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Natural transformations between $T_1^2 T^* M$ and $T^* T_1^2 M$

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Abstract. We determine all natural transformations $T_1^2 T^* \to T^* T_1^2$ where $T_k^r M = J_0^r(\mathbb{R}^k, M)$. We also give a geometric characterization of the canonical isomorphism ψ_2 defined by Cantrijn *et al.* [1] among such natural transformations.

The spaces $T_1^r M$ of one-dimensional velocities of order r are used in the geometric approach to higher-order mechanics. That is why several authors studied the relations between $T_1^r T^* M$ and $T^* T_1^r M$. For example, Modugno and Stefani [7] introduced an intrinsic isomorphism s between the bundles TT^*M and T^*TM. Recently Cantrijn, Crampin, Sarlet and Saunders [1] constructed a canonical isomorphism $\psi_r : T_1^r T^* M \to T^* T_1^r M$, which coincides with s for $r = 1$. From the categorical point of view, ψ_r is a natural equivalence between the functors $T_1^r T^*$ and $T^* T_1^r$, defined on the category $\mathcal{M}f_m$ of m-dimensional manifolds and their local diffeomorphisms. Starting from the isomorphism s, Kolář and Radziszewski [5] determined all natural transformations of TT^* into T^*T. In the present paper we determine all natural transformations $T_1^2 T^* \to T^* T_1^2$ and interpret them geometrically. Further we show that the natural equivalence ψ_2 can be distinguished among all natural transformations by a simple geometric construction.

1. The equations of all natural transformations $T_1^2 T^* \to T^* T_1^2$. We shall use the concept of a natural bundle in the sense of Nijenhuis [8]. Denote by $\mathcal{M}f_m$ the category of m-dimensional manifolds and their local diffeomorphisms, by $\mathcal{FM}$ the category of fibred manifolds and by $B : \mathcal{FM} \to \mathcal{M}f_m$ the base functor. A *natural bundle* over m-manifolds is a covariant functor $F : \mathcal{M}f_m \to \mathcal{FM}$ satisfying $B \circ F = \mathrm{id}$ and the localization condition: for every inclusion of an open subset $i : U \to M$, FU is the restriction to $p_M^{-1}(U)$ of $p_M : FM \to M$ over U and Fi is the inclusion $p_M^{-1}(U) \to FM$. If we replace the category $\mathcal{M}f_m$ by the category $\mathcal{M}f$ of all manifolds and all smooth maps, we obtain the concept of *bundle functor* on the category of all manifolds. A natural bundle $F : \mathcal{M}f_m \to \mathcal{FM}$ is said to be of *order* r

if, for any local diffeomorphisms $f, g : M \to N$ and any $x \in M$, the relation $j^r f(x) = j^r g(x)$ implies $Ff|F_xM = Fg|F_xM$, where F_xM denotes the fibre of FM over $x \in M$.

A *k-dimensional velocity* of order r on a smooth manifold M is an r-jet of $\mathbb{R}^k$ into M with source 0. The space $T^r_k M = J^r_0(\mathbb{R}^k, M)$ of all such velocities is a fibred manifold over M. Every smooth map $f : M \to N$ extends to an $\mathcal{FM}$-morphism $T^r_k f : T^r_k M \to T^r_k N$ defined by $T^r_k f(j^r_0 g) = j^r_0(f \circ g)$. Hence $T^r_k : \mathcal{M}f \to \mathcal{FM}$ is an rth order bundle functor. The simplest example is the functor T^1_1, which coincides with the tangent functor T.

The cotangent bundle T^*M is a vector bundle over the manifold M. Having a local diffeomorphism $f : M \to N$, we define $T^*f : T^*M \to T^*N$ by taking pointwise the inverse map to the dual map $(T_xf)^* : T^*_{f(x)}N \to T^*_x M$, $x \in M$. In this way the cotangent functor T^* is a natural bundle over m-manifolds.

We are going to determine all natural transformations $T^2_1 T^* \to T^* T^2_1$. The canonical coordinates x^i on $\mathbb{R}^m$ induce the additional coordinates p_i on $T^*\mathbb{R}^m$ and $\xi^i = dx^i/dt$, $X^i = d^2x^i/dt^2$, $\pi_i = dp_i/dt$, $P_i = d^2p_i/dt^2$ on $T^2_1 T^*\mathbb{R}^m$. Further, if $y^i = dx^i/dt$, $z^i = d^2x^i/dt^2$ are the induced coordinates on $T^2_1\mathbb{R}^m$, then the expression $\sigma_i\, dx^i + \varrho_i\, dy^i + \tau_i\, dz^i$ determines the additional coordinates σ_i, ϱ_i, τ_i on $T^*T^2_1\mathbb{R}^m$. Set

$$I = p_i\xi^i, \qquad J = p_iX^i + \pi_i\xi^i\,. \tag{1}$$

Let G^r_m be the group of all invertible r-jets of $\mathbb{R}^m$ into $\mathbb{R}^m$ with source and target 0.

Proposition 1. *All natural transformations $T^2_1 T^* \to T^* T^2_1$ are of the form*

$$\begin{aligned}
y^i &= F(I,J)\xi^i\,,\\
z^i &= F^2(I,J)X^i + H(I,J)\xi^i\,,\\
\tau_i &= G(I,J)p_i\,,\\
\varrho_i &= 2F(I,J)G(I,J)\pi_i + M(I,J)p_i\,,\\
\sigma_i &= F^2(I,J)G(I,J)P_i + [F(I,J)M(I,J) + H(I,J)G(I,J)]\pi_i\\
&\quad + N(I,J)p_i
\end{aligned} \tag{2}$$

where F, G, H, M, N are arbitrary smooth functions of two variables and I, J are given by (1).

In the proof of Proposition 1 we shall need the following result, which comes from the book [6]. Let V denote the vector space $\mathbb{R}^m$ with the standard action of the group G^1_m and let

$$V_{k,l} = \underbrace{V \times \ldots \times V}_{k \text{ times}} \times \underbrace{V^* \times \ldots \times V^*}_{l \text{ times}} .$$

Let $\langle\ ,\ \rangle : V \times V^* \to \mathbb{R}$ be the evaluation map $\langle x, y\rangle = y(x)$.

LEMMA. (a) *All G^1_m-equivariant maps $V_{k,l} \to V$ are of the form*

$$\sum_{\beta=1}^{k} g_\beta(\langle x_\alpha, y_\lambda\rangle) x_\beta$$

with any smooth functions $g_\beta : \mathbb{R}^{kl} \to \mathbb{R}$.

(b) *All G^1_m-equivariant maps $V_{k,l} \to V^*$ are of the form*

$$\sum_{\mu=1}^{l} g_\mu(\langle x_\alpha, y_\lambda\rangle) y_\mu$$

with any smooth functions $g_\mu : \mathbb{R}^{kl} \to \mathbb{R}$.

Proof of Proposition 1. According to the general theory [3], if F and G are two rth order natural bundles, then the natural transformations $F \to G$ are in a canonical bijection with the G^r_m-equivariant maps $F_0\mathbb{R}^m \to G_0\mathbb{R}^m$. Hence we have to determine all G^3_m-equivariant maps of $S = (T^2_1 T^*\mathbb{R}^m)_0$ into $Z = (T^*T^2_1\mathbb{R}^m)_0$. Using standard evaluations we find that the action of G^3_m on S is

$$\begin{aligned}
\overline{\xi}^i &= a^i_j \xi^j , \qquad \overline{X}^i = a^i_{jk}\xi^j\xi^k + a^i_j X^j ,\\
\overline{p}_i &= \widetilde{a}^j_i p_j , \qquad \overline{\pi}_i = \widetilde{a}^j_i \pi_j - a^l_{jk}\widetilde{a}^m_l \widetilde{a}^j_i p_m \xi^k ,\\
\overline{P}_i &= \widetilde{a}^j_i P_j - 2a^l_{jk}\widetilde{a}^m_l\widetilde{a}^j_i \pi_m \xi^k - a^r_{klj}\widetilde{a}^j_i\widetilde{a}^t_r \xi^k\xi^l p_t\\
&\quad - a^l_{jk}\widetilde{a}^m_l\widetilde{a}^j_i p_m X^k + 2\widetilde{a}^n_l a^l_{mk}\widetilde{a}^m_r a^r_{sj}\widetilde{a}^j_i \xi^k\xi^s p_n
\end{aligned} \tag{3}$$

where a^i_j, a^i_{jk}, a^i_{jkl} are the canonical coordinates on G^3_m and $\widetilde{a}^j_i$ is the inverse matrix of a^i_j. Taking into account the natural equivalence $\psi_2 : T^2_1 T^*M \to T^*T^2_1 M$ of Cantrijn *et al.* with equations

$$y^i = \xi^i , \quad z^i = X^i , \quad \tau_i = p_i , \quad \varrho_i = 2\pi_i , \quad \sigma_i = P_i , \tag{4}$$

we obtain from (3) the action of G^3_m on Z. The coordinate form of any map $S \to Z$ is

$$\begin{gathered}
y^i = f^i(p,\xi,X,\pi,P), \quad z^i = g^i(p,\xi,X,\pi,P), \quad \sigma_i = h_i(p,\xi,X,\pi,P) ,\\
\varrho_i = l_i(p,\xi,X,\pi,P), \quad \tau_i = t_i(p,\xi,X,\pi,P) .
\end{gathered}$$

First we discuss f^i. The equivariance of f^i with respect to the kernel of the jet projection $G^3_m \to G^2_m$ leads to

$$f^i(p_j,\xi^j,X^j,\pi_j,P_j) = f^i(p_j,\xi^j,X^j,\pi_j,P_j - a^r_{klj}\xi^k\xi^l p_r) .$$

This implies that f^i is independent of P_i. Now it will be useful to distinguish two cases according to the dimension m of the manifold M.

Consider first the case $m \geq 2$. Taking into account the equivariance of f^i with respect to the linear group $G_m^1 \subset G_m^3$ we obtain

$$a_j^i f^j(p_j, \xi^j, X^j, \pi_j) = f^i(\widetilde{a}_j^k p_k, a_k^j \xi^k, a_k^j X^k, \widetilde{a}_j^k \pi_k),$$

so that $f^i(p, \xi, X, \pi)$ is a G_m^1-equivariant map $\mathbb{R}^m \times \mathbb{R}^m \times \mathbb{R}^{m*} \times \mathbb{R}^{m*} \to \mathbb{R}^m$. By our Lemma,

$$f^i(p, \xi, \pi, X) = \varphi(p_j\xi^j, p_jX^j, \pi_j\xi^j, \pi_jX^j)\xi^i + \psi(p_j\xi^j, p_jX^j, \pi_j\xi^j, \pi_jX^j)X^i \tag{5}$$

where φ and ψ are arbitrary two smooth functions of four variables. One calculates easily that the expressions I and J given by (1) are invariants with respect to the group G_m^2. Replace (5) by

$$f^i = \varphi(I, J, p_jX^j - \pi_j\xi^j, \pi_jX^j)\xi^i + \psi(I, J, p_jX^j - \pi_j\xi^j, \pi_jX^j)X^i .$$

Then the equivariance of f^i with respect to the kernel of the jet projection $G_m^2 \to G_m^1$ reads

$$\begin{aligned} &\varphi(I, J, p_jX^j - \pi_j\xi^j, \pi_jX^j)\xi^i + \psi(I, J, p_jX^j - \pi_j\xi^j, \pi_jX^j)X^i \\ &= \varphi(I, J, p_j\overline{X}^j - \overline{\pi}_j\xi^j, \overline{\pi}_j\overline{X}^j)\xi^i + \psi(I, J, p_j\overline{X}^j - \overline{\pi}_j\xi^j, \overline{\pi}_j\overline{X}^j)\overline{X}^i \end{aligned} \tag{6}$$

where $\overline{X}^i = X^i + a_{jk}^i\xi^j\xi^k$ and $\overline{\pi}_i = \pi_i - a_{ik}^j p_j\xi^k$. Setting $\xi = (1, 0, \dots, 0)$, $X = (0, 1, 0, \dots, 0)$ and $i = 1$ in (6) we obtain

$$\begin{aligned} &\varphi(p_1, p_2 + \pi_1, p_2 - \pi_1, \pi_2) \\ &= \varphi(p_1, p_2 + \pi_1, p_2 - \pi_1 + 2a_{11}^j p_j, \pi_2 - a_{21}^j p_j + \pi_j a_{11}^j - a_{11}^k a_{k1}^j p_j) \\ &\quad + \psi(p_1, p_2 + \pi_1, p_2 - \pi_1 + 2a_{11}^j p_j, \pi_2 - a_{21}^j p_j + \pi_j a_{11}^j - a_{11}^k a_{k1}^j p_j)a_{11}^1 . \end{aligned} \tag{7}$$

If all a_{jk}^i except a_{11}^2 and a_{21}^1 are zero, then (7) reads

$$\begin{aligned} &\varphi(p_1, p_2 + \pi_1, p_2 - \pi_1, \pi_2) \\ &= \varphi(p_1, p_2 + \pi_1, p_2 - \pi_1 + 2a_{11}^2 p_2, \pi_2 - a_{21}^1 p_1 + \pi_2 a_{11}^2 - a_{11}^2 a_{21}^1 p_1) . \end{aligned} \tag{8}$$

Putting $a_{11}^2 = 0$ we get

$$\varphi(p_1, p_2 + \pi_1, p_2 - \pi_1, \pi_2) = \varphi(p_1, p_2 + \pi_1, p_2 - \pi_1, \pi_2 - a_{21}^1 p_1) .$$

This implies that φ does not depend on the fourth variable . Then (8) with arbitrary a_{11}^2 gives $\varphi = \varphi(I, J)$.

Further, let $a_{11}^1 = 1$ and let the other a's in (7) be zero. Then

$$0 = \psi(p_1, p_2 + \pi_1, p_2 - \pi_1 + 2p_1, \pi_2 + \pi_1 - p_1) . \tag{9}$$

The components of ψ in (9) are linearly independent functions, so that $\psi = 0$. We have thus deduced that

$$f^i = F(I, J)\xi^i \tag{10}$$

with an arbitrary smooth function $F:\mathbb{R}^2\to\mathbb{R}$.

Quite analogously one can prove that

$$t_i = G(I,J)p_i \tag{11}$$

where G is another smooth function of two variables.

Now write

$$g^i(p,\xi,X,\pi,P) = F^2(I,J)X^i + \overline{g}^i(p,\xi,X,\pi,P)$$

with F taken from (10). Applying the equivariance of g^i with respect to the whole group G^3_m we find

$$\begin{aligned} a^i_{jk}F^2(I,J)\xi^j\xi^k &+ a^i_jF^2(I,J)X^j + a^i_j\overline{g}^j(p,\xi,X,\pi,P) \\ &= F^2(I,J)(a^i_{jk}\xi^j\xi^k + a^i_jX^j) + \overline{g}^i(\overline{p},\overline{\xi},\overline{X},\pi,\overline{P})\,. \end{aligned}$$

We see that $\overline{g}^i$ has the same transformation law as f^i, so that $\overline{g}^i(p,\xi,X,\pi,P) = H(I,J)\xi^i$ and

$$g^i = F^2(I,J)X^i + H(I,J)\xi^i\,. \tag{12}$$

Consider now the map l_i and set

$$l_i(p,\xi,X,\pi,P) = 2F(I,J)G(I,J)\pi_i + \bar{l}_i(p,\xi,X,\pi,P)\,.$$

Using equivariance we get

$$\begin{aligned} 2\widetilde{a}^j_iF(I,J)G(I,J)\pi_j &+ \widetilde{a}^j_i\bar{l}_j(p,\xi,X,\pi,P) - 2a^l_{jk}\widetilde{a}^j_i\widetilde{a}^m_lF(I,J)G(I,J)p_m\xi^k \\ &= 2F(I,J)G(I,J)(\widetilde{a}^j_i\pi_j - a^l_{jk}\widetilde{a}^m_l\widetilde{a}^j_ip_m\xi^k) + \bar{l}_i(\overline{p},\overline{\xi},\overline{X},\overline{\pi},\overline{P})\,. \end{aligned}$$

Quite similarly to (10) and (11) we then deduce $\bar{l}_i(p,\xi,X,\pi,P) = M(I,J)p_i$, so that

$$l_i = 2F(I,J)G(I,J)\pi_i + M(I,J)p_i\,. \tag{13}$$

Finally, assume h_i has the form

$$\begin{aligned} h_i(p,\xi,X,\pi,P) = F^2(I,J)G(I,J)P_i &+ [F(I,J)M(I,J) \\ &+ H(I,J)G(I,J)]\pi_i + \overline{h}_i(p,\xi,X,\pi,P)\,. \end{aligned}$$

Applying the same procedure as for g^i and l_i we obtain $\overline{h}^i(p,\xi,X,\pi,P) = N(I,J)p_i$, i.e.

$$\begin{aligned} h_i = F^2(I,J)G(I,J)P_i &+ [F(I,J)M(I,J) + H(I,J)G(I,J)]\pi_i \\ &+ N(I,J)p_i\,. \end{aligned} \tag{14}$$

Thus, if the dimension m of the manifold M is ≥ 2, then (10)–(14) prove our proposition.

It remains to discuss the case of one-dimensional manifolds. The fact that the map $f(p,\xi,X,\pi,P)$ does not depend on P can be derived in the same way as above. Denote by (a_1, a_2, a_3) the coordinates on G_1^3. We shall only need the following equations of the action of G_1^3 on S and Z:

$$\overline{\xi} = a_1\xi, \quad \overline{p} = \frac{1}{a_1}p, \quad \overline{X} = a_2\xi^2 + a_1X, \quad \overline{y} = a_1y\,,$$
$$\overline{\pi} = \frac{1}{a_1}\pi - \frac{a_2}{a_1^2}p\xi\,.$$

Take any $u \in \mathbb{R}^*$, so that $\overline{u} = \frac{1}{a_1}u$. Then $f(p,\xi,X,\pi)u$ is a G_1^2-invariant function. Let $I = p\xi$, $J = pX + \pi\xi$ and $K = u\xi$.

For any G_1^2-invariant function $F(p,\xi,\pi,X,u)$ define a smooth function $\psi(x,y,z) = F(x,1,y,0,z)$. We claim that

$$F(p,\xi,\pi,X,u) = \psi(I,J,K)\,. \tag{15}$$

Indeed, since $F(p,\xi,\pi,X,u)$ is G_1^2-invariant, in the case $\xi \neq 0$ for $a_1 = \xi$, $a_2 = 0$ we have

$$\psi(\xi p, \xi\pi, \xi u) = F(\xi p, 1, \xi\pi, 0, \xi u) = F(p,\xi,\pi,0,u)\,.$$

Further, set $a_2 = -X/\xi^2$, $a_1 = 1$. Then by invariance $F(p,\xi,\pi,X,u) = F(p,\xi,\pi + Xp/\xi, 0, u) = \psi(\xi p, \xi\pi + Xp, \xi u) = \psi(I,J,K)$. Hence we have proved that (15) holds on the dense subset $\xi \neq 0$, so by continuity it holds everywhere.

Now we complete the proof of our proposition. By (15) we have

$$f(p,\xi,\pi,X)u = \psi(I,J,K)\,.$$

Differentiating this with respect to u we obtain

$$f(p,\xi,\pi,X) = \frac{\partial\psi(I,J,K)}{\partial z}\cdot\xi$$

where z denotes the third variable of $\psi(I,J,K)$. Setting $u = 0$ on the right side we get

$$f(p,\xi,\pi,X) = \varphi(I,J)\cdot\xi$$

where $\varphi(x,y) = \partial\psi(x,y,0)/\partial z$. This implies that for $m = 1$ the map f is of the form (10) as well. One finds easily that (11)–(14) are also true in this case. ■

2. Geometric interpretation. The canonical isomorphism $\psi_2 : T_1^2T^*M \to T^*T_1^2M$ of Cantrijn *et al.* [1] corresponds to the constant values $F = 1$, $G = 1$, $H = 0$, $M = 0$, $N = 0$ in (2). We first give another simple geometric construction of this isomorphism. Gollek introduced a canonical isomorphism $\varkappa : T_k^rT_l^sM \to T_l^sT_k^rM$ which can be viewed as a generalization of the canonical involution $TTM \to TTM$ [2]. Let $q : T^*M \to M$ be the bundle projection and let $\varkappa_2$ be the above isomorphism $TT_1^2M \to T_1^2TM$. The map $\varkappa_2$ has a simple geometric interpretation. Every $C \in TT_1^2M$

is of the form $C = (\partial/\partial t)|_0 j_0^2\gamma(t,\tau)$, where γ is the map $\mathbb{R}\times\mathbb{R}\to M$, $(t,\tau)\mapsto\gamma(t,\tau)$, and j_0^2 means the partial jet with respect to the second variable. Then $\varkappa_2(C)\in T_1^2TM$ is defined by taking the partial jets in opposite order, i.e. $\varkappa_2(C)=j_0^2((\partial/\partial t)|_0\gamma(t,\tau))$. Every $A\in T_1^2T^*M$ is a 2-velocity of a curve $\alpha(t)=(x^i(t),a_i(t))$ in T^*M. Let $v\in T_1^2M$ be the point $T_1^2q(A)$. If $B\in T_vT_1^2M$, then $\varkappa_2(B)$ is a 2-velocity of a curve $\beta(t)=(x^i(t),b^i(t))$ in TM. Hence we can evaluate $\langle\alpha(t),\beta(t)\rangle$ for every t and the expression

$$\frac{d^2}{dt^2}\bigg|_0\langle\alpha(t),\beta(t)\rangle$$

depends only on A and B. Therefore it determines a linear map $T_vT_1^2M\to\mathbb{R}$, i.e. an element of $T^*T_1^2M$.

Now we present a geometric interpretation of the result (2). We shall proceed in four steps.

1. We can define the following multiplication by real numbers on the bundle T_1^2N:

$$k\cdot(x^\alpha,y^\alpha,z^\alpha)=(x^\alpha,ky^\alpha,k^2z^\alpha)\,. \tag{16}$$

There is a canonical inclusion $T_1^2N\to TTN$, $(x^\alpha,y^\alpha,z^\alpha)\mapsto(x^\alpha,y^\alpha,y^\alpha,z^\alpha)$, and the space TTN carries two vector bundle structures. Taking any $(x^\alpha,y^\alpha,y^\alpha,z^\alpha)\in TTN$, we can multiply it by k with respect to the first structure. We obtain

$$(x^\alpha,y^\alpha,ky^\alpha,kz^\alpha)\,. \tag{17}$$

Further, multiplying (17) by k with respect to the second structure gives $(x^\alpha,ky^\alpha,ky^\alpha,k^2z^\alpha)$. This defines the multiplication (16), which we denote by $A\mapsto k\cdot A$. Another way of defining the multiplication (16) on T_1^2N is to use the reparametrization $x^i(t)\mapsto x^i(kt)$.

Take any element $A=(x^i,p_i,\xi^i,X^i,\pi_i,P_i)$ in $T_1^2T^*M$. Evaluating the result of multiplication of A by F, we get $F\cdot A=(x^i,p_i,F\xi^i,F^2X^i,F\pi_i,F^2P_i)$. Next, we can transform this into $T^*T_1^2M$ by means of the canonical transformation ψ_2. The coordinates of $\psi_2(F\cdot A)$ are

$$y^i=F\xi^i\,,\quad z^i=F^2X^i\,,\quad \tau_i=p_i\,,\quad \varrho_i=2F\pi_i\,,\quad \sigma_i=F^2P_i\,.$$

Moreover, multiplying $\psi_2(F\cdot A)$ by G with respect to the vector bundle structure of $T^*T_1^2M$ we obtain an element

$$G\psi_2(F\cdot A) \tag{18}$$

of $T^*T_1^2M$ with coordinates

$$y^i=F\xi^i\,,\quad z^i=F^2X^i\,,\quad \tau_i=Gp_i\,,\quad \varrho_i=2FG\pi_i\,,\quad \sigma_i=F^2GP_i\,.$$

2. The bundle projection $q : T^*M \to M$ determines the projection $r_1 : T_1^2T^*M \to T_1^2M$, $r_1 = T_1^2q$. Further, let $r_2 : T_1^2T^*M \to TT^*M$ be the jet projection and let $r_3 : T_1^2T^*M \to T^*M$ be the bundle projection.

Denote by s the isomorphism $TT^*M \to T^*TM$ of Modugno and Stefani [7]. We recall the coordinate expression of s. Having the canonical coordinates x^i, $\zeta^i = dx^i$ on $T\mathbb{R}^m$, the expression $\alpha_i dx^i + \beta_i d\zeta^i$ determines the additional coordinates α_i, β_i on $T^*T\mathbb{R}^m$. Further, let x^i, p_i, $\xi^i = dx^i$, $\pi_i = dp_i$ be the canonical coordinates on $TT^*\mathbb{R}^m$. Then the equations of the isomorphism s are [5]

$$\zeta^i = \xi^i, \quad \alpha_i = \pi_i, \quad \beta_i = p_i\,.$$

Moreover, there is an inclusion $i : T_1^2M \times_{TM} T^*TM \to T^*T_1^2M$,

$$(x^i, y^i, z^i, \alpha_i, \beta_i) \mapsto (x^i = x^i, y^i = y^i, z^i = z^i, \sigma_i = \alpha_i, \varrho_i = \beta_i, \tau_i = 0)\,.$$

Having an arbitrary element $A = (x^i, p_i, \xi^i, X^i, \pi_i, P_i)$ in $T_1^2T^*M$, we can evaluate $i(r_1F \cdot A, s(r_2F \cdot A))$. Next, multiplying this by the function M on the vector bundle $T^*T_1^2M$ we obtain an element

$$Mi(r_1F \cdot A, s(r_2F \cdot A)) \tag{19}$$

of $T^*T_1^2M$ with coordinates

$$y^i = F\xi^i\,, \quad z^i = F^2X^i\,, \quad \tau_i = 0\,, \quad \varrho_i = Mp_i\,, \quad \sigma_i = FM\pi_i\,.$$

3. Denote by j the inclusion $T_1^2M \times_M T^*M \to T^*T_1^2M$,

$$(x^i, y^i, z^i, \alpha_i) \mapsto (x^i = x^i, y^i = y^i, z^i = z^i, \sigma_i = \alpha_i, \varrho_i = 0, \tau_i = 0)\,.$$

Applying a similar procedure to step 2, we associate to any $A \in T_1^2T^*M$ an element

$$Nj(r_1F \cdot A, r_3F \cdot A) \tag{20}$$

of $T^*T_1^2M$. The coordinate form of (20) is

$$y^i = F\xi^i\,, \quad z^i = F^2X^i\,, \quad \tau_i = 0\,, \quad \varrho_i = 0\,, \quad \sigma_i = Np_i\,.$$

4. It is well known that $T_1^2M \to TM$ is an affine bundle associated to the pullback p_M^*TM of $TM \to M$ over $p_M : TM \to M$. In particular, $T_1^2T^*M \to TT^*M$ is an affine bundle whose associated vector bundle is the pullback of $TT^*M \to T^*M$ over $p_{T^*M} : TT^*M \to T^*M$. Hence we have defined the addition of vectors in TT^*M to points in $T_1^2T^*M$:

$$(x^i, \xi^i, X^i, p_i, \pi_i, P_i) + (x^i, p_i, v^i, u_i) = (x^i, \xi^i, X^i + v^i, p_i, \pi_i, P_i + u_i)\,.$$

Using the canonical isomorphism $\psi_2 : T_1^2T^*M \to T^*T_1^2M$ we can transform this addition in the affine bundle $T_1^2T^*M$ to an addition $\oplus$ in the bundle $T^*T_1^2M$:

$$(x^i, y^i, z^i, \tau_i, \varrho_i, \sigma_i) \oplus (x^i, v^i, \tau_i, u_i) = (x^i, y^i, z^i + v^i, \tau_i, \varrho_i, \sigma_i + u_i)\,.$$

Now we can complete the geometric interpretation of (2). Given an arbitrary $A = (x^i, p_i, \xi^i, X^i, \pi_i, P_i) \in T_1^2 T^* M$ we have constructed geometrically three elements (18), (19) and (20) in $T^* T_1^2 M$. Then their sum

$$B = G\psi_2(F \cdot A) + Mi(r_1 F \cdot A, s(r_2 F \cdot A)) + Nj(r_1 F \cdot A, r_3 F \cdot A)$$

with respect to the vector bundle structure of $T^* T_1^2 M$ has coordinates

$$x^i = x^i\,, \quad y^i = F\xi^i\,, \quad z^i = F^2 X^i\,, \quad \tau_i = Gp_i\,,$$
$$\varrho_i = 2FG\pi_i + Mp_i\,, \quad \sigma_i = F^2 GP_i + FM\pi_i + Np_i\,.$$

Taking further the vector $(x^i, \xi^i, p_i, \pi_i) \in TT^*M$ and multiplying by G in the vector bundle structure $TT^*M \to TM$ we get $(x^i, \xi^i, Gp_i, G\pi_i)$. Moreover, multiplying this by H in $TT^*M \to T^*M$ we obtain $C = (x^i, H\xi^i, Gp_i, HG\pi_i)$. Finally, the sum $B \oplus C$ gives $(x^i, F\xi^i, F^2 X^i + H\xi^i, Gp_i, 2FG\pi_i + Mp_i, F^2 Gp_i + FM\pi_i + Np_i + HG\pi_i)$. This corresponds to (2).

3. A geometric characterization of the isomorphism ψ_2. The natural equivalence $s : TT^*M \to T^*TM$ of Modugno and Stefani can be distinguished among all natural transformations by an explicit geometric construction [5]. We show that a similar result is true for the natural equivalence $\psi_2 : T_1^2 T^* M \to T^* T_1^2 M$ of Cantrijn *et al.*

Every vector field ξ on the manifold M induces the flow prolongation

$$\mathcal{T}_1^2 \xi = \frac{\partial}{\partial t}\bigg|_0 (T_1^2 \exp t\xi)$$

on $T_1^2 M$. Further, if $\omega : M \to T^*M$ is any 1-form on M, then $\langle \omega, \xi \rangle : M \to \mathbb{R}$ and we can construct $T_1^2 \langle \omega, \xi \rangle : T_1^2 M \to T_1^2 \mathbb{R}$. Let $\delta_1 \langle \omega, \xi \rangle$ or $\delta_2 \langle \omega, \xi \rangle$ be the second and third component of the map $T_1^2 \langle \omega, \xi \rangle$, respectively. We have $T_1^2 \omega : T_1^2 M \to T_1^2 T^* M$, so that $\psi_2 T_1^2 \omega : T_1^2 M \to T^* T_1^2 M$ is a 1-form on $T_1^2 M$. Hence we can evaluate $\langle \psi_2 T_1^2 \omega, \mathcal{T}_1^2 \xi \rangle : T_1^2 M \to \mathbb{R}$.

Proposition 2. *ψ_2 is the only natural transformation $T_1^2 T^* \to T^* T_1^2$ over the identity transformation of T_1^2 satisfying*

$$\langle \psi_2 T_1^2 \omega, \mathcal{T}_1^2 \xi \rangle = \delta_2 \langle \omega, \xi \rangle \tag{21}$$

for every vector field ξ and every 1-form ω.

Proof. Let $x^i = x^i$, $p_i = a_i(x)$ be the coordinate expression of ω. Then the coordinate expression of $T_1^2 \omega$ is

$$x^i = x^i, \quad p_i = a_i(x)\,, \quad \xi^i = \xi^i\,, \quad X^i = X^i\,,$$
$$\pi_i = \frac{\partial a_i}{\partial x^j}\xi^j\,, \quad P_i = \frac{\partial^2 a_i}{\partial x^j \partial x^k}\xi^j \xi^k + \frac{\partial a_i}{\partial x^j} X^j\,.$$

Applying transformation (2) with $F=1$, $H=0$ we get

$$x^i = x^i\,, \quad y^i = \xi^i\,, \quad z^i = X^i\,, \quad \tau_i = Ga_i\,, \quad \varrho_i = 2G\frac{\partial a_i}{\partial x^j}\xi^j + Ma_i\,,$$

$$\sigma_i = G\frac{\partial^2 a_i}{\partial x^j \partial x^k}\xi^j\xi^k + G\frac{\partial a_i}{\partial x^j}X^j + M\frac{\partial a_i}{\partial x^j}\xi^j + Na_i\,.$$

The fact that $F=1$, $H=0$ follows from the assumption that our natural transformation is over the identity of T_1^2. Further, the coordinate expression of the flow prolongation $\mathcal{T}_1^2\xi$ is

$$dx^i = b^i(x)\,, \quad dy^i = \frac{\partial b^i}{\partial x^j}\xi^j\,, \quad dz^i = \frac{\partial^2 b^i}{\partial x^j \partial x^k}\xi^j\xi^k + \frac{\partial b^i}{\partial x^j}X^j\,,$$

provided the $b^i(x)$ are the coordinates of a vector field ξ. We can write

$$\delta_1\langle\omega,\xi\rangle = \left(\frac{\partial a_i}{\partial x^j}b^i + a_i\frac{\partial b^i}{\partial x^j}\right)\xi^j\,.$$

Hence (21) reads

$$\begin{aligned} G\left(\frac{\partial^2 a_i}{\partial x^j \partial x^k}\ \xi^j\xi^k + \frac{\partial a_i}{\partial x^j}X^j\right)b^i + M\frac{\partial a_i}{\partial x^j}\xi^j b^i + Na_ib^i + Ma_i\frac{\partial b^i}{\partial x^j}\xi^j \\ + 2G\frac{\partial a_i}{\partial x^j}\xi^j\frac{\partial b^i}{\partial x^k}\xi^k + Ga_i\frac{\partial^2 b^i}{\partial x^j \partial x^k}\xi^j\xi^k + Ga_i\frac{\partial b^i}{\partial x^j}X^j \\ = \frac{\partial^2 a_i}{\partial x^j \partial x^k}b^i\xi^j\xi^k + 2\frac{\partial a_i}{\partial x^j}\frac{\partial b^i}{\partial x^k}\xi^j\xi^k + a_i\frac{\partial^2 b^i}{\partial x^j \partial x^k}\xi^j\xi^k \\ + \frac{\partial a_i}{\partial x^j}b^iX^j + a_i\frac{\partial b^i}{\partial x^j}X^j\,. \end{aligned}$$

This implies $G=1$, $M=0$, $N=0$. ■

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