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ON METRIC PRODUCTS

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0. Introduction. For a given pair of metric spaces $\boldsymbol{X}_i = (X_i, \rho_i)$, $i = 1, 2$, there are various possible product metrics, i.e. metrics which induce the product topology in $X_1 \times X_2$. Evidently, for the multiplicativity of a topological property the choice of a product metric is inessential. But, in general, it is essential for the multiplicativity of a metric property.

Following the idea of Olędzki and Spież [4], we are concerned with metrics induced by functions from $(\mathbb{R}^+)^2$ to $\mathbb{R}^+$. Five families ($\mathcal{F}_0$, $\mathcal{F}_1$, $\widetilde{\mathcal{F}}_1$, $\mathcal{F}_2$, and $\mathcal{F}_2'$) of such functions are defined in Section 1; their role is described in Section 2. The next two sections, 3 and 4, are devoted to $\mathcal{F}$-multiplicativity of different classes of metric spaces for $\mathcal{F}$ being one of the families $\mathcal{F}_1$, $\widetilde{\mathcal{F}}_1$, $\mathcal{F}_2$, and $\mathcal{F}_2'$. It seems interesting that to decide whether a given class $\mathcal{M}$ is f-multiplicative or not, it often suffices to examine the space $(\mathbb{R}^4, \hat{f}(\rho, \rho))$, where ρ is the Euclidean metric in $\mathbb{R}^2$ and $\hat{f}(\rho, \rho)$ is the induced metric in $\mathbb{R}^4$ (compare 4.3 and 4.8).

We use the terminology and notation of [3]; in particular, a space (X, ρ) is said to be *strongly arcwise connected* if any two distinct points $x, y \in X$ can be joined in X by an arc with a finite length; let ρ^* denote the intrinsic metric determined by ρ in a strongly arcwise connected space (X, ρ), i.e. $\rho^*(x, y)$ is the infimum of the lengths of all arcs joining x and y in (X, ρ). By $B_\rho(a, \varepsilon)$ we denote the ball in (X, ρ) with centre a and radius ε, i.e.

$$B_\rho(a, \varepsilon) := \{x \in X; \rho(x, a) < \varepsilon\};$$

by $M_\rho(a, b)$ we denote the set of midpoints of the pair (a, b):

$$M_\rho(a, b) := \{x \in X; \rho(a, x) = \tfrac{1}{2}\rho(a, b) = \rho(x, b)\}.$$

We are concerned with the following classes of metric spaces:

FC — the class of *finitely compact* spaces ($\boldsymbol{X} \in$ FC iff every bounded sequence in $\boldsymbol{X}$ has a convergent subsequence; compare [1]),

GA — the class of *geometrically acceptable* spaces ($(X, \rho) \in$ GA iff (X, ρ) is strongly arcwise connected and ρ^* is topologically equivalent to ρ; compare [2] and [3]), IM — the class of spaces *with intrinsic metrics* ($(X,\rho) \in \mathrm{IM}$ iff $\rho^* = \rho$),

MC — the class of *metrically convex* spaces ($\boldsymbol{X} \in \mathrm{MC}$ iff every pair of points a, b in X can be joined by a metric segment, i.e. by an isometric image of the interval $[0, \rho(a,b)]$; compare [1], [3]),

SMC — the class of *strongly metrically convex* spaces ($\boldsymbol{X} \in \mathrm{SMC}$ iff every pair of points of X can be joined by a unique metric segment),

MidC — the class of *Mid-convex* spaces ($(X,\rho) \in \mathrm{MidC}$ iff $M_\rho(a,b) \neq \emptyset$ for every $a, b \in X$),

SMidC — the class of *strongly Mid-convex* spaces ($(X,\rho) \in \mathrm{SMidC}$ iff $M_\rho(a,b)$ is a singleton for every $a, b \in X$, i.e. M_ρ is an operation),

NL — the class of linear spaces with metric induced by a norm,

SNL — the subclass of NL consisting of spaces with strictly convex balls (i.e. balls with no segments on the boundary).

Let us note the following

0.1. Lemma. $\mathrm{MC} \cap \mathrm{SMidC} = \mathrm{SMC}$.

Proof. The inclusion $\supset$ is evident. We prove $\subset$. Let $\boldsymbol{X} = (X,\rho)$ be a metrically convex and strongly Mid-convex metric space. Let L_1 and L_2 be metric segments in $\boldsymbol{X}$ with endpoints a,b. Then, evidently, there is a set $A \subset L_1 \cap L_2$ which is dense in both arcs L_1 and L_2 (A is obtained by iterating the midpoint operation M_ρ). Thus $L_1 = L_2$. ∎

1. Some sets of real functions. Let $\mathbb{R}^+$ be the set of non-negative reals and let $\sim$ be the proportionality relation in $\mathbb{R}^2$. We shall deal with the following conditions on $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$:

F.0. $|s_i - t_i| \le r_i \le s_i + t_i$ for $i = 1,2 \Rightarrow f(r_1,r_2) \le f(s_1,s_2) + f(t_1,t_2)$ for every $r_i, s_i, t_i \in \mathbb{R}^+$;

F.1. $f(t_1,t_2) = 0 \Leftrightarrow t_1 = t_2 = 0$;

F.2. f is subadditive, i.e. $f(t+s) \le f(t) + f(s)$ for every $t, s \in (\mathbb{R}^+)^2$;

F.2′. f is strictly subadditive, i.e. f is subadditive and

$$f(t+s) = f(t) + f(s) \;\Rightarrow\; t \sim s \text{ for every } t, s \in (\mathbb{R}^+)^2\,;$$

F.3. f is totally increasing, i.e. for every $r = (r_1, r_2)$ and $t = (t_1, t_2)$,

$$r_i \le t_i \text{ for } i = 1,2 \;\Rightarrow\; f(r) \le f(t)\,;$$

F.4.1. f is continuous at $(0,0)$;

F.4.2. f is homogeneous, i.e. for every $t \in (\mathbb{R}^+)^2$ and $\alpha \in \mathbb{R}^+$,

$$f(\alpha t) = \alpha f(t)\,.$$

Let us define five sets of functions:

$\mathcal{F}_0 := \{f : (\mathbb{R}^+)^2 \to \mathbb{R}^+;\ f$ satisfies F.0 and F.1$\}$,
$\mathcal{F}_i := \{f \in \mathcal{F}_0;\ f$ satisfies F.4.i.$\}$ for $i = 1, 2$,
$\widetilde{\mathcal{F}}_1 := \{f : (\mathbb{R}^+)^2 \to \mathbb{R}^+;\ f$ satisfies F.1, F.2, F.3, F.4.1$\}$,
$\mathcal{F}'_2 := \{f : (\mathbb{R}^+)^2 \to \mathbb{R}^+;\ f$ satisfies F.1, F.2$'$, F.3, F.4.2$\}$.

The set $\mathcal{F}_2$ can be characterized as follows:

1.1. $\mathcal{F}_2 = \{f : (\mathbb{R}^+)^2 \to \mathbb{R};\ f$ *satisfies* F.1, F.2, F.3, F.4.2$\}$ (1).

P r o o f. The inclusion $\supset$ is obvious. To verify $\subset$ it suffices to prove

$$\text{F.0} \wedge \text{F.4.2} \Rightarrow \text{F.2} \wedge \text{F.3}\,.$$

Taking $r = s + t$ in F.0, we get F.2. To obtain F.3, we assume $r_i \le u_i$ for $i = 1, 2$ and take $s_i = t_i = \frac{1}{2}u_i$ in F.0. ∎

Using 1.1, we easily obtain

1.2. $\mathcal{F}'_2 \subset \mathcal{F}_2 \subset \widetilde{\mathcal{F}}_1 \subset \mathcal{F}_1 \subset \mathcal{F}_0$.

It can be shown that all the inclusions in 1.2 are proper. We shall need the following three lemmas:

1.3. Lemma. *If* $f \in \mathcal{F}_1$, *then*

(i) f *is continuous*;
(ii) *for every* $(t^{(n)})_{n\in\mathbb{N}}$ *in* $(\mathbb{R}^+)^2$, $\lim_n f(t^{(n)}) = 0 \Rightarrow \lim_n t^{(n)} = (0, 0)$.

P r o o f. (i) By F.0 it follows that

$$|t_i - s_i| \le r_i \le t_i + s_i \quad \text{for } i = 1, 2$$
$$\Rightarrow |f(t_1, t_2) - f(s_1, s_2)| \le f(r_1, r_2) \le f(t_1, t_2) + f(s_1, s_2)\,.$$

Setting $r_i = |t_i - s_i|$, we obtain

$$|f(t_1, t_2) - f(s_1, s_2)| \le f(|t_1 - s_1|, |t_2 - s_2|) \quad \text{for every } (t_1, t_2), (s_1, s_2) \in (\mathbb{R}^+)^2\,. \tag{1}$$

Take $(s_1, s_2) \in (\mathbb{R}^+)^2$ and $\varepsilon > 0$. Since f is continuous at $(0, 0)$, by F.1 there exist $\delta_1, \delta_2 > 0$ such that

$$\forall t_1, t_2 \in \mathbb{R}^+ \quad |t_i - s_i| < \delta_i \text{ for } i = 1, 2 \Rightarrow f(|t_1 - s_1|, |t_2 - s_2|) < \varepsilon\,.$$

Thus (1) yields the continuity at (s_1, s_2).

(ii) Let

$$\lim_n f(t_1^{(n)}, t_2^{(n)}) = 0 \tag{2}$$

(1) By 1.1, $\mathcal{F}_2$ is the set of functions considered in [4], p. 245.

and suppose that $((t_1^{(n)}, t_2^{(n)}))_{n\in\mathbb{N}}$ is not convergent to (0,0). Then we can assume that $(t_1^{(n)})_{n\in\mathbb{N}}$ is either divergent to ∞ or convergent to $t_1 \neq 0$, whence

$$\exists s_1\ \exists n_0\ \forall n > n_0 \quad 0 < s_1 \le 2t_1^{(n)}\,. \tag{3}$$

Thus, by F.0, $f(s_1, 0) \le 2f(t_1^{(n)}, t_2^{(n)})$, which, by (2) and (3), contradicts F.1. ∎

1.4. Lemma. *If f is continuous and subadditive, then the following conditions are equivalent*:

(i) f *is homogeneous*;
(ii) $f(\frac{1}{2}t) = \frac{1}{2}f(t)$ *for every* $t \in (\mathbb{R}^+)^2$.

Proof. The implication (i)⇒(ii) is obvious.

Assume (ii); to prove (i) it suffices to show that for every $\alpha \in \mathbb{R}^+$

$$f(\alpha t) \le \alpha f(t) \quad \text{for } t \in (\mathbb{R}^+)^2\,. \tag{1}$$

Let $k \in \mathbb{N}$; since

$$\frac{1}{k} = \sum_{n=1}^{\infty} \frac{\alpha_n}{2^n} \quad \text{for some } \alpha_n \in \{0,1\},\ n \in \mathbb{N}\,,$$

by F.2 and the continuity of f we obtain (1) for α rational. Using again continuity, we get (1) for every $\alpha \in \mathbb{R}^+$. ∎

1.5. Lemma. *For every $f \in \mathcal{F}_2$ the following conditions are equivalent*:

(i) $f \in \mathcal{F}_2'$;
(ii) $r = s + t \wedge f(s) = f(t) = \frac{1}{2}f(r) \Rightarrow s = t = \frac{1}{2}r$, *for all* $r, s, t \in (\mathbb{R}^+)^2$.

Proof. (i)⇒(ii). Suppose

$$r = s + t \text{ and } f(s) = f(t) = \tfrac{1}{2}f(r)\,. \tag{1}$$

Then $f(s+t) = f(s) + f(t)$, whence, by F.2′,

$$s = \alpha t \quad \text{for some } \alpha \in \mathbb{R}^+\,. \tag{2}$$

If $s = (0,0)$ or $t = (0,0)$, then (ii) holds. Let $s \neq (0,0) \neq t$. By F.4.2 and (2), $f(s) = \alpha f(t)$, whence, by F.1, $\alpha = 1$. Thus, by (i) and (2), $s = t = \frac{1}{2}r$.

(ii)⇒(i). First, notice that (ii) implies

$$(3)_\alpha \qquad r = s + t \wedge f(s) = \alpha f(r) \wedge f(t) = (1-\alpha)f(r) \Rightarrow s = \alpha r \wedge t = (1-\alpha)r$$

for every $\alpha \in [0,1]$.

Indeed, (ii) coincides with $(3)_\alpha$ for $\alpha = \frac{1}{2}$. By F.4.2, $(3)_\alpha \Rightarrow (3)_{\alpha/2}$; evidently $(3)_\alpha \Rightarrow (3)_{1-\alpha}$. Thus $(3)_\alpha$ holds for $\alpha = m/2^n$ for $m, n \in \mathbb{N} \cup \{0\}$, whence it holds for every $\alpha \in [0,1]$ because f is continuous.

By 1.1, it remains to prove

$$f(s+t) = f(s) + f(t) \Rightarrow s \sim t\,. \tag{4}$$

Let $f(s+t) = f(s) + f(t)$ and $r = s+t$. Then $f(s) = \alpha f(r)$ for some $\alpha \in [0,1]$; thus, $(3)_\alpha$ yields $s = \alpha r$ and $t = (1-\alpha)r$, which proves (4). ∎

2. Geometric characterizations of $\mathcal{F}_1$, $\mathcal{F}_2$, and $\mathcal{F}'_2$. Every $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ induces the function $\hat{f}$ which assigns to any pair of metrics ρ_1, ρ_2 in X_1, X_2, respectively, the function

$$\hat{f}(\rho_1, \rho_2) = \rho_f : (X_1 \times X_2)^2 \to \mathbb{R}^+$$

defined by the formula

$$\rho_f((x_1, x_2), (y_1, y_2)) := f(\rho_1(x_1, y_1), \rho_2(x_2, y_2))\,.$$

The following two statements characterize $\mathcal{F}_0$ and $\mathcal{F}_1$:

2.1. Theorem. *For every $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ the following conditions are equivalent*:

(i) $f \in \mathcal{F}_0$;

(ii) *for every pair of metric spaces (X_i, ρ_i), $i = 1, 2$, the function $\hat{f}(\rho_1, \rho_2)$ is a metric in $X_1 \times X_2$*;

(iii) *is ρ is the Euclidean metric in $\mathbb{R}^2$, then $\hat{f}(\rho, \rho)$ is a metric in $\mathbb{R}^4$.*

The proof is routine. ∎

As a consequence of 2.1, 1.2, and 1.3(ii), we obtain

2.2. Theorem. *For every $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ the following conditions are equivalent*:

(i) $f \in \mathcal{F}_1$;

(ii) *for every pair of metric spaces (X_i, ρ_i), $i = 1, 2$, the function $\hat{f}(\rho_1, \rho_2)$ is a product metric in $X_1 \times X_2$*;

(iii) *if ρ is the Euclidean metric in $\mathbb{R}^2$, then $\hat{f}(\rho, \rho)$ is a product metric in $\mathbb{R}^4$.*

The next two statements reflect the role of $\mathcal{F}_2$ and $\mathcal{F}'_2$:

2.3. Theorem. *For every $f \in \mathcal{F}_1$ the following conditions are equivalent*:

(i) $f \in \mathcal{F}_2$;

(ii) *for every pair of metric spaces (X_i, ρ_i), $i = 1, 2$,*

$$M_{\rho_1}(a_1, b_1) \times M_{\rho_2}(a_2, b_2) \subset M_{\hat{f}(\rho_1, \rho_2)}((a_1, a_2), (b_1, b_2))$$

for every $a_i, b_i \in X_i$, $i = 1, 2$;

(iii) *if* ρ *is the Euclidean metric in* $\mathbb{R}$, *then*

$$M_\rho(a_1,b_1)\times M_\rho(a_2,b_2)\subset M_{\hat f(\rho,\rho)}((a_1,a_2),(b_1,b_2))$$

for every $a_i,b_i\in\mathbb{R}$, $i=1,2$.

The proof of the implication (i)$\Rightarrow$(ii) is routine; (ii)$\Rightarrow$(iii) is obvious; (iii)$\Rightarrow$(i) follows from 1.3 and 1.4. ∎

2.4. Theorem. *For every* $f\in\mathcal{F}_1$ *the following conditions are equivalent*:

(i) $f\in\mathcal{F}_2'$;

(ii) *for every pair of metric spaces* (X_i,ρ_i), $i=1,2$,

$$M_{\rho_1}(a_1,b_1)\times M_{\rho_2}(a_2,b_2)=M_{\hat f(\rho_1,\rho_2)}((a_1,a_2),(b_1,b_2))$$

for every $a_i,b_i\in X_i$, $i=1,2$;

(iii) *if* ρ *is the Euclidean metric in* $\mathbb{R}$, *then*

$$M_\rho(a_1,b_1)\times M_\rho(a_2,b_2)=M_{\hat f(\rho,\rho)}((a_1,a_2),(b_1,b_2))$$

for every $a_i,b_i\in\mathbb{R}$, $i=1,2$.

Proof. (i)$\Rightarrow$(ii). Let $\rho_f=\hat f(\rho_1,\rho_2)$, $a=(a_1,a_2)$, $b=(b_1,b_2)$. Since $\mathcal{F}_2'\subset\mathcal{F}_2$, by 2.3 it suffices to prove

$$M_{\rho_f}(a,b)\subset M_{\rho_1}(a_1,b_1)\times M_{\rho_2}(a_2,b_2)\,. \tag{1}$$

We can assume $a\neq b$. Take $x=(x_1,x_2)\in M_{\rho_f}(a,b)$; let $s_i=\rho_i(a_i,x_i)$, $t_i=\rho_i(x_i,b_i)$, $r_i=\rho_i(a_i,b_i)$ for $i=1,2$ and $t=(t_1,t_2)$, $s=(s_1,s_2)$, $r=(r_1,r_2)$. Then $r_i=s_i+t_i$ for $i=1,2$ and $f(s)=f(t)=\frac12 f(r)$, whence, by 1.5, $s=t=\frac12 r$. Thus $x_i\in M_{\rho_i}(a_i,b_i)$, which proves (1).

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i). By 2.3 and 1.5, it suffices to prove

$$r=s+t\wedge f(s)=f(t)=\tfrac12 f(r)\Rightarrow s=t=\tfrac12 r\,, \tag{2}$$

for every $r,s,t\in(\mathbb{R}^+)^2$. Take $r,s,t\in(\mathbb{R}^+)^2$ satisfying the antecedent of (2). For $i=1,2$ there exist $a_i,b_i,c_i\in\mathbb{R}$ such that $\rho(a_i,c_i)=s_i$, $\rho(b_i,c_i)=t_i$, and $\rho(a_i,b_i)=r_i$. Let $a=(a_1,a_2)$, $b=(b_1,b_2)$, $c=(c_1,c_2)$. From the assumption on s, t, r it follows that $c\in M_{\hat f(\rho,\rho)}(a,b)$, whence, by (iii), $c_i\in M_\rho(a_i,b_i)$, which proves (2). ∎

3. On f-multiplicativity of some metric properties. Applying 2.1, for arbitrary $f\in\mathcal{F}_0$ we can define the *f-product* $\boldsymbol{X}_1\times_f\boldsymbol{X}_2$ of metric spaces $\boldsymbol{X}_1$, $\boldsymbol{X}_2$:

If $\boldsymbol{X}_i=(X_i,\rho_i)$ for $i=1,2$, then

$$\boldsymbol{X}_1\times_f\boldsymbol{X}_2:=(X_1\times X_2,\hat f(\rho_1,\rho_2))\,.$$

We are only interested in product metrics. Therefore, we admit the following definitions (compare 2.2):

Let $f \in \mathcal{F}_1$. A class $\mathcal{M}$ of metric spaces is *f-multiplicative* if and only if

$$\boldsymbol{X}_1, \boldsymbol{X}_2 \in \mathcal{M} \Rightarrow \boldsymbol{X}_1 \times_f \boldsymbol{X}_2 \in \mathcal{M} \quad \text{for every pair } (\boldsymbol{X}_1, \boldsymbol{X}_2)\,.$$

Let $\mathcal{F} \subset \mathcal{F}_1$. The class $\mathcal{M}$ is *$\mathcal{F}$-multiplicative* whenever $\mathcal{M}$ is f-multiplicative for every $f \in \mathcal{F}$.

Every class $\mathcal{M}$ determines the maximal subfamily of $\mathcal{F}_1$ for which $\mathcal{M}$ is multiplicative:

$$\mathcal{F}_{\mathcal{M}} := \{f \in \mathcal{F}_1;\ \mathcal{M} \text{ is } f\text{-multiplicative}\}\,.$$

Of course, if $\mathcal{M}$ is a topological invariant, then, by 2.2, $\mathcal{M}$ is $\mathcal{F}_1$-multiplicative if and only if $\mathcal{M}$ is f-multiplicative for $f(t_1,t_2) = \sqrt{(t_1)^2 + (t_2)^2}$.

It is easy to prove that

3.1. *The class of complete metric spaces is $\widetilde{\mathcal{F}}_1$-multiplicative.*

Let us notice that

3.2. *The class* FC *of finitely compact spaces is $\mathcal{F}_2$-multiplicative but not $\widetilde{\mathcal{F}}_1$-multiplicative.*

Proof. To prove that FC is $\mathcal{F}_2$-multiplicative it is enough to show that if A is a bounded set in $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2$, then $A \subset A_1 \times A_2$ for some sets A_i bounded in $\boldsymbol{X}_i$ for $i = 1, 2$. Let

(1) $\quad A \subset B_{\hat{f}(\rho_1,\rho_2)}(a,\alpha) \quad$ for some $a = (a_1,a_2) \in X_1 \times X_2$ and $\alpha > 0$.

If

$$\beta = \alpha \max\{(f(1,0))^{-1}, (f(0,1))^{-1}\} \quad \text{and} \quad A_i = B_{\rho_i}(a_i,\beta) \text{ for } i = 1,2\,,$$

then, by F.3 and F.4.2, for every $t_1, t_2 \in \mathbb{R}^+$

$$t_1 f(1,0) \le f(t_1,t_2) \quad \text{and} \quad t_2 f(0,1) \le f(t_1,t_2)\,,$$

whence, by (1), $A \subset A_1 \times A_2$.

To show that FC is not $\widetilde{\mathcal{F}}_1$-multiplicative, consider f defined by the formula

$$f(t_1,t_2) = t_1 + t_2(1+t_2)^{-1}\,.$$

Evidently $f \in \mathcal{F}_1$. The Euclidean line $\boldsymbol{R} = (\mathbb{R},\rho)$ is finitely compact, while $\boldsymbol{R} \times_f \boldsymbol{R}$ is not; indeed, the sequence $((0,n))_{n\in\mathbb{N}}$ is bounded in $(\mathbb{R}^2, \hat{f}(\rho,\rho))$, but has no convergent subsequence. ∎

In our terminology Theorem 3.7 of Oljdzki and Spież [4] can be formulated as follows:

3.3. *If $f \in \mathcal{F}_2$, then for every pair of metric spaces $\boldsymbol{X}_i = (X_i,\rho_i) \in$* GA, *$i = 1,2$, the function $\hat{f}(\rho_1,\rho_2)$ is a product metric in $X_1 \times X_2$ and*

$$(\hat{f}(\rho_1,\rho_2))^* = \hat{f}(\rho_1^*,\rho_2^*)\,.$$

In fact, they proved the following slightly stronger statement:

3.4. *Let* $\boldsymbol{X}_i = (X_i, \rho_i) \in \mathrm{GA}$ *for* $i = 1, 2$.

(i) *If* $f \in \mathcal{F}_1$ *and* $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2 \in \mathrm{GA}$, *then* $(\hat{f}(\rho_1, \rho_2))^* \geq \hat{f}(\rho_1^*, \rho_2^*)$.

(ii) *If* $f \in \mathcal{F}_2$, *then* $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2 \in \mathrm{GA}$ *and* $(\hat{f}(\rho_1, \rho_2))^* = \hat{f}(\rho_1^*, \rho_2^*)$.

We shall prove

3.5. Proposition. *If* $f \in \widetilde{\mathcal{F}}_1 \cap \mathcal{F}_{\mathrm{GA}}$, *then the following conditions are equivalent*:

(i) $(\hat{f}(\rho_1, \rho_2))^* = \hat{f}(\rho_1^*, \rho_2^*)$ *for every* $(X_i, \rho_i) \in \mathrm{GA}$, $i = 1, 2$;

(ii) *the class* IM *is* f*-multiplicative*;

(iii) *the class* MC *is* f*-multiplicative.*

Proof. The implication (i)$\Rightarrow$(ii) is obvious.

(ii)$\Rightarrow$(i). Assume (ii) and let $\boldsymbol{X}_i = (X_i, \rho_i) \in \mathrm{GA}$ for $i = 1, 2$. Then

$$(\hat{f}(\rho_1^*, \rho_2^*))^* = \hat{f}(\rho_1^*, \rho_2^*)\,. \tag{1}$$

By F.3, $\hat{f}(\rho_1^*, \rho_2^*) = \hat{f}(\rho_1, \rho_2)$, whence

$$(\hat{f}(\rho_1^*, \rho_2^*))^* \geq (\hat{f}(\rho_1, \rho_2))^*\,; \tag{2}$$

by 3.4(i)

$$(\hat{f}(\rho_1, \rho_2))^* \geq \hat{f}(\rho_1^*, \rho_2^*)\,. \tag{3}$$

By (1)–(3), we obtain (i).

In what follows we use the notation $|L|_\rho$ for the length of an arc L in a metric space (X, ρ).

(ii)$\Rightarrow$(iii). Assume (ii) and let $\boldsymbol{X}_i = (X_i, \rho_i) \in \mathrm{MC}$ for $i = 1, 2$. Let $\rho_f = \hat{f}(\rho_1, \rho_2)$. To prove (iii) it suffices to show that for every $a_i, b_i \in X_i$ the points $a = (a_1, a_2)$ and $b = (b_1, b_2)$ can be joined in $\boldsymbol{X}_i \times_f \boldsymbol{X}_2$ by an arc L with $|L|_{\rho_f} = \rho_f(a, b)$.

By the assumption on ρ_i, there exists an arc $L_i \subset X_i$ with endpoints a_i and b_i and with $|L_i|_{\rho_i} = \rho_i(a_i, b_i)$, $i = 1, 2$. Let $\rho_i' = \rho_i|(L_i)^2$, $i = 1, 2$, and $\rho_f' = \hat{f}(\rho_1', \rho_2')$. Then

$$\rho_f' = \rho_f|(L_1 \times L_2)^2\,. \tag{4}$$

Evidently $(L_i, \rho_i') \in \mathrm{MC} \subset \mathrm{IM}$ for $i = 1, 2$, whence, by (ii),

$$(L_1 \times L_2, \rho_f') \in \mathrm{IM}\,. \tag{5}$$

Since $(L_1 \times L_2, \rho_f')$ is compact, by Th. 28.1, p. 70 of [1], condition (5) implies

$$(L_1 \times L_2, \rho_f') \in \mathrm{MC}\,. \tag{6}$$

By (6), there is an arc $L \subset L_1 \times L_2$ joining a and b, with $|L|_{\rho_f'} = \rho_f'(a, b)$. Thus, by (4),

$$|L|_{\rho_f} = |L|_{\rho_f'} = \rho_f'(a, b) = \rho_f(a, b)\,.$$

(iii)$\Rightarrow$(ii). Assume (iii) and let $\boldsymbol{X}_i = (X_i, \rho_i) \in \mathrm{IM}$, i.e. $\rho_i = \rho_i^*$ for $i = 1, 2$. Let $\rho_f = \hat{f}(\rho_1, \rho_2)$. We have to prove that $(\rho_f)^* = \rho_f$. Let $a, b \in X_1 \times X_2$, $a = (a_1, a_2)$, $b = (b_1, b_2)$. It suffices to prove that there is a sequence $(L^{(n)})_{n\in\mathbb{N}}$ of arcs joining a and b in $X_1 \times X_2$ such that

$$\lim_n |L^{(n)}|_{\rho_f} = \rho_f(a, b)\,. \tag{7}$$

Since $\rho_i^* = \rho_i$, there is a sequence $(L_i^{(n)})_{n\in\mathbb{N}}$ of arcs joining a_i and b_i in X_i such that

$$\lim_n |L_i^{(n)}|_{\rho_i} = \rho_i(a_i, b_i), \quad i = 1, 2\,. \tag{8}$$

Let $\rho_i^{(n)} = (\rho_i|(L_i^{(n)})^2)^*$ for $i = 1, 2$, $n \in \mathbb{N}$. Evidently

$$|L_i^{(n)}|_{\rho_i} = \rho_i^{(n)}(a_i, b_i) \quad \text{for } i = 1, 2,\ n \in \mathbb{N}\,. \tag{9}$$

Let

$$\rho_f^{(n)} = \hat{f}(\rho_1^{(n)}, \rho_2^{(n)})\,. \tag{10}$$

By Th. 28.1 of [1], the compactness of $L_i^{(n)}$ implies $(L_i^{(n)}, \rho_i^{(n)}) \in \mathrm{MC}$, whence, by (iii),

$$(L_1^{(n)} \times L_2^{(n)}, \rho_f^{(n)}) \in \mathrm{MC}\,.$$

Let now $L^{(n)}$ be an arc joining a and b in $L_1^{(n)} \times L_2^{(n)}$ such that

$$|L^{(n)}|_{\rho_f^{(n)}} = \rho_f^{(n)}(a, b)\,. \tag{11}$$

Applying in turn (11), (10), 1.2 and 1.3(i), (9), and (8), we obtain

$$\begin{aligned}\lim_n |L^{(n)}|_{\rho_f^{(n)}} &= \lim_n \rho_f^{(n)}(a, b) = \lim_n f(\rho_1^{(n)}(a_1, b_1), \rho_2^{(n)}(a_2, b_2)) \\ &= f(\lim_n \rho_1^{(n)}(a_1, b_1), \lim_n \rho_2^{(n)}(a_2, b_2)) \\ &= f(\lim_n |L_1^{(n)}|_{\rho_1}, \lim_n |L_2^{(n)}|_{\rho_2}) = f(\rho_1(a_1, b_1), \rho_2(a_2, b_2))\,,\end{aligned}$$

i.e.

$$\lim_n |L^{(n)}|_{\rho_f^{(n)}} = \rho_f(a, b)\,. \tag{12}$$

Since $\rho_i^{(n)} \ge \rho_i|(L_i^{(n)})^2$, by F.3 and (10) we infer that

$$\rho_f^{(n)} \ge \hat{f}(\rho_1|(L_1^{(n)})^2, \rho_2|(L_2^{(n)})^2)\,.$$

Hence

$$|L^{(n)}|_{\rho_f} \le |L^{(n)}|_{\rho_f^{(n)}} \quad \text{for every } n \in \mathbb{N}\,. \tag{13}$$

Finally,

$$\rho_f(a, b) \le (\rho_f)^*(a, b) \le \lim_n |L^{(n)}|_{\rho_f}\,. \tag{14}$$

Conditions (12)–(14) imply (7). This completes the proof. ■

Let us now consider the following three examples:

3.6. EXAMPLE. Let $f(t_1,t_2) = \sqrt{t_1} + t_2$ for $t_1,t_2 \in \mathbb{R}^+$. Evidently $f \in \widetilde{\mathcal{F}}_1 - \mathcal{F}_2$. We shall prove that GA is not f-multiplicative.

Let $I = [0,1] \subset \mathbb{R}$ and let ρ be the Euclidean metric. Take $\boldsymbol{X}_1 = (I,\rho)$ and $\boldsymbol{X}_2 = (\{0\},\rho)$. Evidently $\boldsymbol{X}_i \in \mathrm{GA}$ for $i = 1.2$. We have $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2 = (I \times \{0\}, \rho_f)$, where

$$\rho_f((x_1,0),(y_1,0)) = \sqrt{\rho(x_1,y_1)} \quad \text{for } x_1,y_1 \in I\,.$$

The points $(0,0)$ and $(1,0)$ cannot be joined in $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2$ by an arc of finite length. Indeed, let $I_{n,k} = [k/n,(k+1)/n]$ for $n \in \mathbb{N}$ and $k = 0,\ldots,n-1$; then $|I_{n,k}|_{\rho_f} = \sqrt{1/n}$, whence

$$\sum_{k=0}^{n-1} |I_{n,k}|_{\rho_f} = n\sqrt{1/n} = \sqrt{n}\,,$$

and thus $|I \times \{0\}|_{\rho_f}$ is infinite. Therefore $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2$ is not geometrically acceptable. ■

3.7. EXAMPLE. Let $f(t_1,t_2) = \sqrt{t_1+t_2}$ for $t_1,t_2 \in \mathbb{R}^+$. It is easy to check that $f \in \widetilde{\mathcal{F}}_1 - \mathcal{F}_2$. We shall prove that IM, MC, and MidC are not f-multiplicative.

Let ρ be the Euclidean metric in [0,1]; let $\boldsymbol{X}_i = ([0,1],\rho)$ for $i = 1,2$ and let $\rho_f = \hat{f}(\rho,\rho)$. Clearly $\boldsymbol{X}_1$ and $\boldsymbol{X}_2$ are convex, whence ρ is an intrinsic metric. On the other hand, $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2$ is not convex; moreover, $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2$ is not Mid-convex, because for every $x \in [0,1]^2$, if $\rho_f(a,x)+\rho_f(x,b) = \rho_f(a,b)$, then $x = a$ or $x = b$. Since $\boldsymbol{X}_1 \times_f \boldsymbol{X}_2$ is compact, by Th. 28.1 of [1] it follows that ρ_f is not an intrinsic metric. ■

3.8. EXAMPLE. Let $f(t_1,t_2) = t_1+t_2$ for $t_1,t_2 \in \mathbb{R}^+$. Then $f \in \mathcal{F}_2 - \mathcal{F}_2'$. Clearly the Euclidean line $\boldsymbol{R}$ is strongly Mid-convex (it is even strongly convex), while $\boldsymbol{R} \times_f \boldsymbol{R}$ is not. ■

We complete this section with two corollaries.

3.9. COROLLARY. *The classes* GA, IM, MC, *and* MidC *are* $\mathcal{F}_2$*-multiplicative but not* $\mathcal{F}_1$*-multiplicative.*

P r o o f. For the class GA the statement follows from 3.4(ii) and 3.6; for IM and MC it follows from 3.4(ii), 3.5, and 3.7; for MidC it follows from 2.3 and 3.7. ■

3.10. COROLLARY. *The classes* SMidC *and* SMC *are* $\mathcal{F}_2'$*-multiplicative but not* $\mathcal{F}_2$*-multiplicative.*

P r o o f. For the class SMidC we use 2.4 and 3.8; for SMC we use 0.1, 3.8, and 3.9. ■

4. Products of normed linear spaces. We are now concerned with normed linear spaces. Every $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ induces a function $\check{f}$ which assigns to any pair of norms $\| \|_1$, $\| \|_2$ in linear spaces E_1, E_2, respectively, the function

$$\check{f}(\| \|_1, \| \|_2) = \| \|_f : E_1 \times E_2 \to \mathbb{R}^+$$

defined by the formula

$$\|(x_1, x_2)\|_f := f(\|x_1\|_1, \|x_2\|_2)\,.$$

Evidently

4.1. *If $(\boldsymbol{E}_i, \| \|_i)$ is a normed linear space and ρ_i is the metric induced by the norm $\| \|_i$ for $i = 1, 2$, then for every $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ and $x, y \in E_1 \times E_2$*

$$\hat{f}(\rho_1, \rho_2)(x, y) = \|x - y\|_f\,.$$

As a direct consequence of 4.1 we obtain

4.2. *Let ρ_i be the metric induced by a norm $\| \|_i$ in $\boldsymbol{E}_i$, $i = 1, 2$. For every $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$*

(i) *if $\check{f}(\| \|_1, \| \|_2)$ is a norm in $\boldsymbol{E}_1 \times \boldsymbol{E}_2$, then $\hat{f}(\rho_1, \rho_2)$ is the metric induced by this norm*;

(ii) *if f satisfies* F.4.2 *and $\hat{f}(\rho_1, \rho_2)$ is a metric in $E_1 \times E_2$, then $\check{f}(\| \|_1, \| \|_2)$ is the norm which induces this metric.*

We can now characterize $\mathcal{F}_2$ as follows:

4.3. Theorem. *For every $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ the following conditions are equivalent*:

(i) $f \in \mathcal{F}_2$;

(ii) *the class* NL *is f-multiplicative*;

(iii) *if ρ is the Euclidean metric in $\mathbb{R}^2$, then $\hat{f}(\rho, \rho)$ is induced by a norm in $\mathbb{R}^4$.*

P r o o f. The implication (i)$\Rightarrow$(ii) follows from 2.1 and 4.2(ii).

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i). By 2.1, $f \in \mathcal{F}_0$; it remains to verify F.4.2. Let $\rho_f = \hat{f}(\rho, \rho)$. By assumption, ρ_f is induced by a norm $\| \|$ in $\mathbb{R}^4$. Take $(t_1, t_2) \in (\mathbb{R}^+)^2$ and let $0 = (0, \ldots, 0) \in \mathbb{R}^4$. Then $t_i = \rho((0, 0), x_i)$ for some $x_i \in \mathbb{R}^2$, $i = 1, 2$, and for any $\alpha \in \mathbb{R}^+$

$$\begin{aligned} f(\alpha(t_1, t_2)) &= f(\rho((0,0), \alpha x_1), \rho((0,0), \alpha x_2)) = \rho_f(0, \alpha(x_1, x_2)) \\ &= \|\alpha(x_1, x_2)\| = \alpha f(t_1, t_2)\,. \end{aligned}$$

This proves F.4.2. ■

By 4.3, the family $\mathcal{F}_2$ coincides with the family of all functions for which NL is multiplicative:

4.4. COROLLARY. $\mathcal{F}_2 = \mathcal{F}_{\mathrm{NL}}$.

We are now going to prove the analogue of 4.4 for $\mathcal{F}_2'$ and the class SNL. Let us start with two simple lemmas:

4.5. LEMMA. *If ρ is induced by a norm in a linear space E, then $M_\rho(a,b)$ is affine convex for every $a, b \in E$.*

Proof. First notice that in (E,ρ)

(1) every closed, affine Mid-convex set is affine convex.

By the continuity of ρ,

(2) for every a, b the set $M_\rho(a,b)$ is closed ([2]).

Thus, it suffices to prove that for every $a, b \in E$ the set $M_\rho(a,b)$ is affine Mid-convex, i.e.

$$c_1, c_2 \in M_\rho(a,b) \Rightarrow \tfrac{1}{2}(c_1 + c_2) \in M_\rho(a,b)\,. \tag{3}$$

The proof of (3) is left to the reader. ∎

4.6. LEMMA. *If ρ is induced by a norm in a linear space E, then translations and central symmetries are isometries of (E,ρ).*

Let us now establish

4.7. PROPOSITION. *For every normed linear space $(\boldsymbol{E}, \|\ \|)$ and the metric ρ induced by $\|\ \|$ the following conditions are equivalent*:

(i) *balls are strictly convex*;
(ii) *the space (E,ρ) is strongly convex.*

Proof. (i)⇒(ii). Clearly (E,ρ) is metrically convex, since every affine segment is a metric segment. Thus, by 0.1, it suffices to prove

$$\forall a, b \in E \quad M_\rho(a,b) \text{ is a singleton}\,. \tag{1}$$

Suppose there are a, b, c_1, c_2 such that $a \neq b$, $c_1 \neq c_2$, and $c_i \in M_\rho(a,b)$ for $i = 1, 2$. Then, by 4.5, $\Delta(c_1,c_2) \subset M_\rho(a,b)$. Let $\alpha = \rho(b, c_i)$. Then $\Delta(c_1,c_2) \subset \partial B_\rho(b,\alpha)$, contrary to (i).

(ii)⇒(i). By 4.6, it suffices to prove that there exists a strictly convex ball. Let $B_0 = B_\rho(a,1)$ for some $a \in E$. Suppose that B_0 is not strictly convex, i.e. there are distinct points p, q with $\Delta(p,q) \subset \partial B_0$. Let $r = \frac{1}{2}(p+q)$; take the symmetry σ_r with respect to r and let $b = \sigma_r(a)$. Then, by 4.6, $\sigma_r(B_0) = B_\rho(b,1)$. It is easy to check that $p, q \in M_\rho(a,b)$, contrary to (ii). ∎

([2]) Condition (2) holds in an arbitrary metric space.

We are now ready to prove

4.8. THEOREM. *For every* $f : (\mathbb{R}^+)^2 \to \mathbb{R}^+$ *the following conditions are equivalent*:

(i) $f \in \mathcal{F}_2'$;

(ii) *the class* SNL *is* f*-multiplicative*;

(iii) *if* ρ *is the Euclidean metric in* $\mathbb{R}^2$, *then* $(\mathbb{R}^4, \hat{f}(\rho, \rho)) \in \mathrm{SNL}$.

Proof. Applying 4.7 and 3.10 we obtain the implication (i)$\Rightarrow$(ii).

(ii)$\Rightarrow$(iii) is obvious.

(iii)$\Rightarrow$(i). Assume (iii). By 4.7, the metric $\hat{f}(\rho, \rho)$ is strongly convex, whence for every $a, b \in \mathbb{R}^4$

$$M_{\hat{f}(\rho,\rho)}(a, b) = \{\tfrac{1}{2}(a + b)\}\,. \tag{1}$$

Let $a = (a_1, a_2)$, $b = (b_1, b_2)$, $a_i, b_i \in \mathbb{R}^2$ for $i = 1, 2$. Clearly, $M_\rho(a_i, b_i) = \{\frac{1}{2}(a_i + b_i)\}$ for $i = 1, 2$, which, together with (1), implies

$$M_\rho(a_1, b_1) \times M_\rho(a_2, b_2) = M_{\hat{f}(\rho,\rho)}(a, b)\,. \tag{2}$$

Since, by 4.3, $f \in \mathcal{F}_2$, and thus, by 1.2, $f \in \mathcal{F}_1$, from 2.4 and (2) it follows that $f \in \mathcal{F}_2'$. ■

By 4.8, the family $\mathcal{F}_2'$ coincides with the family of all functions for which SNL is multiplicative:

4.9. COROLLARY. $\mathcal{F}_2' = \mathcal{F}_{\mathrm{SNL}}$.

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