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ARE EC-SPACES AE(METRIZABLE)?

BY

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1. The appealing conjecture that equiconnected spaces are AE(metrizable) remains unanswered. (Recall that a space X is *equiconnected* (abbrev. EC) provided that there exists a continuous function $\lambda : X \times X \times I \to X$, where $I = [0,1]$, such that $\lambda(x,y,0) = x$, $\lambda(x,y,1) = y$ and $\lambda(x,x,t) = x$, for all $x, y \in X$ and $t \in I$. (λ is called an *equiconnecting function* for X.) We will discuss the significance of the preceding conjecture, recall some partial answers and provide a new partial answer, and conclude with some new thoughts which may help with its solution.

The significance of answering the question "are equiconnected spaces *absolute extensor spaces for metrizable spaces* (i.e. AE(metrizable))?" lies in the fact that a positive answer to this question will easily imply the following (for details, see [6]):

(i) linear topological spaces are AE(metrizable),

(ii) compact strongly convex metric spaces are AE(metrizable),

(iii) many groups of homeomorphisms (including the group $H_\delta(B^n)$ of homeomorphisms of the euclidean n-ball which leave the boundary fixed) are homeomorphic to the Hilbert space ℓ_2.

Next, let us discuss the known partial answers to the conjecture at hand. Throughout, we will use the terminology of Michael [10].

THEOREM 1.1. *If L is an equiconnected metrizable space with* $\dim L < \infty$ *then L is an AE*(*metrizable*).

Proof. This follows from Theorems 2.4 and 3.1 of Dugundji [8].

THEOREM 1.2. *Let X be a stratifiable space*, *L an equiconnected space*, *A a closed subset of X and $f : A \to L$ a continuous function. If* $\dim(X - A) < \infty$ *then there exists a continuous extension $\bar{f} : X \to L$ of f.*

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P r o o f. This follows from Theorem 4.2 of [3] and [4].

THEOREM 1.3. *Let L be an equiconnected space with an equiconnecting function λ which satisfies the following condition: for each $x \in L$ and each neighborhood U of x there exists a neighborhood V of x such that $\lambda(U \times V \times I) \subset U$. Then L is an AE(stratifiable).*

P r o o f. This follows from Theorems 3.1 and 4.1 of [3] and [4].

THEOREM 1.4. *If L is equiconnected then L is an AE(CW-complexes of Whitehead).*

P r o o f. This follows from Theorems 3.2 and 4.3 of [3] and [4].

THEOREM 1.5. *Let L be an equiconnected space, X a stratifiable space, A a closed separable metrizable subspace of X and $f : A \to L$ a continuous function. If $\dim A < \infty$ then there exists a continuous extension $\bar{f} : X \to L$ of f.*

P r o o f. Say $\dim A = n$. Then, by Theorem IV.8 of [13], there exists an embedding $j : A \to E^{2n+1}$. Since E^{2n+1} is an AE(metrizable) space, continuously extend j to $g : X \to E^{2n+1}$. Then, by Lemma 4.2 of [10], there exists a continuous function $h : X \to F = E^{2n+1} \times I - (E^{2n+1} - j(A)) \times \{0\}$ such that $h|A = j = g|A$.

We are finally ready to define the map $\bar{f}$, as follows: By Theorems 3.2 and 4.2 of [3] and [4], let $\hat{f} : F \to L$ be a continuous extension of the map $fj^{-1} : j(A) \to L$. Let $\bar{f} = \hat{f}h$. (Clearly, $\bar{f} : X \to L$ and $\bar{f}|A = fj^{-1}j = f$.)

Note that if A is not separable then the best embedding result known to us (i.e. Theorem VI.10 of [13]) does not guarantee that A can be embedded in a finite-dimensional AE(metrizable) space.

Theorems 1.2 and 1.5 suggest the following question.

QUESTION 1. Let X be metrizable and A a closed subset of X. Is there a stratifiable space Y such that A is (embedded as) a closed subset of Y, $\dim(Y - A) < \infty$ and the identity function $i : A \to A$ extends to a continuous function $\bar{i} : X \to Y$?

A positive answer to the preceding question proves that equiconnected spaces are AE(metrizable) as follows: Let A be a closed subset of a metrizable space X, E an equiconnected space and $f : A \to E$ a continuous function. Continuously extend f to $\bar{f} : Y \to E$, by Theorem 1.2. Note that $\hat{f} = \bar{f}\,\bar{i}$ is the desired extension.

THEOREM 1.6. *If convex subsets of linear topological spaces over the reals, with vector bases which are k_ω-spaces, are AE(stratifiable) then equiconnected k_ω-spaces are AE(stratifiable).*

Proof. Let B be an equiconnected k_ω-space. By Theorem 2.1 in the Appendix, B can be embedded as a closed linearly independent subset of a locally convex linear topological space $L = M(B)$; clearly, without loss of generality, we may assume that $\mathrm{lin}B$ (i.e. the linear subspace of L spanned by B) equals L.

Next, note that the space $L_w = \sum_n L_n(B)$, described in the Appendix, is a linear topological space, by Theorem 2.2 in the Appendix; furthermore, B is (embedded as) a closed linearly independent subset of L_w.

Finally, we prove that B is a continuous retract of $(\mathrm{conv}\, B)_w$ (i.e. the convex hull of B as a subspace of L_w): Using the terminology of Propositions 2.4 and 2.5 in the Appendix, we define a map $r : (\mathrm{conv}\, B)_w \to B$ by

$$r\Big(\sum_{i=1}^{n} t_i b_i\Big) = h_n((b_{\mu(1)}, \dots, b_{\mu(n)}), (t_{\mu(1)}, \dots, t_{\mu(n)})),$$

where $(b_1, \dots, b_n) \in B_*^n$ and $(t_{\mu(1)}, \dots, t_{\mu(n)})$ means that $(t_1, \dots, t_n)$ is reordered the same way that $(b_1, \dots, b_n)$ is reordered by $(b_{\mu(1)}, \dots, b_{\mu(n)})$ (note that the coordinates of $(t_1, \dots, t_n)$ may not be distinct). The map r is well-defined, because B is linearly independent.

In order to prove that r is continuous, let us first note that $(\mathrm{conv}\, B)_w = \sum_n \mathrm{conv}_n\, B$, with $\mathrm{conv}\,_n B = L_n(B) \cap \mathrm{conv}\, B$. Therefore, we need only prove that each $r_n = r|\mathrm{conv}_n B$ is continuous, which we do by using induction. Assuming that $r_1, \dots, r_{n-1}$ are continuous (clearly, r_1 is continuous), let us prove that r_n is continuous.

It is easily seen that r_n is continuous at each point of $\mathrm{conv}_n B - \mathrm{conv}_{n-1} B = E$. Indeed, pick $q = \sum_{i=1}^n t_i b_i \in E$. Then all $t_i \neq 0$. Let V be any neighborhood of $r(q) = h_n((b_{\mu(1)}, \dots, b_{\mu(n)}),\ (t_{\mu(1)}, \dots, t_{\mu(n)}))$. By continuity of h_n, pick a neighborhood $N = (N_{\mu(1)} \times \dots \times N_{\mu(n)}) \times (V_{\mu(1)} \times \dots \times V_{\mu(n)})$ of $((b_{\mu(1)}, \dots, b_{\mu(n)}),\ (t_{\mu(1)}, \dots, t_{\mu(n)}))$ such that $N_{\mu(1)} \times \dots \times N_{\mu(n)} \subset O_n$ (see Proposition 2.5 in the Appendix), $V_{\mu(1)} \times \dots \times V_{\mu(n)} \subset P_{n-1}$, $0 \notin \bigcup_{i=1}^n V_{\mu(i)}$ and $h_n(N) \subset V$. Then $U = \{\sum_{i=1}^n s_i b_i' \,|\, b_i' \in N_{\mu(i)}$ and $s_i \in V_{\mu(i)}$, for $i = 1, \dots, n\}$ is a neighborhood of q in L such that $r(U \cap \mathrm{conv}_n B) \subset V$.

It is also easily seen that r_n is continuous at each point w in the boundary of $\mathrm{conv}_{n-1} B$ (as a subspace of $\mathrm{conv}_n B$). Indeed, let $w = \sum_{i=1}^j t_i b_i$, with all $t_i \neq 0$ and $j < n-1$. Let V be any neighborhood of $y = r_j(w) = h_j((b_{\mu(1)}, \dots, b_{\mu(j)}), (t_{\mu(1)}, \dots, t_{\mu(j)}))$. Pick any $\overline{b} = (b_{\mu(1)}, \dots, b_{\mu(j)}, b_{j+1}, \dots, b_n) \in O_n$. Letting $\overline{t} = (t_{\mu(1)}, \dots, t_{\mu(j)}, 0, \dots, 0)$, pick a neighborhood $(N_{\mu(1)} \times \dots \times N_{\mu(n)}) \times (V_{\mu(1)} \times \dots \times V_{\mu(n)})$ of $(\overline{b}, \overline{t})$ in $L^n \times \mathbb{R}^n$ such that $h((N_{\mu(1)} \times \dots \times N_{\mu(n)}) \times ((V_{\mu(1)} \times \dots \times V_{\mu(n)}) \cap P_{n-1})) \subset V$ (see Proposition 2.4 in the Appendix). Then, letting $M = \{\sum_{i=1}^n s_i b_i' \,|\, b_i' \in N_{\mu(i)}$ and $s_i \in V_{\mu(i)}$, for $i = 1, \dots, n\}$, we find that M is a neighborhood of w such that $r_n(M \cap \mathrm{conv}_n B) \subset V$. This shows that r_n is continuous at w.

From the preceding two paragraphs we finally conclude that r_n is continuous, which completes the proof.

Theorem 1.6 raises some interesting questions.

QUESTION 2. When is a closed convex subset of a linear topological space L a continuous retract of L?

From Dugundji's Extension Theorem one immediately sees that closed convex subsets of a metrizable locally convex linear topological space L are continuous retracts of L. The answer in general appears quite difficult. The results of [7] may help answer this question for completely metrizable linear topological spaces.

2. Appendix. Michael [11] has proved that every metric space can be embedded isometrically as a closed, *linearly independent* subset of a normed linear space, while Arens and Eells [1] have proved that any Tikhonov space can be embedded as a closed, but not linearly independent, subset of a locally convex linear space. Fortunately, a modification of their embedding along the lines of Michael's technique yields the stronger and quite useful result that follows.

THEOREM 2.1. *Every Tikhonov space X can be embedded as a closed, linearly independent subset of a locally convex linear topological space $M(X)$. If X is metric then $M(X)$ is a normed linear space and the embedding is isometric.*

Proof. Let (Y,τ) be a Tikhonov space and let $X = Y \cup \{x_0\}$, for some $x_0 \notin Y$. The topology of X is the one generated by $\tau \cup \{\{x_0\}\}$; clearly, X is also a Tikhonov space.

Using the same construction of [1], let $M(X)$ be the set of all real-valued functions m on X such that $m(y) = 0$ for all but finitely many $y \in Y$ and $\sum_{y\in X} m(y) = 0$; for convenience, letting $m(y) = \lambda_y$ for $m(y) \neq 0$, m is represented as a linear combination $m = \sum_{\lambda_y \neq 0} \lambda_y y$ with $\sum_{\lambda_y\neq 0} \lambda_y = 0$. It is proved in [1] that

(i) $M(X)$, with the usual addition and scalar multiplication of real-valued functions, can be given a topology $\mathcal{L}_0$ such that $(M(X),\mathcal{L}_0)$ is a locally convex linear topological space,

(ii) X is embedded as a closed subspace of $M(X)$ by the map $\psi : X \to M(X)$ defined by $\psi(x) = x - x_0$; furthermore, $B = \{x - x_0 \mid x \in X - \{x_0\}\}$ is a vector base for $M(X)$.

From (ii) we immediately see that $\psi(Y) = \{x - x_0 \mid x \in Y\}$ is linearly independent (indeed, $\psi(Y) = B$); furthermore, $\psi(Y)$ is a closed subset of $M(X)$, since $\psi(Y)$ is a closed subset of $\psi(X)$. This completes the proof.

For any linear topological space $(L,\mathcal{T})$ over a field F and nonempty subset B of L, and $n \in \omega$, let $L_n(B) = \{\sum_{i=1}^n r_i b_i \mid b_i \in B \text{ and } r_i \in F\}$. Also, let $\operatorname{lin} B = \bigcup\{L_n(B) \mid n \in \omega\}$ be the linear subspace of L spanned by B. For any linear topological space (L,τ) and vector base B for L, let L_w denote the set L with the weak topology over $\{L_n(B) \mid n \in \omega\}$, i.e. $L_w = \sum_n L_n(B)$ or, equivalently, L_w has the quotient topology generated by the natural map $q : \bigvee_n L_n(B) \to L$, where $\bigvee_n L_n(B)$ denotes the disjoint topological union of $\{(L_n(B),\ \tau \mid L_n(B)) \mid n \in \omega\}$; note that each $L_n(B) \subset L_w$ retains its original topology as a subspace of L.

For compact metric spaces, a different proof of the following result is essentially contained in the proof of Proposition VIII.5.2 of [2].

Let us first recall that a Hausdorff space which is a union of an increasing sequence $\{X_n\}$ of compact subspaces is said to be a *k_ω-space* if the natural map $q : \bigvee_n X_n \to X$, from the disjoint topological union of the X_n, is a quotient map (i.e. $X = \sum_n X_n$). From results of [12], one immediately sees that finite products and quotient images of k_ω-spaces are k_ω-spaces.

THEOREM 2.2. *Let L be a linear topological space over the real* (*or complex*) *numbers with a vector base B which is a k_ω-space. Then L_w is a linear topological space.*

Proof. Since B and $\mathbb{R}$ are k_ω-spaces, one immediately finds that each $L_n(B)$ is a k_ω-space (since the natural map $m : \prod_{i=1}^n (\mathbb{R}\times B) \to L_n(B)$, defined by $m((t_1,b_1),\ldots,(t_n,b_n)) = t_1b_1 + \ldots + t_nb_n$, is a quotient (indeed, open and continuous) map). Therefore, from the following diagram

$$\begin{array}{ccc} \bigvee\limits_n L_n(B) \times \bigvee\limits_n L_n(B) & \xrightarrow{q\times q} & L_w \times L_w \\ & \searrow^{\psi} & \downarrow + \\ & & L_w \end{array}$$

where the map ψ is also addition on each $L_n(B)\times L_m(B)$, we conclude that addition in L_w is continuous, because $q\times q$ is a quotient map.

Similarly, the fact that q is a quotient map and $\mathbb{R}$ is locally compact implies that scalar multiplication is continuous (because $q\times 1 : (\bigvee_n L_n(B))\times \mathbb{R} \to L_w \times \mathbb{R}$ is a quotient map, by Theorem XII.4.1 of [9]). Consequently, L_w is a linear topological space, which completes the proof.

The work that follows is needed for the proof of Theorem 1.6 and consists of refinements of the work in [3]. For convenience, let us recall that

(i) for any set X and $n = 1,2,\ldots,$ $X^{n+1} = \prod_{i=1}^n X$,
(ii) for $(x_1,\ldots,x_{n+1}) = x \in X^{n+1}$, $\hat{x} = (x_1,\ldots,x_n) \in X^n$,
(iii) if $(t_1,\ldots,t_{n+1}) = t \in P_n$ (the unit n-simplex in E^{n+1}) and $t_{n+1} \neq 1$ then $(t_1/(1-t_{n+1}),\ldots,t_n/(1-t_{n+1})) = \hat{t} \in P_{n-1}$, for $n = 1,2,\ldots,$

(iv) if $\lambda : L\times L\times I \to L$ is an equiconnecting function then $h_1 : L\times\{1\} \to L$ is defined by $h_1(x,1) = x$ and, for $n = 2,3,\ldots$, $h_n : L^n \times P_{n-1} \to L$ is defined by

$$h_{n+1}(x,t) = \begin{cases} x_{n+1} & \text{if } t_{n+1} = 1, \\ \lambda(h_n(\hat{x},\hat{t}), x_{n+1}, t_{n+1}) & \text{if } t_{n+1} \neq 1. \end{cases}$$

The following lemma is needed for the next very crucial proposition.

LEMMA 2.3. *A function $f : X \to Y$ is continuous at $x \in X$ iff each net $\{x_\nu\}_{\nu\in\Gamma}$ in X which converges to x has a subnet $\{x_\alpha\}_{\alpha\in\Lambda}$ such that* $\lim_\alpha f(x_\alpha) = f(x)$.

Proof. The "only if" part is well-known (indeed, $\lim_\nu f(x_\nu) = f(x)$).

The "if" part: Suppose that f is not continuous at x. Then there exists a net $\{x_\nu\}_{\nu\in\Gamma}$ in X such that $\lim_\nu x_\nu = x$ but $\{f(x_\nu)\}_{\nu\in\Gamma}$ does not converge to $f(x)$. Hence, there exists a neighborhood V of $f(x)$ and a subnet $\{f(x_\beta)\}_{\beta\in\Theta}$ of $\{f(x_\nu)\}_{\nu\in\Gamma}$ such that $\{f(x_\beta) \mid \beta \in \Theta\} \cap V = \emptyset$. Since $\lim_\beta x_\beta = x$, by hypothesis there exists a subnet $\{x_\alpha\}_{\alpha\in\Lambda}$ of $\{x_\beta\}_{\beta\in\Theta}$ (hence, a subnet of $\{x_\nu\}_{\nu\in\Gamma}$) such that $\lim_\alpha f(x_\alpha) = f(x)$, a contradiction (since $\{f(x_\alpha) \mid \alpha \in \Lambda\} \cap V = \emptyset$).

PROPOSITION 2.4. *If $\lambda : L \times L \times I \to L$ is an equiconnecting function then the functions $h_1, h_2, \ldots$ are continuous and satisfy conditions* (a), (b) *and* (d) *of Definition* 2.2 *of* [3].

Proof. Clearly h_1 is continuous (and $h_2 = \lambda$). By induction, let us assume that h_j is continuous for $j \le n$ and let us prove that $h_{n+1} : L^{n+1} \times P_n \to L$ is continuous. (First note that we already know from the proof of Theorem 3.1 in [3] that each h_{n+1} is continuous in the second variable.) Let us prove that h_{n+1} is continuous at each $(x,t) \in L^{n+1} \times P_n$ by considering two cases.

Case 1: $t = (t_1,\ldots,t_{n+1})$ with $t_{n+1} \neq 1$. Pick a neighborhood N_t of t in P_n such that, for each $s \in N_t$, $s_{n+1} \neq 1$. Then, letting $h'_{n+1} = h_{n+1}|L^{n+1} \times N_t$, we find that $h'_{n+1}(x,s) = \lambda(h_n(\hat{x},\hat{s}), x_{n+1}, s_{n+1})$. Since h_n and λ are continuous, we immediately conclude that h'_{n+1} is continuous. This proves that h_{n+1} is continuous at any $(x,t) \in L^{n+1} \times P_n$ such that $t_{n+1} \neq 1$.

Case 2: $t = (0,\ldots,0,1)$. Then $h_{n+1}(x,t) = x_{n+1}$. Let $\{(x_\alpha,t_\alpha)\}_{\alpha\in\Gamma}$ be a net in $L^{n+1} \times P_n$ which converges to (x,t); say $(x_\alpha,t_\alpha) = ((x_1^\alpha, \ldots, x_{n+1}^\alpha), (t_1^\alpha,\ldots,t_{n+1}^\alpha))$.

Next, let us recall that

$$h_{n+1}(x_\alpha,t_\alpha) = \begin{cases} x_{n+1}^\alpha & \text{if } t_{n+1}^\alpha = 1, \\ \lambda(h_n((x_1^\alpha,\ldots,x_n^\alpha),\hat{t}_\alpha), x_{n+1}^\alpha, t_{n+1}^\alpha) & \text{if } t_{n+1}^\alpha \neq 1, \end{cases}$$

where

$$\hat{t}_\alpha = \left(\frac{t_1^\alpha}{1 - t_{n+1}^\alpha}, \cdots, \frac{t_n^\alpha}{1 - t_{n+1}^\alpha} \right),$$

and note the following:

(i) If there exists a subnet $\{(x_\beta, t_\beta)\}_{\beta \in \Theta}$ of $\{(x_\alpha, t_\alpha)\}_{\alpha \in \Gamma}$ such that $h_{n+1}(x_\beta, t_\beta) = x_{n+1}^\beta$, for each $\beta \in \Theta$, then $\lim_\beta h_{n+1}(x_\beta, t_\beta) = \lim_\beta x_{n+1}^\beta = x_{n+1}$.

(ii) If there exists a subnet $\{(x_\gamma, t_\gamma)\}_{\gamma \in \Lambda}$ of $\{(x_\alpha, t_\alpha)\}_{\alpha \in \Gamma}$ such that $h_{n+1}(x_\gamma, t_\gamma) = \lambda(h_n((x_1^\gamma, \ldots, x_n^\gamma), \hat{t}_\gamma), x_{n+1}^\gamma, t_{n+1}^\gamma)$, for each $\gamma \in \Lambda$, then let us pick a convergent subnet $\{\hat{t}_\beta\}_{\beta \in \Theta}$ of $\{\hat{t}_\gamma\}_{\gamma \in \Lambda}$ in P_{n-1}; say, $\lim_\beta \hat{t}_\beta = (t_1, \ldots, t_n) \in P_{n-1}$. It follows that

$$\begin{aligned}\lim_\beta h_{n+1}(x_\beta, t_\beta) &= \lim_\beta \lambda(h_n(\hat{x}_\beta, \hat{t}_\beta), x_{n+1}^\beta, t_{n+1}^\beta) \\ &= \lambda(h_n(\hat{x}, (t_1, \ldots, t_n)), x_{n+1}, 1) = x_{n+1}\end{aligned}$$

(because, by inductive hypothesis, we know that h_n is continuous).

We immediately conclude from (i) and (ii) that any net $\{(x_\alpha, t_\alpha)\}_{\alpha \in \Lambda}$ which converges to (x, t) has a subnet $\{(x_\beta, t_\beta)\}_{\beta \in \Theta}$ such that $\lim_\beta h_{n+1}(x_\beta, t_\beta) = x_{n+1} = h_{n+1}(x, t)$. By Lemma 2.3, h_{n+1} is continuous at (x, t).

Cases 1 and 2 show that h_{n+1} is continuous. The fact that the h_n satisfy conditions (a), (b) and (d) of Definition 2.2 of [3] is proved in Theorem 3.1 of [3] (of course, the continuity of the h_n is much more stronger than (b)).

For any space X and positive integer n, let X_*^n denote the subspace of the cartesian product X^n which consists of all points in X^n with distinct coordinates. It is clear that if X is Hausdorff then X_*^n is an open subspace of X^n. Let T denote the relation on X^n defined by $(x_1, \ldots, x_n) T (y_1, \ldots, y_n)$ if and only if there exists $\sigma \in S_n$ (the symmetric group on $\{1, \ldots, n\}$) such that $(y_1, \ldots, y_n) = (x_{\sigma(1)}, \ldots, x_{\sigma(n)})$. It is well-known that the quotient map $\nu_n : X^n \to X^n/T$ is an open and closed (i.e. clopen) map. Let $\mu_n = \nu_n | X_*^n$; since X_*^n is an open inverse set under ν_n, we immediately see that μ_n is also clopen.

For metric spaces, the following result is essentially due to V. Klee, by very different methods (cf. Ex. A on p. 271 of [2]).

Proposition 2.5. *Let X be a Hausdorff space. For $n = 1, 2, \ldots$ there exists a clopen subspace O_n of X_*^n such that*

(i) *$\mu_n | O_n$ is a homeomorphism,*
(ii) *$\mu_n(O_n) = \mu_n(X_*^n)$,*
(iii) *$(x_1, \ldots, x_n) \in O_n$ implies that $(x_1, \ldots, x_{n-1}) \in O_{n-1}$.*

Proof. By induction, assume that the subspaces $O_1, \ldots, O_{n-1}$ have been found so that (i)–(iii) are satisfied, and let us define O_n: Let $\mathcal{S}$ be the collection of all open subsets S of X^n_* such that $\mu_n|S$ is one-to-one (therefore, a homeomorphism, because μ_n is an open map) and S satisfies (iii). Note that $\mathcal{S} \neq \emptyset$. (Pick $(x_1, \ldots, x_n) \in X^n_*$ such that $(x_1, \ldots, x_n) \in O_{n-1}$ and open neighborhoods N_k of x_k, $k = 1, \ldots, n$, such that $N_i \cap N_j = \emptyset$ whenever $i \neq j$, and $N_1 \times \ldots \times N_{n-1} \in O_{n-1}$. Then $N_1 \times \ldots \times N_n \in \mathcal{S}$.) Partially order $\mathcal{S}$ by inclusion and let $\mathcal{N}$ be a nest in $\mathcal{S}$. Clearly, $\bigcup \mathcal{N} \in \mathcal{S}$; therefore, by Zorn's Lemma, let O_n be a maximal element of $\mathcal{S}$. Clearly, O_n satisfies (iii), and O_n satisfies (i) because O_n is open (so $\mu_n|O_n$ is an open one-to-one map).

O_n satisfies (ii): Suppose not. Then there exists $x = (x_1, \ldots, x_n) \in X^n_*$ such that $x \in O_n^- - O_n$ and $\mu_n(x) \notin \mu_n(O_n)$. (Simply pick $y \in \mu_n(X^n_*) - \mu_n(O_n)$ such that $y \in \mu_n(O_n)^-$. Then $\mu^{-1}(y) \cap O_n^- \neq \emptyset$, because μ_n is a closed map.) It follows that $(x_1, \ldots, x_{n-1}) \in O_{n-1}$. Pick a net $\{x_\beta = (x_{\beta 1}, \ldots, x_{\beta n})\}_{\beta \in \Lambda}$ in O_n such that $\lim_\beta x_\beta = x$. Then $\lim_\beta \hat{x}_\beta = \hat{x} = (x_1, \ldots, x_{n-1})$. Since O_{n-1} is a closed subspace of X^{n-1}_*, we get $\hat{x} \in O_{n-1}$.

Since $(x_1, \ldots, x_{n-1}) \in O_{n-1}$, there exist open neighborhoods N_i of x_i, $i = 1, \ldots, n$, such that $N_1 \times \ldots \times N_{n-1} \subset O_{n-1}$ and $N_i \cap N_j = \emptyset$ whenever $i \neq j$. Therefore, for each $(z_1, \ldots, z_n) \in (N_1 \times \ldots \times N_n) \cap O_n$, $(z_1, \ldots, z_{n-1}) \in O_{n-1}$. Hence, letting $O'_n = O_n \cup (N_1 \times \ldots \times N_n)$, we conclude that $O'_n \in \mathcal{S}$ and O_n is a proper subset of O'_n, which contradicts the maximality of O_n; hence, O_n satisfies (ii).

In order to complete the proof, we need only show that O_n is also a closed subspace of X^n_*: Suppose not. Pick $(z_1, \ldots, z_n) \in X^n_*$ such that $(z_1, \ldots, z_n) \in O_n^- - O_n$. Pick an open neighborhood $N_1 \times \ldots \times N_n$ of $(z_1, \ldots, z_n)$ such that $N_i \cap N_j = \emptyset$ whenever $i \neq j$. Letting $O'_n = O_n \cup (N_1 \times \ldots \times N_n)$, we easily deduce that $\mu_n|O'_n$ is a one-to-one map; however, since O_n satisfies (ii), this is impossible, a contradiction which completes the proof.

DEFINITION 2.6. For each $(x_1, \ldots, x_n) \in X^n_*$, $(x_{\mu(1)}, \ldots, x_{\mu(n)})$ will denote the point of O_n such that $\mu_n(x_1, \ldots, x_n) = \mu_n(x_{\mu(1)}, \ldots, x_{\mu(n)})$. This defines a function $p : X^n_* \to O_n$ by $p(x_1, \ldots, x_n) = (x_{\mu(1)}, \ldots, x_{\mu(n)})$.

LEMMA 2.7. *The function $p : X^n_* \to O_n$ is an open continuous map.*

Proof. Simply note that $p(N_1 \times \ldots \times N_n) = N_{\mu(1)} \times \ldots \times N_{\mu(n)}$, for any open subsets $N_1, \ldots, N_n$ of X such that $N_i \cap N_j = \emptyset$ whenever $i \neq j$.

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