_l\,,$$
$$O^\beta_\alpha R^k_j C^l_{\beta k} = C'^{\,k}_{\alpha j}\, R^l_k\,.$$

Now, we multiply both sides by $(\kappa R^T \kappa)^s_l$:

$$O^\beta_\alpha R^k_j C^l_{\beta k} (\kappa R^T \kappa)^s_l = C'^{\,k}_{\alpha j}\, R^l_k (\kappa R^T \kappa)^s_l = C'^{\,k}_{\alpha j}\, \delta^s_k = C'^{\,s}_{\alpha j}\,,$$
$$O^\beta_\alpha [R C_\beta \kappa R^T \kappa]^s_j = C'^{\,s}_{\alpha j}\,,$$
$$O^\beta_\alpha R C_\beta R^{-1} = C'_\alpha\,,$$

as required. It is easy to see that if the matrices (C_α) satisfy (5) then so do the (C'_α). ■

COROLLARY 4. *The general formula describing the admissible matrices* γ'_α *satisfying* (8) *is*

$$\gamma'_\alpha = O^\beta_\alpha R \gamma_\beta R^{-1}\,, \tag{28}$$

where $R \in O(\kappa)$, $O \in O(\widehat{\eta})$.

COROLLARY 5. *If* (n^α) *is an arbitrary system of numbers satisfying* (13) *and* γ_α, $\alpha = 1, \ldots, p$, $\alpha \neq t$, *is an arbitrary system of matrices determined by* (7)–(9) *then*, *changing the base in the space* (V, κ) *by means of an orthogonal transformation* $R \in O(\kappa)$, *we have the following formula for the admissible matrices* $J' \in O(k,l)$ *satisfying* $(J')^2 = -I_n$, $n = k + l$:

$$J' = RJR^{-1}\,,$$

where $J = in^\alpha \gamma_\alpha$.

Now, fix matrices γ_α, $\alpha = 1, \ldots, p$, $\alpha \neq t$, and a system of $p-1$ real numbers (n^α) satisfying (13). Denote by $\mathrm{Or}(J^0) := \{M \in M(n); M = RJ^0R^{-1},\ R \in O(\kappa)\}$ the $O(\kappa)$-orbit of the matrix J^0. Further, let $\mathrm{Or}(J)$ denote the $O(\kappa)$-orbit of $J = in^\alpha\gamma_\alpha$. Let us compute the moments of J^0 and J. We have

$$\operatorname{Tr} J^{2k} = \operatorname{Tr}(J^2)^k = \operatorname{Tr}(-I_n)^k = (-1)^k \operatorname{Tr} I_n = n(-1)^k\,,$$
$$\operatorname{Tr}(J^0)^{2k} = \operatorname{Tr}(J^{02})^k = \operatorname{Tr}(-I_n)^k = n(-1)^k\,, \qquad \text{for } k = 1, \ldots, n/2\,.$$

Analogously, by Corollary 3, we have

$$\operatorname{Tr} J^{2k+1} = \operatorname{Tr}(J^{2k} \cdot J) = \operatorname{Tr}(-J) = 0$$

and, since J^0 is antisymmetric,

$$\operatorname{Tr}(J^0)^{2k+1} = \operatorname{Tr}(-J^0) = 0\,.$$

The matrices J and J^0 have the same moments so they belong to the same orbit of $O(\kappa)$:

$$\mathrm{Or}(J^0) = \mathrm{Or}(J)\,.$$

LEMMA 6. *Let* n *and* p *be positive integers determined by Lemma* 1, $n > 1$. *Then*, *to any system* (n^α) *of* $p-1$ *real numbers satisfying* (13) *we can associate a system* γ_α, $\alpha = 1, \ldots, p$, $\alpha \neq t$, *of imaginary* $n \times n$*-matrices satisfying* (8) *so that*

$$in^\alpha \gamma_\alpha = J^0\,. \tag{29}$$

P r o o f. By the considerations preceding Lemma 6, for any system (n^α) of $p-1$ real numbers satisfying (13) and for any system γ_α of imaginary $n \times n$-matrices satisfying (8) the matrices $J = in^\alpha\gamma_\alpha$ and J^0 belong to the same $O(\kappa)$-orbit. Consequently, by the transitivity of the action of $O(\kappa)$ in this orbit, for each system (n^α) in question there exists an orthogonal transformation of one matrix to the other and so the proof is complete. ∎

Let us pose the following problem:

PROBLEM 3. Describe the orbit $O(\kappa) \cdot J^0$.

Let Ω and Ω' belong to $O(\kappa) \cdot J^0$. Then $\Omega = AJ^0A^{-1}$, $\Omega' = BJ^0B^{-1}$, where $A, B \in O(\kappa)$. Notice that

$$(\Omega = \Omega') \Leftrightarrow [(A^{-1}B)J^0(A^{-1}B)^{-1} = J^0] .$$

Introduce the following relation in $O(\kappa)$:

$$(A \sim B) \Leftrightarrow [(A^{-1}B)J^0(A^{-1}B)^{-1} = J^0] .$$

It is clear that this is an equivalence relation. Then the set of different matrices Ω in the orbit $O(\kappa) \cdot J^0$ is isomorphic to the group $O(\kappa)/\sim \equiv O(\kappa)/S(J^0)$, where $S(J^0) := \{A \in O(\kappa) : AJ^0A^{-1} = J^0\}$ is the stability group of J^0.

Let us recall that the endomorphism $\widetilde{n}$ is represented in the basis (e_j) by the matrix

$$J = in^\alpha \gamma_\alpha , \tag{30}$$

where

$$J = RJ^0R^{-1} \tag{31}$$

for some $R \in O(\kappa)$.

DEFINITION. The endomorphism $\widetilde{n}$ described by (4), (8), (12) and (13) will be called a *supercomplex structure* on (V, κ).

This definition is motivated by

LEMMA 7. *If a supercomplex structure* $\widetilde{n}$ *exists, then*

$$\begin{aligned} (Re)_{2j} &= J(Re)_{2j-1} = \widetilde{n}(Re)_{2j-1} , \\ (Re)_{2j-1} &= -J(Re)_{2j} = -\widetilde{n}(Re)_{2j} \end{aligned} \tag{32}$$

for some $R \in O(\kappa)$.

Proof. This is a straightforward consequence of Corollaries 4 and 5, Lemma 6, and (11), (13), (30). ■

DEFINITION. $[(V, \kappa), J, \widetilde{n}, \cdot, \boldsymbol{E}]$ is a complex vector space $[(V, \kappa), J, \cdot\ ]$ equipped with a supercomplex structure $(J, \widetilde{n})$ and a Hurwitz type vector space $\boldsymbol{E}$ of endomorphisms $E : V \to V$ satisfying

$$(q + is) \cdot f = fq + (Jf)s \qquad \text{for } f \in V \text{ and } q, s \in \mathbb{R} . \tag{33}$$

(By the definition it has to satisfy also the relations (32), (11), (13), and (14).)

THEOREM 3. *Consider a pseudo-Euclidean Hurwitz pair* $(V(\kappa), S(\eta))$ *of bidimension* (n, p), $n > 1$, *and some orthonormal bases* (e_j) *in* V *and* (ε_α) *in* S. *Let* (n^α) *be an arbitrary system of real numbers* (13) *and* (γ_α) *a system* *of imaginary $n \times n$-matrices* (8)–(9) *with the property* (29), *which is possible under the assumption that* $\kappa = \mathrm{diag}(\underbrace{1,\ldots,1}_{k=2k'},\underbrace{-1,\ldots,-1}_{l=2l'})$, $k', l' \neq 0$. *Suppose that f is an arbitrary vector in V and let $\sum_{j=1}^{n} e_j f_{\mathbb{R}}^j$ be its decomposition (in V). Then this decomposition can be rearranged into the form*

$$f = \sum_{j=1}^{n/2} (Re)_{2j-1} f^{2j-1}, \quad \text{where } f^{2j-1} = E_0 f_{\mathbb{R}}^{2j-1} + \widetilde{n} f_{\mathbb{R}}^{2j}, \tag{34}$$

or

$$f = \sum_{j=1}^{n/2} (Re)_{2j} f^{2j}, \qquad \text{where } f^{2j} = E_0 f_{\mathbb{R}}^{2j} - \widetilde{n} f_{\mathbb{R}}^{2j-1}, \tag{35}$$

for some $R \in O(\kappa)$, where $\widetilde{n} = \sum_{\alpha=1,\alpha\neq t}^{p} n^\alpha E_\alpha$.

Proof. The problem whose solution is formulated in Theorem 3 is well-posed by Lemma 1, (11), (13), Theorem 2 and Lemma 6. By (11) and (13),

$$\widetilde{n} e_j = n^\alpha (i\gamma_{j\alpha}^k e_k) = (in^\alpha \gamma_\alpha)_j^k e_k = J_j^k e_k .$$

By Lemma 6, $\widetilde{n}(Re)_j = (J^0)_j^k e_k$. Using Lemma 7, we get

$$\begin{aligned} \widetilde{n}(Re)_{2j-1} &= (J^0)_{2j-1}^k (Re)_k = (Re)_{2j}, \\ \widetilde{n}(Re)_{2j} &= (J^0)_{2j}^k (Re)_k = -(Re)_{2j-1}. \end{aligned} \tag{36}$$

Thus, for every $f = \sum_{j=1}^{n} (Re)_j f_{\mathbb{R}}^j$ we get

$$\begin{aligned} f &= \sum_{j=1}^{n/2} [(Re)_{2j-1} f_{\mathbb{R}}^{2j-1} + (Re)_{2j} f_{\mathbb{R}}^{2j}] \\ &= \sum_{j=1}^{n/2} [(Re)_{2j-1} f_{\mathbb{R}}^{2j-1} + \widetilde{n}(Re)_{2j+1} f_{\mathbb{R}}^{2j}] = \sum_{j=1}^{n/2} (Re)_{2j-1} f^{2j-1}, \end{aligned}$$

where $f^{2j-1} := E_0 f_{\mathbb{R}}^{2j-1} + \widetilde{n} f_{\mathbb{R}}^{2j}$.

Analogously, we obtain (35). The uniqueness of these decompositions is a clear consequence of the uniqueness of $f = \sum_{j=1}^{n} e_j f_{\mathbb{R}}^j$.

From (34) and (35) we also deduce

LEMMA 8. *If* $\kappa = \mathrm{diag}(\underbrace{1,\ldots,1}_{k=2k'},\underbrace{-1,\ldots,-1}_{l=2l'})$, *where* $k', l' \neq 0$, *then by Theorem* 3 *the decompositions* (34) *and* (35) *for* $f \in V$ *generate the decom-*

positions

$$V = \bigoplus_{j=1}^{n/2} C_j(E_0, \widetilde{n}, J) \tag{37}$$

or

$$V = \bigoplus_{j=1}^{n/2} \widetilde{C}_j(E_0, \widetilde{n}, J)\,, \tag{38}$$

where $C_j(E_0, \widetilde{n}, J)$ *and* $\widetilde{C}_j(E_0, \widetilde{n}, J)$ *are complex one-dimensional subspaces of* V, *generated by* e_{2j-1} *and* e_{2j}, *respectively*, *for* $j = 1, \ldots, n/2$. *Their dependence on* E_0, $\widetilde{n}$ *and* J *is determined by* (11), (13), *and* (29).

On the other hand, with the help of the complex structure J we can introduce the complex scalar product $(\,,) : V \times V \to \mathbb{C}$ as follows:

$$(f, g) = (f, g)_{\mathbb{R}} + i(Jf, g)_{\mathbb{R}} \quad \text{for } f, g \in V \tag{39}$$

(provided κ, the metric of V, satisfies the assumption of Lemma 8), where $(\,,)_{\mathbb{R}}$ denotes the usual (real) scalar product in $V : (f, g)_{\mathbb{R}} := \sum_{i=1}^{n} f^i g^i$ for $f = f^i e_i$, $g = g^i e_i$. Then we have

PROPOSITION 1. *The complex scalar product* $(\,,)$ *has the properties*

$$(f, g) = \overline{(g, f)}, \qquad (f, g + h) = (f, g) + (f, h) \quad \textit{for } f, g, h \in V\,, \tag{40}$$

$$(f, zg) = z(f, g), \qquad (f, f) = \|f\|^2 \quad \textit{for } f, g \in V \textit{ and } z \in \mathbb{C}\,, \tag{41}$$

$$(f, g) = \sum_{j=1}^{n/2} \overline{f^j_{\mathbb{C}}} g^j_{\mathbb{C}} \quad \textit{for } f, g \in V\,, \tag{42}$$

where the bar denotes complex conjugation and

$$f^j_{\mathbb{C}} = f^{2j-1}_{\mathbb{R}} + i f^{2j}_{\mathbb{R}}, \qquad g^j_{\mathbb{C}} = g^{2j-1}_{\mathbb{R}} + i g^{2j}_{\mathbb{R}}, \qquad j = 1, \ldots n/2 \tag{43}$$

Proof. (40) and (41) follow from (30) and (31) and from the definition of $(\,,\,)$ and $(\,,\,)_{\mathbb{R}}$. Indeed,

$$\begin{aligned}
(g, f) &= (g, f)_{\mathbb{R}} + i(Jg, f)_{\mathbb{R}} = (f, g)_{\mathbb{R}} + i(\widetilde{n}g, f)_{\mathbb{R}} = (f, g)_{\mathbb{R}} - n^{\alpha}(\gamma_\alpha g, f) \\
&= (f, g)_{\mathbb{R}} - n^{\alpha} \sum_{k=1}^{n} (\gamma_\alpha g)_k f_k = (f, g)_{\mathbb{R}} - n^{\alpha} \sum_{k=1}^{n} \Big(\sum_{m=1}^{n} \gamma^m_{\alpha k} g_m \Big) f_k \\
&= (f, g)_{\mathbb{R}} - n^{\alpha} \sum_{m=1}^{n} \sum_{k=1}^{n} g_m (-\gamma^k_{\alpha m} f_k) \\
&= (f, g)_{\mathbb{R}} + n^{\alpha} \sum_{m=1}^{n} g_m (\gamma_\alpha f)_m = (f, g)_{\mathbb{R}} + n^{\alpha}(g, \gamma_\alpha f)_{\mathbb{R}} \\
&= (f, g)_{\mathbb{R}} - i n^{\alpha}(g, E_\alpha f)_{\mathbb{R}} = (f, g)_{\mathbb{R}} - i(g, \widetilde{n} f)_{\mathbb{R}}
\end{aligned}$$

$$= (f,g)_{\mathbb{R}} - i(g,Jf)_{\mathbb{R}} = (f,g)_{\mathbb{R}} - i(Jf,g)_{\mathbb{R}} = \overline{(f,g)}\,.$$

In particular,

$$(f,f) = (f,f)_{\mathbb{R}} + i(Jf,f)_{\mathbb{R}} = \overline{(f,f)} = (f,f)_{\mathbb{R}} - i(Jf,f)_{\mathbb{R}}\,.$$

Hence $(Jf,f)_{\mathbb{R}} = 0$ and $(f,f) = (f,f)_{\mathbb{R}} = \|f\|^2$. The remaining equalities in (40) and (41) are obvious.

To prove (42) we take (36):

$$\begin{aligned}
(f,g) &= (f,g)_{\mathbb{R}} + i(Jf,g)_{\mathbb{R}} = \sum_{k=1}^{n} f^k g^k + i(\widetilde{n}f,g)_{\mathbb{R}} \\
&= \sum_{k=1}^{n}(f^k g^k + i(\widetilde{n}(f^k e_k),g)_{\mathbb{R}}) = \sum_{k=1}^{n} f^k g^k + i\sum_{j=1}^{n/2}(f^{2j-1}\widetilde{n}(e_{2j-1}) \\
&\qquad + f^{2j}\widetilde{n}(e_{2j}),g)_{\mathbb{R}} = \sum_{k=1}^{n} f^k g^k + i\sum_{j=1}^{n/2}(f^{2j-1}e_{2j} - f^{2j}e_{2j-1},g)_{\mathbb{R}} \\
&= \sum_{k=1}^{n} f^k g^k + i\sum_{j=1}^{n/2}(f^{2j-1}g^{2j} - f^{2j}g^{2j-1}) \\
&= \sum_{j=1}^{n/2}[f^{2j-1}(g^{2j-1} + ig^{2j}) + f^{2j}(g^{2j} - ig^{2j-1})] \\
&= \sum_{j=1}^{n/2}(f^{2j-1} - if^{2j})(g^{2j-1} + ig^{2j}) = \sum_{j=1}^{n/2}\overline{f^j_{\mathbb{C}}}g^j_{\mathbb{C}}\,,
\end{aligned}$$

where $f^j_{\mathbb{C}}$ and $g^j_{\mathbb{C}}$ are defined by (43). ■

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Reçu par la Rédaction le 5.9.1990