

Applications of certain linear operators in the theory of analytic functions

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Abstract. The object of the present paper is to illustrate the usefulness, in the theory of analytic functions, of various linear operators which are defined in terms of (for example) fractional derivatives and fractional integrals, Hadamard product or convolution, and so on.

1. Introduction and definitions. Let \mathcal{A} denote the class of functions normalized by

$$(1.1) \quad f(z) = z + \sum_{n=2}^{\infty} a_n z^n,$$

which are *analytic* in the *open* unit disk

$$\mathcal{U} = \{z : |z| < 1\}.$$

Also let \mathcal{S} denote the class of all functions in \mathcal{A} which are *univalent* in \mathcal{U} . We denote by $\mathcal{S}^*(\alpha)$ and $\mathcal{K}(\alpha)$ the subclasses of \mathcal{S} consisting of all functions which are, respectively, *starlike* and *convex of order α* in \mathcal{U} ($0 \leq \alpha < 1$), that is,

$$(1.2) \quad \mathcal{S}^*(\alpha) = \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re} \left(\frac{zf'(z)}{f(z)} \right) > \alpha \ (0 \leq \alpha < 1; z \in \mathcal{U}) \right\},$$

$$(1.3) \quad \mathcal{K}(\alpha) = \left\{ f : f \in \mathcal{S} \text{ and } \operatorname{Re} \left(1 + \frac{zf''(z)}{f'(z)} \right) > \alpha \ (0 \leq \alpha < 1; z \in \mathcal{U}) \right\}.$$

It follows readily from (1.2) and (1.3) that

$$(1.4) \quad f(z) \in \mathcal{K}(\alpha) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\alpha) \quad (0 \leq \alpha < 1),$$

whose special case, when $\alpha = 0$, is the familiar Alexander theorem (cf., e.g., [2, p. 43, Theorem 2.12]). We note also that

$$(1.5) \quad \mathcal{K}(\alpha) \subset \mathcal{S}^*(\alpha) \subset \mathcal{S} \quad (0 \leq \alpha < 1),$$

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$$(1.6) \quad \mathcal{S}^*(\alpha) \subseteq \mathcal{S}^*(0) \equiv \mathcal{S}^* \quad (0 \leq \alpha < 1),$$

$$(1.7) \quad \mathcal{K}(\alpha) \subseteq \mathcal{K}(0) \equiv \mathcal{K} \quad (0 \leq \alpha < 1).$$

For the functions $f_j(z)$ defined by

$$f_j(z) = \sum_{n=0}^{\infty} a_{j,n+1} z^{n+1} \quad (j = 1, 2),$$

we denote by $f_1 * f_2(z)$ the Hadamard product or convolution of the functions $f_1(z)$ and $f_2(z)$, that is,

$$(1.8) \quad f_1 * f_2(z) = \sum_{n=0}^{\infty} a_{1,n+1} a_{2,n+1} z^{n+1}.$$

Thus, following Ruscheweyh [9], a function $f(z) \in \mathcal{A}$ is said to be *prestarlike of order α* ($\alpha \leq 1$) if and only if

$$(1.9) \quad \begin{cases} \frac{z}{(1-z)^{2(1-\alpha)}} * f(z) \in \mathcal{S}^*(\alpha) & (\alpha < 1), \\ \operatorname{Re} \left(\frac{f(z)}{z} \right) > \frac{1}{2} & (\alpha = 1), \end{cases}$$

and we denote by $\mathcal{R}(\alpha)$ the subclass of \mathcal{A} consisting of all prestarlike functions of order α in \mathcal{U} .

The various linear operators (whose usefulness, in the theory of such subclasses of analytic functions as those defined above, will be considered in this paper) include the *Carlson–Shaffer operator* $\mathcal{L}(a, c)$ defined by (cf. [1])

$$(1.10) \quad \mathcal{L}(a, c)f(z) = \varphi(a, c; z) * f(z) \quad (f(z) \in \mathcal{U}),$$

where $\varphi(a, c; z)$ is an incomplete Beta function defined by

$$(1.11) \quad \varphi(a, c; z) = \sum_{n=0}^{\infty} \frac{(a)_n}{(c)_n} z^{n+1} = z {}_2F_1(1, a; c; z) \\ (c \neq 0, -1, -2, \dots; z \in \mathcal{U}),$$

in terms of the Pochhammer symbol $(\lambda)_n$ given by

$$(1.12) \quad (\lambda)_n = \frac{\Gamma(\lambda + n)}{\Gamma(\lambda)} = \begin{cases} 1 & (n = 0) \\ \lambda(\lambda + 1) \dots (\lambda + n - 1) & (n \in \mathbb{N} = \{1, 2, 3, \dots\}), \end{cases}$$

and of the *Gaussian case* $p - 1 = q = 1$ of the *generalized hypergeometric function* ${}_pF_q(z)$ defined, for complex parameters α_j ($j = 1, \dots, p$) and β_j ($\neq 0, -1, -2, \dots$) ($j = 1, \dots, q$), by

$$(1.13) \quad {}_pF_q(z) \equiv {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \\ = {}_pF_q \left[\begin{matrix} \alpha_1, \dots, \alpha_p; \\ \beta_1, \dots, \beta_q; \end{matrix} \middle| z \right]$$

$$= \sum_{n=0}^{\infty} \frac{(\alpha_1)_n \dots (\alpha_p)_n z^n}{(\beta_1)_n \dots (\beta_q)_n n!} \quad (p \leq q + 1)$$

($p \leq q$ and $|z| < \infty$; $p = q + 1$ and $z \in \mathcal{U}$).

We note in passing that

$$(1.14) \quad z {}_pF_q(\alpha_1, \dots, \alpha_p; \beta_1, \dots, \beta_q; z) \in \mathcal{A}.$$

The Carlson–Shaffer operator $\mathcal{L}(a, c)$ maps \mathcal{A} onto itself. Moreover, if $a \neq 0, -1, -2, \dots$, then $\mathcal{L}(c, a)$ is an inverse of $\mathcal{L}(a, c)$. Observe also that (cf. [8, p. 1067])

$$(1.15) \quad \mathcal{K}(\alpha) = \mathcal{L}(1, 2)\mathcal{S}^*(\alpha); \quad \mathcal{S}^*(\alpha) = \mathcal{L}(2, 1)\mathcal{K}(\alpha) \quad (0 \leq \alpha < 1).$$

Next we recall the *generalized Bernardi–Libera–Livingston integral operator* \mathcal{J}_γ defined by

$$(1.16) \quad \mathcal{J}_\gamma f(z) = \frac{\gamma + 1}{z^\gamma} \int_0^z t^{\gamma-1} f(t) dt \quad (\gamma > -1; f(z) \in \mathcal{A}),$$

which, for various further constraints on the parameter γ , was used recently by several authors (see, e.g., [15, pp. 66, 154, 181, and 338]).

Numerous operators of *fractional calculus* (that is, *fractional integral* and *fractional derivative*) have indeed been studied in the literature rather extensively (cf., e.g., [3, Chapter 13], [4], [5], [10], [11], [13, p. 28 *et seq.*], [14, Chapter 5], and [15]). We choose to recall here the following interesting generalization of the fractional derivative operator D_z^λ (considered by Owa [6]):

DEFINITION (cf. [7] and [12]). For real numbers λ, μ , and ν , the *fractional derivative operator* $J_{0,z}^{\lambda,\mu,\nu}$ is defined by

$$(1.17) \quad J_{0,z}^{\lambda,\mu,\nu} f(z) = \frac{1}{\Gamma(1-\lambda)} \frac{d}{dz} \left\{ z^{\lambda-\mu} \int_0^z (z-\zeta)^{-\lambda} \right. \\ \left. \times {}_2F_1(\mu-\lambda, -\nu; 1-\lambda; 1-\zeta/z) f(\zeta) d\zeta \right\} \\ (0 \leq \lambda < 1; \kappa > \max\{0, \mu-\nu-1\} - 1),$$

where $f(z)$ is an analytic function in a simply-connected region of the z -plane containing the origin, with the order

$$f(z) = O(|z|^\kappa) \quad (z \rightarrow 0),$$

and the multiplicity of $(z-\zeta)^{-\lambda}$ is removed by requiring $\log(z-\zeta)$ to be real when $z-\zeta > 0$.

It follows readily from the definition (1.17) that

$$(1.18) \quad J_{0,z}^{\lambda,\lambda,\nu} f(z) = D_z^\lambda f(z) \quad (0 \leq \lambda < 1),$$

where the fractional calculus operator D_z^λ is, in fact, defined for all values of λ (see, e.g., [15, p. 343]).

2. Applications involving univalent and starlike generalized hypergeometric functions. We begin by recalling the following inclusion theorem for the generalized hypergeometric function, which was proven by Owa and Srivastava [8] by applying the Carlson–Shaffer operator $\mathcal{L}(a, c)$:

THEOREM 1. *For the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.13), let*

$$(2.1) \quad \left| \frac{z {}_pF_q''(z)}{{}_pF_q'(z)} \right| < (1 - \alpha)^{-1} \left(1 - \frac{3}{2}\alpha + \alpha^2 \right) \\ (0 \leq \alpha \leq 1/2; \alpha_1 \dots \alpha_p \neq 0; z \in \mathcal{U}).$$

Then

$$(2.2) \quad z {}_{p+1}F_{q+1}(\alpha_1 + 1, \dots, \alpha_p + 1, 1; \beta_1 + 1, \dots, \beta_q + 1, 2; z) \in \mathcal{S}^*(\alpha) \\ (0 \leq \alpha \leq \frac{1}{2}).$$

Now, in view of the fact that

$$(2.3) \quad f(z) \in \mathcal{S}^* \Rightarrow \mathcal{J}_\gamma f(z) \in \mathcal{S}^* \quad (0 \leq \gamma \leq 1),$$

Theorem 1 (with $\alpha = 0$) and the definition (1.16) would lead us eventually to

THEOREM 2. *For the generalized hypergeometric function ${}_pF_q(z)$ defined by (1.13), let*

$$(2.4) \quad \left| \frac{z {}_pF_q''(z)}{{}_pF_q'(z)} \right| < 1 \quad (\alpha_1 \dots \alpha_p \neq 0; z \in \mathcal{U}).$$

Then

$$(2.5) \quad z {}_{p+s+1}F_{q+s+1} \left[\begin{matrix} \alpha_1 + 1, \dots, \alpha_p + 1, 1, \gamma_1 + 1, \dots, \gamma_s + 1; \\ \beta_1 + 1, \dots, \beta_q + 1, 2, \gamma_1 + 2, \dots, \gamma_s + 2; \end{matrix} z \right] \in \mathcal{S}^* \\ (0 \leq \gamma_j \leq 1; j = 1, \dots, s).$$

From among various special cases of Theorem 2, which are worthy of note, we mention here the case when $\gamma_j = 1$ ($j = 1, \dots, s$). The assertion (2.5) reduces, in this special case, to the inclusion relation:

$$(2.6) \quad z {}_{p+s}F_{q+s} \left[\begin{matrix} \alpha_1 + 1, \dots, \alpha_p + 1, 1, 2, \dots, 2; \\ \beta_1 + 1, \dots, \beta_q + 1, 3, 3, \dots, 3; \end{matrix} z \right] \in \mathcal{S}^*,$$

which holds true under the relevant hypotheses of Theorem 2.

3. An application involving the classes $\mathcal{R}(\alpha)$ and $\mathcal{K}(\alpha)$. It is easily seen from the definition (1.9) for the class $\mathcal{R}(\alpha)$ that

$$(3.1) \quad \mathcal{R}(\alpha) = \mathcal{L}(1, 2 - 2\alpha)\mathcal{S}^*(\alpha) \quad (\alpha < 1)$$

and

$$(3.2) \quad \mathcal{R}(1) = \{f : f \in \mathcal{A} \text{ and } \operatorname{Re}(f(z)/z) > \frac{1}{2} \ (z \in \mathcal{U})\}.$$

With a view to presenting a connection theorem involving the classes $\mathcal{R}(\alpha)$ and $\mathcal{K}(\alpha)$, we find it to be convenient to introduce the operator $\Omega_z^{\lambda, \mu, \nu}$ defined by

$$(3.3) \quad \Omega_z^{\lambda, \mu, \nu} f(z) = \frac{\Gamma(2 - \mu)\Gamma(3 - \lambda + \nu)}{\Gamma(3 - \mu + \nu)} z^\mu J_{0,z}^{\lambda, \mu, \nu} f(z) \quad (f(z) \in \mathcal{A}),$$

where $J_{0,z}^{\lambda, \mu, \nu}$ denotes the fractional derivative operator defined already by (1.17). Since

$$(3.4) \quad J_{0,z}^{\lambda, \mu, \nu} z^\rho = \frac{\Gamma(\rho + 1)(\rho - \mu + \nu + 2)}{\Gamma(\rho - \mu + 1)\Gamma(\rho - \lambda + \nu + 2)} z^{\rho - \mu} \quad (\rho + 2 > \mu - \nu),$$

which stems naturally from the definition (1.17), it is not difficult to relate the operators $\Omega_z^{\lambda, \mu, \nu}$ and $\mathcal{L}(a, c)$ as follows:

$$(3.5) \quad \Omega_z^{\lambda, \mu, \nu} f(z) = \mathcal{L}(2, 2 - \mu)\mathcal{L}(3 - \mu + \nu, 3 - \lambda + \nu)f(z) \\ (0 \leq \lambda < 1; \mu - \nu < 3; f(z) \in \mathcal{A}).$$

Making use of the relationships (3.5) and (1.15), and the following result (due essentially to Carlson and Shaffer [1]):

$$(3.6) \quad \mathcal{L}(2 - 2\beta, 2 - 2\alpha)\mathcal{S}^*(\alpha) \subset \mathcal{S}^*(\beta) \subset \mathcal{S}^*(\alpha) \quad (0 \leq \alpha \leq \beta < 1),$$

it can be shown that

$$(3.7) \quad \mathcal{L}(3 - \lambda + \nu, 3 - \mu + \nu)\Omega_z^{\lambda, \mu, \nu}\mathcal{K}(1/2) \subset \mathcal{S}^*(1/2) \\ (0 \leq \lambda < 1; \mu - \nu < 3; 0 \leq \mu < 1).$$

Finally, rewriting a special case of (3.6) in the form:

$$(3.8) \quad \mathcal{L}(1, 2 - \alpha)\mathcal{S}^*(\alpha/2) \subset \mathcal{S}^*(1/2) \subset \mathcal{S}^*(\alpha/2) \quad (0 \leq \alpha < 2),$$

we obtain the following connection theorem involving the classes $\mathcal{R}(\alpha)$ and $\mathcal{K}(\alpha)$:

THEOREM 3. For the classes $\mathcal{K}(\alpha)$ and $\mathcal{R}(\alpha)$ defined by (1.3) and (1.9), respectively,

$$(3.9) \quad \mathcal{L}(3 - \lambda + \nu, 3 - \mu + \nu)\Omega_z^{\lambda, \mu, \nu}\mathcal{K}(\mu/2) = \mathcal{R}(\mu/2) \\ (0 \leq \lambda < 1; \mu - \nu < 3; 0 \leq \mu < 2).$$

For $\mu = 0$ and $\mu = 1$, the assertion (3.9) of Theorem 3 simplifies considerably. The details may be omitted.

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