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SOME ADDITIVE PROPERTIES OF SPECIAL SETS OF REALS

BY

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D. H. Fremlin and J. Jasiński [4] have proved a relative consistency of the existence of a very thin set of reals. In this context they have asked (private communication) the following question: Given a universally null set $X \subseteq \mathbb{R}$ and a Borel measure μ on $\mathbb{R}$, does it follow that there exists a Borel set $B \subseteq \mathbb{R}$ covering X such that for every $t \in \mathbb{R}$, $\mu(B + t) = 0$? Note that the answer is in the affirmative if X has strong measure zero (due to the uniform continuity of Borel measures). In Theorem 1 we provide a negative answer to this question.

The thin set of Fremlin and Jasiński mentioned above preserves many properties of thinness under linear sums. Fremlin and Jasiński asked whether its linear sums with any universally null set are universally null. In Theorem 2 we show that the answer is in the negative.

For the definition of strong measure zero sets and basic properties of other special sets considered in this paper see A. W. Miller [9] or J. B. Brown and G. V. Cox [1]. Recall only that a set $X \subseteq \mathbb{R}^k$ is *universally null* ($X \in \beta$) if for every Borel measure μ (continuous probability measure on the family of all Borel subsets of $\mathbb{R}^k$), $\mu^*(X) = 0$. For $X, Y \subseteq \mathbb{R}$ we set $X + Y = \{x + y : x \in X \wedge y \in Y\}$ and $X - Y = \{x - y : x \in X \wedge y \in Y\}$.

Most of the results of this note are based on von Neumann's theorem that there exists a perfect set of reals which is linearly independent over the rationals [11]. The basic technical lemma is the following:

LEMMA 1. *Let C and D be F_σ (resp. compact) subsets of $\mathbb{R}$. Suppose that $X \subseteq C$ and $(C - X) \cap (D - D) = \{0\}$. Then*

(a) $+ : X \times D \to X + D$ *is a Borel isomorphism* (*resp. homeomorphism*),

(b) *if $y_x = d_x + x$, $d_x \in D$, $x \in X$, then $Y = \{y_x : x \in X\}$ is the preimage of X by a one-to-one Borel* (*resp. continuous*) *function and $\{d_x : x \in X\} \subseteq Y - X$; in particular, if X is universally null, so is Y.*

Proof. (a) Clearly $+$ is continuous on $C \times D$. Since $(C - X) \cap (D - D) =$

$\{0\}$ we have

$$\begin{aligned} X\times D \subseteq &\{(c,d)\in C\times D: \\ &(\forall (c',d')\in C\times D)((c',d')\neq (c,d)\Rightarrow +(c',d')\neq +(c,d))\}. \end{aligned}$$

So, $+$ is one-to-one on $X\times D$. Moreover, for any $F\subseteq C\times D$,

$$+[F\cap (X\times D)] = +[F]\cap +[X\times D] = +[F]\cap (X+D).$$

If F is F_σ (resp. closed) then so is $+[F]$. It follows that $+$ sends relative F_σ (resp. closed) subsets of $X\times D$ to relative F_σ (resp. closed) subsets of $X+D$.

(b) Note that by (a), $y_x \mapsto (x,d_x)$ is a Borel (resp. continuous) function. Also $(x,d_x)\mapsto x$, being a projection, is continuous.

THEOREM 1. *Assume that there exists a universally null set X with $|X| = \mathfrak{c}$. Then there exists a Borel measure μ on $\mathbb{R}$ and a universally null set $Y\subseteq\mathbb{R}$ such that whenever B is a Borel set covering Y, then $\mu(B+t)=1$ for some $t\in\mathbb{R}$.*

Proof. Let C and D be disjoint, perfect subsets of $\mathbb{R}$ such that $C\cup D$ is linearly independent over the rationals (von Neumann [11]). We can assume that $X\subseteq C$. Let B_x, $x\in X$, be all Borel sets. For every $x\in X$, choose $y_x\in C+x$ so that $y_x\notin B_x$ whenever $(C+x)\setminus B_x\neq\emptyset$. By Lemma 1, $Y=\{y_x : x\in X\}$ is universally null. Also, $y_x\in B_x$ for any Borel set $B_x\supseteq Y$, so $(C+x)\setminus B_x=\emptyset$ by our choice of y_x, and hence $C\subseteq B_x-x$. It follows that if for a Borel measure μ we have $\mu(C)=1$ then μ satisfies the conclusion of the theorem.

THEOREM 2. *Let $X\subseteq\mathbb{R}$, $|X|=\mathfrak{c}$, be a universally null set for which there exists a meagre F_σ set $C\supseteq X$ such that $C-X$ is meagre. Then there exists a universally null set $Y\subset\mathbb{R}$ such that $X+Y$ is not universally null.*

Proof. By a theorem of Mycielski [10], if $G\subseteq\mathbb{R}$ is a dense G_δ with $0\in G$ then there exists a perfect set $D\subseteq\mathbb{R}$ such that $D-D\subseteq G$. So, in our case, we can find a perfect set D such that $(C-X)\cap(D-D)=\{0\}$. Let $\{d_x : x\in X\}=D$. By Lemma 1, $Y=\{x+d_x : x\in X\}$ is universally null and $D\subseteq Y-X$. So $Y-X$ is not universally null.

THEOREM 3. *Assume Martin's Axiom. For every $X\subseteq\mathbb{R}$ with $|X|=\mathfrak{c}$ there exists a universally null set $Y\subseteq\mathbb{R}$ such that $X+Y$ is not universally null.*

Proof. Suppose that $X\subseteq\mathbb{R}$ with $|X|=\mathfrak{c}$ is such that $X+Y$ is universally null for every universally null set $Y\subseteq\mathbb{R}$. Let μ_α, $\alpha<\mathfrak{c}$, be all Borel measures on $\mathbb{R}$. For each α, let P_α^ξ, $\xi<\mathfrak{c}$, be all dense G_δ sets of μ_α measure zero, and let $Q_\alpha^\xi=\bigcap_{\eta\le\xi}P_\alpha^\eta$.

CLAIM 1. *For each α there is $\xi(\alpha)$ such that $X + Q_\alpha^{\xi(\alpha)} \subseteq P_\alpha^{\xi(\alpha)}$.*

Proof. Fix α. If $X + Q_\alpha^\xi \not\subseteq P_\alpha^\xi$, then choose $y_\xi \in Q_\alpha^\xi$ such that $X + y_\xi \not\subseteq P_\alpha^\xi$. If this can be done for all $\xi < \mathfrak{c}$, then $Y = \{y_\xi : \xi < \mathfrak{c}\}$ is a generalized Lusin set, hence a universally null set. Also no P_α^ξ covers $X+Y$, so $\mu_\alpha^*(X + Y) > 0$ and $X + Y$ is not universally null, a contradiction.

CLAIM 2. *There are α and $t \in \mathbb{R}$ such that $|X \setminus (P_\alpha^{\xi(\alpha)} + t)| = \mathfrak{c}$.*

Proof. Suppose not, and let $\mathbb{R} = \{t_\alpha : \alpha < \mathfrak{c}\}$. Let $x_\alpha \in X \cap (t_\alpha + \bigcap_{\beta\le\alpha} P_\beta^{\xi(\beta)})$ and $y_\alpha = t_\alpha - x_\alpha$. Then $-y_\alpha \in \bigcap_{\beta\le\alpha} P_\beta^{\xi(\beta)}$, so $Y = \{y_\alpha : \alpha < \mathfrak{c}\}$ is universally null. Also $X + Y = \mathbb{R}$, so $X + Y$ is not universally null, contrary to our assumption.

Now, fix α and t as in Claim 2. Let $X_0 = X \setminus (P_\alpha^{\xi(\alpha)} + t)$ and $C = \mathbb{R} \setminus (P_\alpha^{\xi(\alpha)} + t)$. Then $|X_0| = \mathfrak{c}$, $X_0 \subseteq C$, C is a meagre F_σ set and $C - X_0 \subseteq \mathbb{R} \setminus (Q_\alpha^{\xi(\alpha)} + t)$ is also meagre. So, by Theorem 2, there is a universally null set Y such that $X_0 + Y$, and hence $X + Y$, is not universally null, a contradiction.

Our next aim is to prove the existence of some special subspaces of $\mathbb{R}$. Similar problems where investigated independently by W. F. Pfeffer and K. Prikry [12]. Let us recall some definitions.

Let $X \subseteq \mathbb{R}^k$. X is called a *λ-set* ($X \in \lambda$) if every countable subset of X is a relative G_δ. X is called *always of the first category* ($X \in \mathcal{K}^*$) if for any perfect set P the set $X \cap P$ is meagre in P. $X \in \widetilde{\mathcal{K}}^*$ if for every $Y \subseteq \mathbb{R}^n$ such that there exists a one-to-one Borel function $f : Y \to X$, we have $Y \in \mathcal{K}^*$.

For $W \subseteq \mathbb{R}$ let $W^{(n)} = \{(w_1, \ldots, w_n) \in W^n : w_i < w_j \text{ for } i < j \le n\}$.

LEMMA 2. *Let $Z \subseteq \mathbb{R}$ be a perfect set linearly independent over $\mathbb{Q}$ and let $\tau = (q_1, \ldots, q_n)$ be a finite sequence of non-zero rational numbers. Then the function $f_\tau : Z^{(n)} \to \mathbb{R}$, $f_\tau(z_1, \ldots, z_n) = \sum_{i=1}^n q_i z_i$, is continuous, one-to-one, and for every F_σ set $F \subseteq Z^{(n)}$ the set $f_\tau(F)$ is an F_σ in $\mathbb{R}$.*

Proof. The continuity of f_τ is obvious. The linear independence of Z implies that f_τ is one-to-one. The last assertion follows from the continuity of f_τ and σ-compactness of $Z^{(n)}$.

For $X \subseteq \mathbb{R}$ let $((X))$ be the linear space over $\mathbb{Q}$ generated by X.

THEOREM 4. *If $Z \subseteq \mathbb{R}$ is a perfect set linearly independent over $\mathbb{Q}$ then for every $X \subseteq Z$ the following hold*:

1) *If $X \in \beta$ then $((X)) \in \beta$.*
2) *If $X \in \lambda$ then $((X)) \in \lambda$.*
3) *If $X \in \widetilde{\mathcal{K}}^*$ and $|X| \le \omega_1$ then $((X)) \in \widetilde{\mathcal{K}}^*$.*

P r o o f. Observe that if $X \subseteq Z$ then

$$((X)) = \bigcup_{n\in\omega\setminus\{0\}} \bigcup_{\tau\in(\mathbb{Q}\setminus\{0\})^n} f_\tau(X^{(n)}) \cup \{0\}.$$

If $X \in \beta$ then $X^{(n)} \in \beta$, and since f_τ is a Borel isomorphism, $f_\tau(X^{(n)}) \in \beta$. Thus also $((X)) \in \beta$.

If $X \in \lambda$ then $X^{(n)} \in \lambda$, and, by Lemma 2, $f_\tau(X^{(n)}) \in \lambda$. Notice that $f_\tau(X^{(n)}) \subseteq f_\tau(Z^{(n)})$ and whenever $\sigma \in (\mathbb{Q}\setminus\{0\})^k$ and $\tau \in (\mathbb{Q}\setminus\{0\})^n$, $\sigma \neq \tau$ implies $f_\tau(Z^{(n)}) \cap f_\sigma(Z^{(k)}) = \emptyset$. In this case a countable union of λ-sets is a λ-set. Thus $((X)) \in \lambda$.

If $X \in \widetilde{\mathcal{K}}^*$ then $X^{(n)} \in \widetilde{\mathcal{K}}^*$ (see E. Grzegorek [7]). By Lemma 2, $f_\tau(X^{(n)}) \in \widetilde{\mathcal{K}}^*$, and as $\widetilde{\mathcal{K}}^*$ is a σ-ideal we have $((X)) \in \widetilde{\mathcal{K}}^*$.

The following theorem is a version of a theorem of Erdős, Kunen and Mauldin [2]. Our theorem is weaker but no hypothesis besides ZFC is required.

THEOREM 5. *There exist universally null*, *linear spaces* $X, Y \subseteq \mathbb{R}$ *over* $\mathbb{Q}$ *such that* $X \cap Y = \{0\}$ *and* $X + Y$ *is not universally null.*

P r o o f. Let C_1 and D_1 be disjoint, perfect subsets of $\mathbb{R}$ such that $C_1 \cup D_1$ is linearly independent over the rationals (von Neumann [11]). By a result of Grzegorek [6], there are sets $X_1 \subseteq C_1$ and $Z_1 \subseteq D_1$ such that $|X_1| = |Z_1|$, X_1 is universally null and Z_1 is not universally null. Let $g : X_1 \to Z_1$ be a bijection. Let $C = ((C_1))$, $D = ((D_1))$, $X = ((X_1))$, $Z = ((Z_1))$. Then C and D are F_σ sets, $(C - C) \cap (D - D) = \{0\}$ and $X \subseteq C$, $Z \subseteq D$. Moreover, g can be extended to a linear isomorphism between X and Z. By Lemma 2, X is universally null. Let $Y = \{x + g(x) : x \in X\}$. Then Y is a linear space over the rationals, $X \cap Y = \{0\}$, and, by Lemma 1, Y is universally null. Also $Y - X \supseteq Z$, so $Y - X$ is not universally null.

We conclude with a number of results saying that $\mathbb{R}$ may be expressed as a linear sum of some special sets and Lebesgue null sets.

Let $\mathcal{S} \subseteq \bigcup_{n\in\omega\setminus\{0\}} P(\mathbb{R}^n)$. We say that $\mathcal{S}$ has *property* $(*)$ if whenever $Y \in \mathcal{S}$ and $f : X \to Y$ is a one-to-one continuous function, then $X \in \mathcal{S}$. Observe that β, λ, $\widetilde{\mathcal{K}}^*$, $\beta \cap \lambda$, and $\widetilde{\mathcal{K}}^* \cap \beta$ have property $(*)$.

Let m be the Lebesgue measure on $\mathbb{R}$.

THEOREM 6. *Let* $Z \subseteq \mathbb{R}$ *be a perfect set linearly independent over* $\mathbb{Q}$, *let* C *and* D *be perfect disjoint compact subsets of* Z *and let* $\mathcal{S}$ *be a family with property* $(*)$. *Suppose there exist* $T \subseteq D$, $G \subseteq \mathbb{R}$ *with* $m(G) = 0$, $T + G = \mathbb{R}$, *and a set of reals* $X \in \mathcal{S}$ *with* $|X| = |T|$. *Then there exists a set of reals* $Y \in \mathcal{S}$ *and a set* $V \subseteq \mathbb{R}$ *with* $m(V) = 0$ *such that* $Y + V = \mathbb{R}$.

Recall that a set $Y \subseteq \mathbb{R}$ does not have strong measure zero iff there exists a meagre set $V \subseteq \mathbb{R}$ such that $Y + V = \mathbb{R}$ (see F. Galvin, J. Mycielski and R. Solovay [5]). The corollaries below show that certain special sets are not necessarily of strong first category (see [9], p. 210).

The proof of the following lemma is similar to the proof of Lemma 9 of P. Erdős, K. Kunen and R. Mauldin [2].

LEMMA 3. *For every* $H \subseteq \mathbb{R}$ *with* $m(H) = 0$ *and for every perfect set* E *there exists a perfect set* $E_1 \subseteq E$ *such that* $m(H - E_1) = 0$.

Proof of Theorem 6. Let $E_1 \subseteq C$ be such that $m(G - E_1) = 0$. We may assume that $X \subseteq E_1$. Let $g : X \stackrel{\text{onto}}{\to} T$. As $\mathcal{S}$ has property $(*)$, $\operatorname{graph}(g) \in \mathcal{S}$. The function $h : E_1 \times D \to E_1 + D$, $h(e, d) = e + d$, is a homeomorphism (Lemma 1). Let $Y = h(\operatorname{graph}(g))$ and $V = G - E_1$. Clearly $Y \in \mathcal{S}$ and $m(V) = 0$. We will show that $V + Y = \mathbb{R}$. If $z \in \mathbb{R}$ then there are $t \in T$ and $a \in G$ such that $z = t + a$. There is $x \in X$ such that $g(x) = t$, so $x + t \in Y$. Obviously $a - x \in G - E_1 = V$, thus $(a - x) + (x + t) = z \in V + Y$.

LEMMA 4 (P. Erdős, K. Kunen and R. Mauldin). *If* $T \subseteq \mathbb{R}$ *is not always of first category*, *then there is a set* G *with* $m(G) = 0$ *such that* $T + G = \mathbb{R}$.

Proof. See the proof of Theorem 3 of [2].

COROLLARY 1. *There are* $Y \in \widetilde{\mathcal{K}}^*$ *and* $G \subseteq \mathbb{R}$ *with* $m(G) = 0$ *such that* $Y + G = \mathbb{R}$.

Proof. By a theorem of E. Grzegorek [7] there are sets $T \notin \mathcal{K}^*$ and $X \in \widetilde{\mathcal{K}}^*$ such that $|T| = |X|$. We can assume that $T \subseteq D$. The statement now follows from Lemma 4 and Theorem 6.

Recall that $X \subseteq \mathbb{R}^k$ is called a *Q-set* if every subset of X is a relative F_σ.

COROLLARY 2. *It is consistent that there exist a* Q*-set* $Y \subseteq \mathbb{R}$ *and* $G \subseteq \mathbb{R}$ *with* $m(G) = 0$ *such that* $Y + G = \mathbb{R}$.

Proof. Notice that the family of Q-sets has property $(*)$. W. G. Fleissner and A. W. Miller [3] proved that it is consistent that there exist a Q-set X and a Lusin set T_0 with $|T_0| < |X|$. It follows that there are a set $T \subseteq D$ of second category in D and a Q-set X such that $|T| < |X|$.

Remark. Similar results for the families λ, $\mathcal{K}^* \cap \beta$, $\lambda \cap \beta$, etc. may be obtained under the assumption of CH or MA.

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